RENEWAL PROCESSES
AND
POISSON PROCESSES

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Anno Accademico 1997-1998
Renewal Processes

A renewal process is a point process characterized by the fact that the successive inter-arrival times $\tau_1, \tau_2, \tau_3$ etc... are distributed with the same cdf $F(t)$ with density $f(t)$ and expected value $E[\tau] = m_1$:

$$F(t) = \Pr\{\tau \leq t\} ; \quad f(t) = \frac{d F(t)}{d t}$$

$$E[\tau] = m_1$$

Let us define the corresponding Laplace transforms:

$$f^*(s) = \mathcal{L}[f(t)] ; \quad F^*(s) = \mathcal{L}[F(t)]$$

The $\tau_i$ form a sequence of independent identically distributed random variables.

Let $s_k$ denote the time up to the $k$ arrival:

$$s_k = \sum_{i=1}^{k} \tau_i$$

Let $F_k(t)$ denote the cdf of $s_k$, and $f_k(t)$ its density.

$$F_k(t) = \Pr\{s_k \leq t\} ; \quad f_k(t) = \frac{d F_k(t)}{d t}$$

In terms of Laplace transforms, we obtain:

$$F_k^*(s) = \frac{1}{s} [f^*(s)]^k ; \quad f_k^*(s) = [f^*(s)]^k$$
Renewal Processes - number of arrivals

Let $N(t)$ denote the number of arrivals in the interval $0 - t$.

$$N(t) < k \text{ if and only if } s_k > t$$

from which:

$$Pr\{N(t) < k\} = Pr\{s_k > t\} = 1 - F_k(t) \quad (2)$$

$$Pr\{N(t) = k\} = Pr\{N(t) < k + 1\} - Pr\{N(t) < k\} = F_k(t) - F_{k+1}(t) \quad (3)$$

Expected number of arrivals in $(0 - t)$, $H(t)$:

$$H(t) = E[N(t)] = \sum_{k=0}^{\infty} k Pr\{N(t) = k\} \quad (4)$$

$$= \sum_{k=0}^{\infty} k [F_k(t) - F_{k+1}] = \sum_{k=1}^{\infty} F_k(t)$$

In Laplace transforms:

$$H^*(s) = \sum_{k=1}^{\infty} F_k^*(s) = \frac{1}{s} \sum_{k=1}^{\infty} f^k(s)$$
The Renewal Equation

The renewal density $h(t)$ is defined as:

$$h(t) = \lim_{\Delta t \to 0} \frac{Pr\{\text{one or more events occur in } (t - t + \Delta t)\}}{\Delta t}$$

The probability that the $k$-th event occurs in $(t - t + \Delta t)$ is:

$$Pr\{k - \text{th event occurs in } (t - t + \Delta t)\} = f_k(t) + O(\Delta t)$$

Hence:

$$h(t) = \sum_{k=1}^{\infty} f_k(t) = \frac{dH(t)}{dt}$$

In Laplace transform:

$$h^*(s) = \sum_{k=1}^{\infty} f_k^*(s)$$

$$= f^*(s) + [f^*(s)]^2 + \ldots + [f^*(s)]^k + \ldots$$

$$= \frac{f^*(s)}{1 - f^*(s)}$$

From which we obtain the renewal equation in LT and time domain:

$$h^*(s) = f^*(s) + h^*(s) \cdot f^*(s)$$

$$h(t) = f(t) + \int_0^t h(t - u) \cdot f(u) \, du$$
The Fundamental Renewal Equation

In terms of the renewal function $H(t)$, the renewal equation can be derived in the following way.

In Laplace transform:

$$H^*(s) = \frac{h^*(s)}{s} = \frac{1}{s} \frac{f^*(s)}{1 - f^*(s)}$$

In a similar way, we can write:

$$H^*(s) = \sum_{k=1}^{\infty} F_k^*(s) = \frac{1}{s} \sum_{k=1}^{\infty} f^*^k(s)$$

$$= \frac{1}{s} \left\{ f^*(s) + f^*^2(s) + \ldots + f^*^k(s) + \ldots \right\} = \frac{1}{s} \frac{f^*(s)}{1 - f^*(s)}$$

From the above we derive:

$$H^*(s) = \frac{f^*(s)}{s} + H^*(s) f^*(s)$$

In time domain:

$$H(t) = F(t) + \int_0^t H(t-u) \cdot f(u) \, du$$

This is known as the fundamental renewal equation.
Poisson Process

A Poisson process is a renewal process in which the inter-arrival times are exponentially distributed with parameter $\lambda$.

$$f(t) = \lambda e^{-\lambda t} \implies f^*(s) = \frac{\lambda}{s + \lambda}$$

The cdf and density of the time up to the $k$-th arrival $s_k$ are in LT:

$$f_k^*(s) = \left(\frac{\lambda}{s + \lambda}\right)^k ; \quad F_k^*(s) = \frac{\lambda^k}{s(s + \lambda)^k}$$

$$f_1(t) = \mathcal{L}^{-1}\left[\frac{\lambda}{s + \lambda}\right] = \lambda e^{-\lambda t}$$

$$f_2(t) = \mathcal{L}^{-1}\left[\frac{\lambda^2}{(s + \lambda)^2}\right] = \lambda^2 t e^{-\lambda t}$$

$$\cdots \cdots \cdots$$

$$f_k(t) = \mathcal{L}^{-1}\left[\frac{\lambda^k}{(s + \lambda)^k}\right] = \frac{\lambda (\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}$$

$$F_k(t) = \int_0^t f_k(u) \, du = 1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$
Poisson Process

Let us define:

\[ P_k(t) = Pr\{N(t) = k\} = F_k(t) - F_{k+1}(t) \]

Taking Laplace transforms:

\[ P^*_k(s) = \frac{\lambda^k}{s(s + \lambda)^k} - \frac{\lambda^{k+1}}{s(s + \lambda)^{k+1}} = \frac{\lambda^k}{(s + \lambda)^{k+1}} \]

Inverting again in the time domain, we obtain the Poisson distribution:

\[ P_k(t) = Pr\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \]
Poisson Distribution

\[ P_k(t) = \Pr\{N(t) = k\} \]

\[ \lambda t = 1 \]

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<th>( k )</th>
<th>( P_k )</th>
</tr>
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<td>0.368</td>
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<tr>
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<td>2</td>
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<td>6</td>
<td>( 5.1 \cdot 10^{-4} )</td>
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<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>10</td>
<td>( 1.01 \cdot 10^{-7} )</td>
</tr>
</tbody>
</table>
Poisson Distribution

\[ \lambda t = 4 \]

\[ P_k(t) = Pr\{N(t) = k\} \]

\[
\begin{array}{c|c}
  k & P_k \\
  \hline 
  0 & 0.0183 \\
  1 & 0.073 \\
  2 & 0.146 \\
  3 & 0.195 \\
  4 & 0.195 \\
  5 & 0.156 \\
  6 & 0.104 \\
  7 & 0.059 \\
  8 & 0.029 \\
  9 & 0.013 \\
  10 & 0.005 \\
\end{array}
\]
Poisson Process

Expected number of events

\[ H(t) = E[N(t)] = \sum_{k=0}^{\infty} k \Pr\{N(t) = k\} = \sum_{k=0}^{\infty} k P_k(t) \]

\[ = e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k - 1)!} \]

\[ = \lambda t \cdot e^{-\lambda t} (1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots) = \lambda t \]

An alternative derivation in Laplace transforms:

\[ H^*(s) = E^*[N(s)] = \frac{1}{s} \frac{f^*(s)}{1 - f^*(s)} \]

Since in the Poisson process:

\[ f^*(s) = \frac{\lambda}{s + \lambda} \quad \implies \quad E^*[N(s)] = \frac{\lambda}{s^2} \quad \text{(5)} \]

\[ \implies \quad h^*(s) = \frac{\lambda}{s} \]

We obtain:

\[ H(t) = E[N(t)] = \lambda t \]

\[ h(t) = \lambda \]
Buffer design in a Poisson Process

Jobs arrive according to a Poisson process of rate $\lambda$ and must be stored in a buffer during an interval $T$.

The design problem consists in evaluating the number of slots in a buffer such that the probability of refusing an incoming job in the interval $T$ is less than a prescribed (small) risk $\alpha$ \quad (0 \leq \alpha \leq 1).

Let $N(T)$ be the number of arrivals in the interval $T$ and let $K$ be the number of slots to be determined.

The design problem can be formulated as:

\[
Pr\{N(T) > K\} \leq \alpha
\]

\[
Pr\{N(T) \leq K\} > 1 - \alpha
\]

\[
Pr\{N(T) \leq K\} = \sum_{j=0}^{K} \left( \frac{\lambda T}{j!} \right)^j e^{-\lambda T} > 1 - \alpha
\]

The value of $K$ is the smallest integer that satisfies the above equation.
Poisson Process
An alternative derivation

A counting process \( N(t) \) is a stochastic point process identified by a sequence of random points in time. The state of the process at time \( t \) is defined by the number of arrivals \( N(t) \) in the interval \( (0 - t) \).

Let \( E_i(t) \) be the event \( N(t) = i \), i.e. \( i \) arrivals in \((0 - t)\).

According to the figure, the event \( E_i(t + \Delta t) \) can be decomposed into the sequence of independent and mutually exclusive events:

\[
E_i(t + \Delta t) = \{E_i(t), \text{ no arrivals in } \Delta t\}
\]

\[
+ \{E_{i-1}(t), \text{ 1 arrival in } \Delta t\}
\]

\[
+ \{E_{i-2}(t), \text{ 2 arrivals in } \Delta t\}
\]

\[
+ \cdots
\]
Poisson Process
An alternative derivation

By the theorem of the total probability, we can write:

\[
Pr\{N(t + \Delta t) = i\} = Pr\{\text{no arrivals in } \Delta t \mid N(t) = i\} \cdot Pr\{N(t) = i\} \\
+ Pr\{1 \text{ arrival in } \Delta t \mid N(t) = i - 1\} \cdot Pr\{N(t) = i - 1\} \\
+ Pr\{2 \text{ arrivals in } \Delta t \mid N(t) = i - 2\} \cdot Pr\{N(t) = i - 2\} \\
+ \cdots
\]
Poisson Process
An alternative derivation

A stochastic point process $N(t)$ is a Poisson process, if the probability of having one event in any interval $dt$ is constant and equal to $\lambda$.

By the theorem of the total probability, we can write for $i > 0$:

$$Pr\{N(t + \Delta t) = i | N(t) = i\} = 1 - \lambda \Delta t + O(\Delta t)$$

$$Pr\{N(t + \Delta t) = i | N(t) = i - 1\} = \lambda \Delta t + O(\Delta t)$$

Where:

$$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0$$

For $i = 0$, we can write:

$$Pr\{N(t + \Delta t) = 0 | N(t) = 0\} = 1 - \lambda \Delta t + O(\Delta t)$$
Poisson Process

An alternative derivation

Let us define:

\[ P_i(t) = Pr\{N(t) = i\} \]

According to the above relations we can write:

\[
\begin{cases}
    P_0(t + \Delta t) = (1 - \lambda \Delta t) P_0(t) & i = 0 \\
    P_i(t + \Delta t) = (1 - \lambda \Delta t) P_i(t) + \lambda \Delta t P_{i-1}(t) & i > 0
\end{cases}
\]

\[
\begin{cases}
    \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) & i = 0 \\
    \frac{P_i(t + \Delta t) - P_i(t)}{\Delta t} = -\lambda P_i(t) + \lambda P_{i-1}(t) & i > 0
\end{cases}
\]

Taking the limit \( \Delta t \longrightarrow 0 \), a Poisson process is characterized by the following set of linear differential equations:

\[
\begin{cases}
    \frac{d P_0(t)}{dt} = -\lambda P_0(t) & i = 0 \\
    \frac{d P_i(t)}{dt} = -\lambda P_i(t) + \lambda P_{i-1}(t) & i > 0
\end{cases}
\]

with initial conditions:

\[
\begin{cases}
    P_0(0) = 1 & i = 0 \\
    P_i(0) = 0 & i > 0
\end{cases}
\]
Poisson Process
An alternative derivation

Taking Laplace transforms of the time domain differential equations:

\[
\begin{align*}
    s P_0^*(s) - 1 &= -\lambda P_0^*(s) & i = 0 \\
    s P_i^*(s) &= -\lambda P_i^*(s) + \lambda P_{i-1}^*(s) & i > 0
\end{align*}
\]

\[
P_0^*(s) = \frac{1}{s + \lambda}
\]

\[
P_1^*(s) = \frac{\lambda}{s + \lambda} \cdot P_0^*(s) = \frac{\lambda}{(s + \lambda)^2}
\]

\[
\begin{align*}
    \ldots \\
    P_i^*(s) &= \frac{\lambda}{s + \lambda} \cdot P_{i-1}^*(s) = \frac{\lambda^i}{(s + \lambda)^{i+1}} \\
    \ldots
\end{align*}
\]

From the above equations we derive again the Poisson distribution:

\[
P_k(t) = Pr\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\]
Superposition of Poisson Processes

A superposition of Poisson processes is obtained by cumulating the occurrences of \( n \) independent sources of Poisson processes with parameters \( \lambda_1, \lambda_2, \ldots, \lambda_n \), respectively.

The cumulated process is still a Poisson process:

\[
Prob\{ N(t + \Delta t) = k + 1 \mid N(t) = k \} = \lambda_1 \Delta t + \lambda_2 \Delta t + \ldots + \lambda_n \Delta t + O(t)
\]

The parameter of the cumulated Poisson process is given by:

\[
\lambda = \sum_{i=1}^{n} \lambda_i
\]
Decomposition of Poisson Processes

A Poisson process with parameter $\lambda$ is split into $n$ independent branches according to a probability distribution $p_1, p_2, \ldots, p_n$ with:

$$\sum_{i=1}^{n} p_i = 1$$

By decomposing the original Poisson process of parameter $\lambda$, $n$ Poisson processes are generated: $N_1(t), N_2(t), \ldots, N_n(t)$, with parameters: $\lambda p_1, \lambda p_2, \ldots, \lambda p_n$.

$$Pr\{N_i(t + \Delta t) = k + 1 \mid N_i(t) = k\} = p_i \lambda \Delta t + O(\Delta t)$$
Alternating Renewal Processes

The process is constituted by a sequence of Type I variables $X'$ with density $f_1(x)$ followed by a Type II variables $X''$ with density $f_2(x)$. The process starts with probability 1 with a Type I variable.

If we look at the sequence formed by the occurrence of the Type II variables, the process is an ordinary renewal process with inter-arrival time ($X' + X''$).

The mean number of Type II occurrences satisfies (in LT):

$$H^*_2(s) = \frac{f_1^*(s) \cdot f_2^*(s)}{s \{ 1 - f_1^*(s) \cdot f_2^*(s) \}}$$

For the Type I occurrences we have a modified renewal process, for which the expected number of renewals is:

$$H^*_1(s) = \frac{f_1^*(s)}{s \{ 1 - f_1^*(s) \cdot f_2^*(s) \}}$$

In both cases:

$$h^*_i(s) = s \cdot H^*_i(s) \quad i = 1, 2$$
Alternating Renewal Processes

Let be:

$\pi_1(t)$ - Probability Type I variable occurs at time $t$

$\pi_2(t)$ - Probability Type II variable occurs at time $t$

Type I is in use at time $t$ if:

a) - No Type I event occurs in $(0 - t)$;

b) - A Type II event occurs in $u - u + du$ ($u < t$), and no Type I events occur in $(t - u)$:

$$\pi_1(t) = [1 - F_1(t)] + \int_0^t h_2(u) [1 - F_1(t - u)] du$$

In Laplace transform:

$$\pi_1^*(s) = \frac{1 - f_1^*(s)}{s} + \frac{f_1^*(s) \cdot f_2^*(s)}{1 - f_1^*(s) \cdot f_2^*(s)} \frac{1 - f_1^*(s)}{s}$$

$$\pi_1^*(s) = \frac{1 - f_1^*(s)}{s \{ 1 - f_1^*(s) \cdot f_2^*(s) \}}$$

Note also that:

$$\pi_1^*(s) = H_2^*(s) - H_1^*(s) + \frac{1}{s}$$

$$\pi_1(t) = H_2(t) - H_1(t) + 1$$
Alternating Poisson Process

*Type I* variable is exponential with rate $\lambda$;
*Type II* variable is exponential with rate $\mu$.

\[
\pi_1^*(s) = \frac{s + \mu}{s(s + \lambda + \mu)}
= \frac{\mu}{\lambda + \mu} \cdot \frac{1}{s} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{s + \lambda + \mu}
\]

\[
\pi_1(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu) t}
\]

\[
\pi_2^*(s) = \frac{\lambda}{s(s + \lambda + \mu)}
\]

\[
\pi_2(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu) t}
\]

\[
\lim_{t \to \infty} \pi_1(t) = \frac{\mu}{\lambda + \mu} ; \quad \lim_{t \to \infty} \pi_2(t) = \frac{\lambda}{\lambda + \mu}
\]