Proceedings Int. AMSE Conf. "Modelling & Simulation", Athens, June 27-29, 1984, Vol. 1.1, p. 173-192

DISCRETE STATE STOCHASTIC SYSTEMS WITH PHASE TYPE DISTRIBUTED TRANSITION TIMES

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Abstract

Stochastic system operation is very commonly represented by a labeled directed transition graph, whose nodes denote the possible distinct system states and whose labeled arcs indicate the activities that cause the state to change after an associated stochastic delay.

If the stochastic delays are not exponentially distributed, the underlying stochastic process becomes non-Markov, and its analytical or numerical solution is extremely complex. When however the distribution functions are of Phase type (PH), under quite general assumptions, it is possible to build up a homogeneous Markov process defined over an extended but still discrete state space, whose state probabilities are very simply related to the state probabilities of the original graph.

This paper is primarily aimed to show how to expand the labeled graph, given the individual PH type distributions modelling the stochastic delay associated with each arc.

I.- INTRODUCTION

In various areas of systems engineering, such as queuing theory, performance analysis and reliability, it is common practice to represent the operation of a stochastic system by a labeled directed graph. The nodes of the graph denote the possible distinct system states and the labeled arcs indicate the activities or events that cause the state to change after an associated stochastic delay.

In the simplest case of exponentially distributed delay times, the underlying stochastic process is a continuous time homogeneous Markov process, whose properties are well established. In this case the transition graph provides a complete description of the process, once the time independent transition rates of the exponential distributions are given.

With general distributions, however the graph representation is often not sufficient to uniquely determine the system behaviour versus time, but additional assumptions must be invoked.

Indeed by assuming that after each change of state the process evolution does not depend on the past but only on the current state and on the time already elapsed in that state we get a Markov renewal process, or semi-Markov process, for which general formulas are again available [1, 2].

Yet, the assumption which leads to a semi-Markov process is deceptive in a large number of actual systems; in particular, when there are activities that evolve in parallel such that the completion of one of them causes a state change but does not have any influence on the execution of the other ones. In these situations the process depends not only on the current state but also on the time each activity has already worked before the last transition; we refer to these processes as age memory processes, since each activity must keep track of its age throughout the state transitions.

A classical way for coping with this type of non-Markov processes is to attach to each event a supplementary variable [3] which accounts for its age up to the current time.

By defining a new state space for the process as the cartesian product of the original discrete state space and of the continuous domains of the supple

mentary variables the process becomes markovian and the equations for the state probabilities can be formally written down. However the analytical or numerical solution of these equations turns out to be extremely complex.

A different method, also known as stage device, is based on the use of a special class of distribution functions, called Phase type (PH) distributions. PH distributions [4] are those of the time till absorption of discrete state continuous time homogeneous Markov processes with at least one absorbing state.

In fact it has been shown in [5] that PH distributions can be considered as a mathematical artifice for discretizing the supplementary variables such that we can find a still discrete (even if expanded) state space over which the original non-Markov process becomes homogeneous Markov.

This paper is primarily aimed to show how stochastic processes with PH distributions can be handled in practice, and in particular:

i) how the expanded state space can be automatically built up starting from the original labeled transition graph, and given the PH distributions associated to each activity;

ii) what kind of actual situations can be conveniently modeled and analyzed by mean of this technique;

iii) numerical advantages and limitations of the technique.

II.- THE CLASS OF DISTRIBUTIONS OF PHASE TYPE (PH)

PH distributions [4] are the cumulative distribution functions (Cdf) of the time till absorption of continuous time homogeneous Markov processes with discrete state space and with at least one absorbing state.

We assume in the following that the Markov process has ν transient states and a single absorbing state numbered $\nu+1$; ν is called the order of the PH distribution.

Let us denote by A the $(\nu+1 \times \nu+1)$ time independent transition rate matrix of the process, whose generic labeled transition graph is reported in Fig. 1.

Clearly, we have

$$a_{ik} \geq 0 \qquad i \neq k \qquad \sum_{i=1}^{\nu+1} a_{ik} = 0 \qquad \forall k \tag{1}$$

and furthermore, since $\nu+1$ is absorbing,

$$a_{i,\nu+1} = 0 \qquad \forall i \tag{2}$$

We adhere to the convention of representing probability vectors by column vectors, so that a_{ik} is the transition rate from state k to state i.

Let R be the initial probability vector satisfying the conditions

$$r_i \geq 0 \qquad \forall i \qquad \sum_{i=1}^{\nu+1} r_i = 1 \tag{3}$$

If $r_{\nu+1} > 0$ the distribution will have an atom of mass $r_{\nu+1}$ at the origin. We shall exclude this possibility in the following since it would mean, in our case, that one activity has a non zero probability of completing in zero time.

From the above the state probability vector $P(t) = \left[p_1(t), \ldots, p_{\nu+1}(t)\right]^T$ is given by the forward equation

$$\dot{P}(t) = AP(t) \tag{4}$$

with initial condition $P(0) = R$; solving eq. (4) we get

$$P(t) = e^{At} R \tag{5}$$

The Cdf of the time till absorption is given by:

$$F(t) = p_{\nu+1}(t) = 1 - \sum_{i=1}^{\nu} p_i(t) \tag{6}$$

Finally, if $a_{\nu+1}$ is the $(\nu+1)$-th row of A, the corresponding density is given by:

$$f(t) = \dot{p}_{\nu+1}(t) = a_{\nu+1} P(t) \tag{7}$$

A PH distribution can be equivalently described by its analytical expression (6) or by the labeled transition graph of Fig. 1 representing the Markov process from which eq. (6) originates; in applied system modelling it is convenient to use the graphical description.

The properties of PH distributions have been investigated by a number

of authors [4, 6, 7, 8, 9, 10] and the simplest subclass of PH distributions are commonly used in applied stochastic modeling. In fact the exponential distribution is the PH distribution of order $\nu = 1$; the Erlang distribution of order ν is the PH distribution of a series of ν identical exponential stages with a final absorbing state, and the Hyperexponential distribution derives from a Markov model with ν states in parallel, connected to a single absorbing state.

Cox [6] proved that PH distributions of order ν admit of a canonical representation of the form displayed in Fig. 2 (each state is only connected to the subsequent one and to the final one) with complex valued transition rates. However the converse is not true since no manageable necessary and sufficient conditions are known for establishing whether a model like the one of Fig. 2 yields a probability distribution upon application of eq. 6.

Cumani [7] has further shown that when the model of Fig. 2 is a true homogeneous Markov model (i.e. with real non negative transition rates) it can represent all and only the PH distributions of order ν described by acyclic graphs (called triangular PH distributions or TPH since the states can always be ordered in such a way that the transition rate matrix A becomes lower triangular)

The following points emphasize the most useful properties of PH distributions in applied stochastic modelling:

i) TPH (and thus PH) distributions can approximate as closely as desired any Cdf as ν increases [7]

ii) Satisfactory fits are however obtained with low ν [11]; the strongest limitation is that the minimum squared coefficient of variation of a TPH distribution of order ν is $1/\nu$ (achieved by the Erlang of order ν). No analogous result is known for PH distributions even though it is known that this limit can be slightly overcome [6].

iii) The use of the canonical representation (Fig. 2) of a PH (or TPH) distribution keeps at the minimum the number of free parameters (i.e. the number of arcs in the associated graph)

III.- NON EXPONENTIAL STOCHASTIC MODELS

Let us suppose that the system operation is represented within a discrete state space and that the corresponding labeled directed graph is given.

Fig. 3 is, for instance, the state transition diagram of the failure process of a two component system. From state 1 (two components up) we pass either to state 2 upon failure of component 1 or to state 3 upon failure of component 2. The arc labels denote in this case whose component failure causes the state change.

Let us further suppose that from the system specifications we are able to inferring the Cdf's of the random times associated with each activity when considered alone.

We denote by $G_k(t)$ the Cdf of activity a_k

Looking back to the example of Fig. 3, we mean that the a priori knowledge about the system is restricted to the time-to-failure distributions of the individual components.

If from the transition graph we know that more than one arc depart from a node, in order to univocally determine the future system behaviour, further assumptions must be set up regarding the time the system sojourns in that node and the choice of the next path.

A very usual assumption, which is inherent in homogeneous Markov modelling, is that the sojourn time in state S_i is a random variable distributed as the minimum of the random variables associated to the arcs outgoing from S_i, and thus the process chooses as its future jump the one which comes stochastically first in time.

With reference to Fig. 4, if the m activities outgoing from S_o with Cdf $G_k(t)$ $(k = 1, 2, \ldots, m)$, are s-independent, from probability theory [12] we know that the Cdf $G_o(t)$ of the minimum of m random variables is given by:

$$1 - G_o(t) = \prod_{k=1}^{m} \left[1 - G_k(t)\right] \qquad (8)$$

The probability that the next move is toward state k and occurs before epoch t has the following expression:

$$\text{Prob}\,(S_o \longrightarrow S_k \text{ occurs at } x \leq t \mid S_o \text{ at } t = 0) = \int_0^t \prod_{\substack{j=1 \\ j \neq k}}^{m} \left[1 - G_j(x)\right] f_k(x)\,dx \qquad (9)$$

where $f_k(x)$ is the density of $G_k(x)$ (supposed differentiable).

Other policies for the choice of the future path can be devised but the one which leads to eqs. (8) and (9) (and which we call race policy) is the more significant in a wide variety of real world systems.

For what concerns the type of memory the process should hold after each change of state two alternative assumptions are usually considered in stochastic modelling:

Assumption 1: (Semi-Markov): each activity starts a new as the state changes (i.e., changes of state correspond to renewal epochs)

Assumption 2: (age memory): each activity keeps track of its age (i.e. of the time it has been enabled since its last completion)

The consequences on the process behaviour of the two different assumptions are often disregarded since the memoryless property of the commonly used exponential distribution hides the effect of recording the age of the activity.

Assumption 1 leads to semi-Markov processes, whose solution with general distributions is still available in closed form in the Laplace transform domain [2].

This assumption, however, does not properly correspond to the actual operation of many physical systems.

Take again, for instance, the two component system of Fig. 3. Assumption 1 would mean that as state 2 is entered component 2 behaves as if it were as good as new. But clearly if component 2 wears-out, its failure probability in state 2 is given by its residual life distribution given it has already worked for a time equal to the lifetime of component 1. And this is exactly what we mean under assumption 2.

Given that state S_o is entered at $t = 0$, under assumption 2 we know the age x_k of each activity a_k. Thus we can calculate the Cdf of the residual life distribution of a_k which is given by:

$$G_k(t \mid x_k) = \frac{G_k(t + x_k) - G_k(x_k)}{1 - G_k(x_k)} \qquad (10)$$

Substituting the m Cdf's obtained from eq. (10) into eqs. (8) and (9) we model the situation in which the sojourn in S_o is determined by the minimum of the residual lifes of the m activities.

By defining a new state space partly discrete (the distinct states of the original graph) and partly continuous (the cartesian product of the supplementary variables x_k) the process becomes a Markov process [5] whose equations can be formally written down. However the solution of these equations is extremely complex even in simple systems.

IV.- STOCHASTIC MODELS WITH PH DISTRIBUTED TRANSITION TIMES

We now suppose that each individual Cdf $G_k(t)$ is of Phase type, and that the corresponding transition graph is given. Let $X(t)$ be the random process defined by these distributions on the original system transition graph, whose state space is denoted by Γ.

We show that there exists an expanded discrete state space Ω on which a homogeneous vector Markov process $Y(t)$ can be constructed, and that this new process is equivalent to the original process $X(t)$ in the sense that there is a one-to-one correspondence between states in Γ and proper, disjoint subsets of Ω so that the probability of $X(t)$ in the states of Γ equals the probability of $Y(t)$ being in the corresponding subsets of Ω.

Moreover we show that a quite simple algorithm can be devised for building up the expanded state space, both in the case of the semi-Markov assumption and in the case of the age memory assumption.

We begin by showing how to expand a single node with more than one outgoing arc. For the sake of simplicity we refer again to Fig. 4.

Let $F^{(k)}(t)$ be the Cdf, of order ν_k, related to activity a_k; let $A^{(k)}$ be the corresponding Markov transition rate matrix and $P^{(k)}$ the probability vector. From eq. (6) the survival function $Sf^{(k)}$ of a_k is:

$$Sf^{(k)}(t) = 1 - F^{(k)}(t) = \sum_{i=1}^{\nu_k} p_i^{(k)}(t) \qquad (11)$$

and the corresponding density is:

$$f^{(k)}(t) = a^{(k)}_{\nu+1} P^{(k)}(t) = \sum_{i=1}^{\nu_k} a^{(k)}_{\nu+1,i} p^{(k)}_i(t) \qquad (12)$$

where $a^{(k)}_{\nu+1}$ is the $(\nu+1)$-th row of $A^{(k)}$

Let us now consider the system graphically depicted in Fig. 5, and made up of the m activities, evolving in parallel. The state of this system is an m-vector $z(t) = \left[z_1(t), z_2(t), \ldots, z_m(t)\right]$ where each z_k is 0 or 1 according to whether the corresponding activity is in S_{ok} or S_{1k}. Under the correspondence:

$$\begin{aligned} X(t) = S_o \triangleq z(t) &= \left[0, 0, \ldots, 0\right] \\ X(t) = S_k \triangleq z(t) &= \left[0, 0, \ldots, 1, \ldots, 0\right] \quad (1 \text{ in position } k),\ k = 1, 2, \ldots, m \end{aligned} \qquad (13)$$

it is clear that $z(t)$ is equivalent, in the above sense, to $X(t)$ up to, and including, the epoch of first passage from S_o to a state in $\{S_1, S_2, \ldots, S_m\}$ (notice that the probability of two activities ending simultaneously is zero).

On the other hand, it is not difficult to construct a Markov process $Y(t)$ equivalent to $z(t)$ for all t. To this end, if $H_k = \{1, 2, \ldots, \nu_k+1\}$ is the state space of the PH distribution associated to α_k, the state space Ω of $Y(t)$ is given by the cartesian product

$$\Omega = H_1 \times H_2 \times \ldots \times H_m \qquad (14)$$

Each state in Ω can be characterized by a vector $U = \left[u_1, u_2, \ldots, u_m\right]$ where for all k, $u_k \in \left[1, \nu_k+1\right]$. The Markov transition graph of $Y(t)$ is constructed by connecting any two states $\omega_1, \omega_2 \in \Omega$, differing only in one component of U (say $u_j : u_{j1}, u_{j2}$) by the corresponding arc (if it exists) between states u_{j1} and u_{j2} in the PH model of activity α_j.

With $m = 2$ and assuming α_1 and α_2 distributed with a canonical PH of order 2 ($\nu_1 = \nu_2 = 2$), the expanded state space Ω, with the proper state connections, is reported in Fig. 6. The labels inside the states represent the values of the vector U.

An elegant and algorithmically compact way for constructing the transition rate matrix of $Y(t)$ requires the use of Kronecker algebra for matrices.

We first define the two basic operations (product and sum) of Kronecker algebra, if B is a (pxq) matrix and C a (rxl) matrix, their Kronecker product is

the (prxqs) matrix D:

$$D = B \otimes C = \begin{bmatrix} b_{11}C & \ldots\ldots\ldots & b_{1m}C \\ \vdots & & \vdots \\ b_{n1}C & \ldots\ldots\ldots & b_{nm}C \end{bmatrix} \qquad (15)$$

If B is a square (pxp) matrix and C a square (qxq) matrix, their Kronecker sum is the (pqxpq) matrix D:

$$D = B \oplus C = (B \otimes I_{qq}) + (I_{pp} \otimes C) \qquad (16)$$

where I_{pp}, I_{qq} are identity matrices of orders p, q respectively. Both operations are associative.

Following Amoia et al. [13], we define the state probability vector of $Y(t)$ over Ω as:

$$Q(t) = P^{(1)}(t) \otimes P^{(2)}(t) \otimes \ldots P^{(m)}(t) \qquad (17)$$

Notice that the generic entry of $Q(t)$, corresponding to state $U = \left[u_1, u_2, \ldots, u_m\right]$, has the form:

$$Q_u(t) = P^{(1)}_{u_1}(t)\, P^{(2)}_{u_2}(t) \ldots P^{(m)}_{u_m}(t) \qquad (18)$$

and therefore, due to the assumed s-independence of the activities, it is equal to the joint probability of α_1 being in state u_1, α_2 being in state u_2, etc. in the system of Fig. 5.

It is not hard to show [13], using eq. (4), that $Q(t)$ satisfies the standard forward equation

$$\dot{Q}(t) = A\, Q(t) \qquad (19)$$

with

$$A = A^{(1)} \oplus A^{(2)} \oplus \ldots \oplus A^{(m)} \qquad (20)$$

which can be shown to correspond to the expanded graph constructed as above.

Let us now consider the following partition of Ω:

$$\Omega_o = \{U \mid 1 \le u_k \le \nu_1, \quad k = 1, 2, \ldots, m\}$$
$$\Omega_k = \{U \mid u_k = \nu_{k+1}, \ 1 \le u_j \le \nu_j, \ j \neq k\}, \quad k = 1, 2, \ldots, m \tag{21}$$
$$\Omega_p = \Omega - (\Omega_o \cup \Omega_1 \cup \ldots \cup \Omega_m)$$

and construct a modified process $\bar{Y}(t)$ on Ω by making absorbing all states in Ω_1, Ω_2, ..., Ω_k and Ω_p (the modified transition graph of the example of Fig. 6 is reported in Fig. 7) The state probability vector $\bar{Q}(t)$ of $\bar{Y}(t)$ satisfies:

$$\dot{\bar{Q}}(t) = \bar{A}\,\bar{Q}(t) \tag{22}$$

where $\bar{A}$ is obtained from eq. (20) by setting to zero the appropriate entries. By suitably renumbering the states in Ω, eq. (22) can be written in partitioned form:

$$\begin{bmatrix} \dot{\bar{Q}}_o \\ \dot{\bar{Q}}_1 \\ \vdots \\ \dot{\bar{Q}}_m \\ \dot{\bar{Q}}_P \end{bmatrix} = \begin{bmatrix} A_{oo} & 0 \\ A_{1o} & 0 \\ \vdots & \\ A_{mo} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Q}_o \\ \bar{Q}_1 \\ \vdots \\ \bar{Q}_m \\ \bar{Q}_P \end{bmatrix} \tag{23}$$

We now prove the following

Theorem. The Markov process $\bar{Y}(t)$ defined on the expanded state space Ω is equivalent to the original process $X(t)$ under the correspondence

$$S_o \triangleq \Omega_o$$
$$S_k \triangleq \Omega_k, \quad k = 1, 2, \ldots, m \tag{24}$$

Proof. Since S_1, S_2, ..., S_m are absorbing states for $X(t)$ and so are the sets Ω_1 Ω_2, ..., Ω_m for $\bar{Y}(t)$, it will suffice to prove that Prob. $\{\bar{Y}(t) \in \Omega_o\}$ equals eq. (8) and that the probability of $X(t)$ jumping from S_o to S_k before epoch t equals eq. (9).

We begin by noting that the partition A_{oo} of $\bar{A}$ is by definition identical to the corresponding partition of A (eq. 20) so that $\bar{Q}_o(t) = Q_o(t)$ for all t. But

$$\text{Prob } \{\bar{Y}(t) \in \Omega_o\} = \mathcal{Sf}_{\Omega_o}(t) = \sum_{\omega \in \Omega_o} Q_\omega(t) \tag{25}$$

Remembering eq. (18), after a simple manipulation eq. (25) can be written as

$$\mathcal{Sf}_{\Omega_o} = \prod_{k=1}^{m} \sum_{i=1}^{\nu_j} p_i^{(k)} \tag{26}$$

which in view of eq. (11) becomes

$$\mathcal{Sf}_{\Omega_o} = \prod_{k=1}^{m} \mathcal{Sf}^{(k)} \tag{27}$$

and proves the first part of the theorem.

To prove the second part, we must show that Prob $\{\bar{Y}(t) \in \Omega_k\}$ equals expression (9). Without loss of generality, we can assume k = 1. Let us partition the probability vectors $P^{(i)}$ of the individual activities as

$$P^{(i)} = \begin{bmatrix} P_G^{(i)} \\ \cdots \\ P_{\nu+1}^{(i)} \end{bmatrix} \tag{28}$$

Then $\bar{Q}_1$ of eq. (23) can be written as

$$\bar{Q}_1 = P_{\nu+1}^{(1)} \otimes M \tag{29}$$

where

$$M = P_G^{(2)} \otimes P_G^{(3)} \otimes \ldots \otimes P_G^{(m)}$$

is a vector with $N_1 = \nu_2 \times \nu_3 \times \ldots \times \nu_m$ entries (and so is $\bar{Q}_1$). Similarly,

$$\bar{Q}_o = P_G^{(1)} \otimes M \tag{30}$$

Now, by definition

$$\text{Prob } \{\bar{Y}(t) \in \Omega_1\} = \sum_{j=1}^{N_1} \bar{Q}_{1j}(t) = P_{\Omega_1}(t) \tag{31}$$

The state evolution equation for $\bar{Q}_1$, from eq. (23) can be rewritten as

$$\frac{d}{dt}\bar{Q}_1(t) = (\bar{a}_{\nu+1}^{(1)} \otimes I_{N1,N1})\,(P_G^{(1)} \otimes M) \tag{32}$$

where $\bar{a}_{\nu+1}^{(1)}$ is the $(\nu+1)$-th row of $A^{(1)}$, which, after summation of the entries of $\bar{Q}_1$ and some manipulation, yields

$$\frac{d}{dt} P_{\Omega_1}(t) = \sum_{j=1}^{\nu_1} a^{(1)}_{\nu+1,j} p^{(1)}_j \sum_{i=1}^{N_1} M_i \qquad (33)$$

By noting that

$$\sum_{i=1}^{N_1} M_i(t) = \prod_{k=2}^{m} Sf^{(k)}(t) \qquad (34)$$

and

$$\sum_{i=1}^{\nu_1} a^{(1)}_{\nu+1,j} p^{(1)}_j = f^{(1)}(t) \qquad (35)$$

we finally have, for the density of the probability of absorption into Ω_1

$$\frac{d}{dt} P_{\Omega_1}(t) = f^{(1)}(t) \prod_{k=2}^{m} Sf^{(k)}(t) \qquad (36)$$

which, upon integration, coincides with eq. (9) and completes the proof.

In order to go on with the expansion of the original labeled transition graph, we assume the states $S_1 \ldots S_k$ as initial nodes and proceed along the same lines examined for node S_o. However the following should be remarked.

If the process is semi Markov we lump the absorbing states Ω_k into a single state which is the starting point for the expansion of S_k.

If, on the contrary, the age memory policy is at hand we keep the states in Ω_k distinct such that each of them remembers the state that each activity a_i ($i \neq k$) has reached in its own PH model at the epoch of the jump from S_o to S_k.

V – EXAMPLE

The expansion algorithm is illustrated on a simple two component series system with a single repair crew whose labeled graph is reported in Fig. 8: in S_o the two components are up while S_1 and S_2 represent the failure of component 1 or 2, respectively. Labels 1f and 2f denote the failure of 1 or 2 and 1r and 2r repair.

Very often when repair is undertaken the system is switched off, and when the system is started up again, only the repaired component is as good as new, and the others behave following their residual life distribution.

If the life time Cdf's of the two components are Erlang of order 2 and repair is exponential, the expanded graph is reported in Fig. 9. The subsets Ω_o Ω_1 Ω_2 corresponding to the original states S_o, S_1 and S_2 are encircled by a dotted line.

It is graphically evident that from Ω_1 the only possible action is the repair of component 1 which however does not modify the state attained by component 2.

A deeper analysis of the above system, together with some other examples of actual system models with PH distributions are reported in [11].

VI.– DISCUSSION AND CONCLUSIONS

The technique examined in the paper is proposed as a tool for approximating stochastic processes with generally distributed transition times, by means of continuous time discrete state homogeneous Markov processes, under the very broad assumptions stated in Sect. III.

The price we should pay is obviously the increase in the model size. From eq. (14) it is recognized that the model size grows geometrically with the order of the individual PH distributions.

For example if each component of the system of Fig. 3 has two states (an up and a down state) the system has $2^n = 4$ states; if each component is modeled with a 3– state PH distribution the number of system states becomes $3^n = 9$.

A natural question arises: what is the maximum size of a homogeneous Markov process we are able to solve? Clearly the answer depends on many factors but as a rough indication we can quote the results reported in two recent papers on the subject [14, 15]. Both papers, which exploit the sparsity of the transition rate matrix, but use quite different numerical integration techniques, claim to be able to solve systems with up to about 10^5 states.

Moreover work is in progress for properly partitioning the state space, (typically in the presence of stiffness) in order to get approximate solutions over subsets of the original state space.

Thus it can be expected that the development of ever more powerful computing machines and the contemporary refinement of software techniques will make it possible to resort to the proposed discretization techniques in highly dimensioned problems.

As a second issue in favour of this technique we want to stress how easily it can be used in applied modelling. In fact what we only need to know is the system labeled graph, that can be obtained directly or by resorting to more powerful graphical techniques as for instance Petri Nets [16].

Furthermore we need to know the Cdf's of the random variables associated with the different activities. All the subsequent analytical treatment, which consists in:

- the state expansion
- the solution of the final homogeneous Markov process
- the evaluation of the relevant process measures,

is done in a completely automated way by a computer program. The mathematics associated to the technique is thus transparent to the analyst who only has to provide the original labeled graph.

Moreover when the Cdf's of the various activities are state dependent, the analytical solution by means of supplementary variables arises very intricated theoretical questions which have not received deep attention in the literature.

In the state expansion this problem does not increase the complexity of the algorithm, since the state dependence is simply modelled by properly altering the numerical values of the transition rates in the corresponding PH distributions.

The proposed discretization technique seems thus to trade off successfully between the need of having powerful and flexible models and their analytical tractability.

REFERENCES

[1] R. PYKE: "Markov renewal processes: definitions and preliminary properties" Annals Math. Statistics, 32, 1231-1242 (1961).

[2] R. PYKE: "Markov renewal processes with finitely many states", Annals Math Statistics, 32, 1243-1259 (1961).

[3] D.R. COX, H.D. MILLER: "The theory of stochastic processes", Chapman and Hall, London (1965).

[4] H.F. NEUTS: "Probability distributions of Phase type", Purdue University, Report no. 374 (1974).

[5] D.R. COX: "The analysis of non-markovian stochastic processes by the inclusion of supplementary variables", Proc. Cambridge Phyl. Soc., 51, 433-440 (1955).

[6] D.R. COX: "A use of complex probabilities in the theory of stochastic processes", Proc. Cambridge Phyl. Soc., 51, 313-319 (1955).

[7] A. CUMANI: "On the canonical representation of homogeneous Markov processes modelling failure-time distributions", Microelectron Reliab., 22, 583-602 (1982).

[8] C. SINGH, R. BILLINTON, S.T. LEE: "The method of stages for non-Markov models", IEEE Trans. Reliab., R-26, 135-137 (1977).

[9] D. ASSAF, B. LEVIKSON: "Closure of phase type distributions under operations arising in reliability theory", Annals Probab., 10, 265-269 (1982).

[10] A. BOBBIO, A. CUMANI: "A Markov approach to wear-out modelling", Microelectron Reliab., 23, 113-119 (1983).

[11] A. BOBBIO, A. CUMANI: "Modelling wear-out by multistate homogeneous Markov processes", In Reliability in Electrical and Electronic components and systems (Proc. EUROCON '82), E. Lauger and J. Møltoft eds., North Holland p.c., Amsterdam (1982).

[12] A. PAPOULIS: "Probability, random variables and stochastic processes", Mc Graw Hill, New York (1965).

[13] V. AMOIA, G. DE MICHELI, M. SANTOMAURO: "Computer-oriented formulation of transition rate matrices via Kronecker algebra", IEEE Trans. Reliab., R-30, 123-132 (1981).

[14] D.R. MILLER: "Reliability calculations using randomization for markovian fault-tolerant computing systems", Proc. Int. Sym. Fault Tolerant Computing FTCS-13, 284-289 (1983).

[15] C.A. CLAROTTI, F. PICCHIA, V. COMINCIOLI, L. CUERRI, C. DE COLA: "MARKAN: an ad-hoc designed code for integrating Markov stiff equations", Proc. 4-th EUREDATA Conference, paper 18.2, Venice (1982).

[16] M. AJMONE MARSAN, A. BOBBIO, G. CONTE, A. CUMANI: "Performance analysis of degradable multiprocessor systems using generalized stochastic Petri nets", IEEE Computer Society, Distributed processing technical Committee Newsletter, 6, SI-1, Jun. (1984).

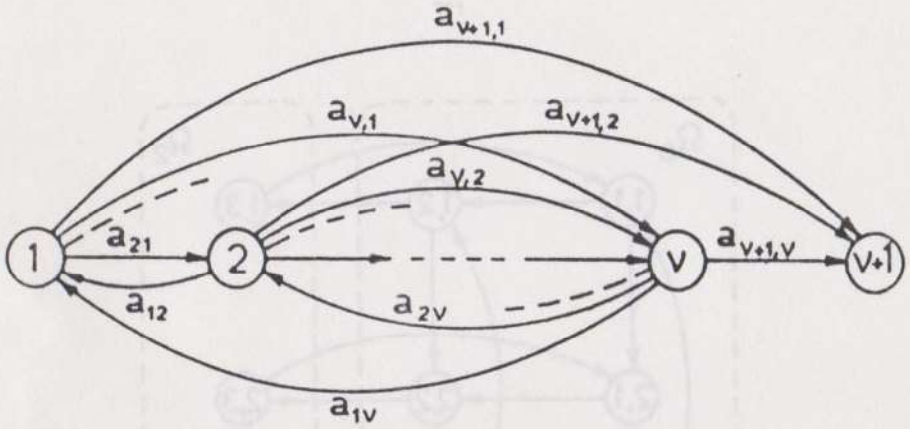

Fig. 1 – Transition graph of a PH distribution.

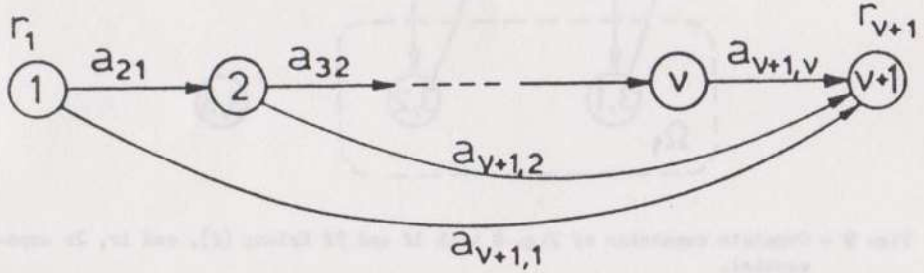

Fig. 2 – Canonical form for PH distributions.

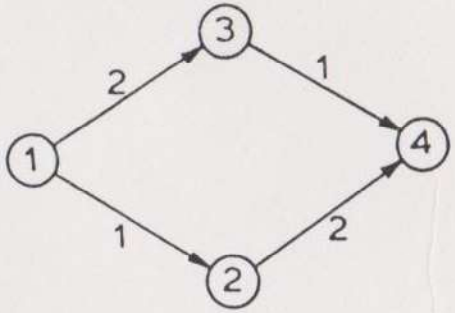

Fig. 3 – State transition graph of a system with two components subject to failure.
Node (state) labels: 1 = C1, C2 up; 2 = C1 down, C2 up; 3 = C1 up, C2 down; 4 = C1, C2 down.
Arc (activity) labels: 1 = C1 failure; 2 = C2 failure.

Fig. 4 – Subgraph of a generic graph restricted to the first jump out of node S_o

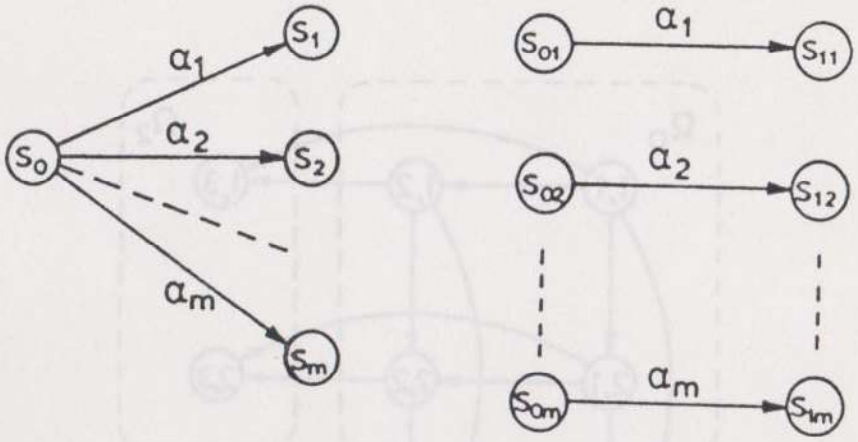

Fig. 5 – Parallel system graph equivalent to that of Fig. 4 up to the first jump.

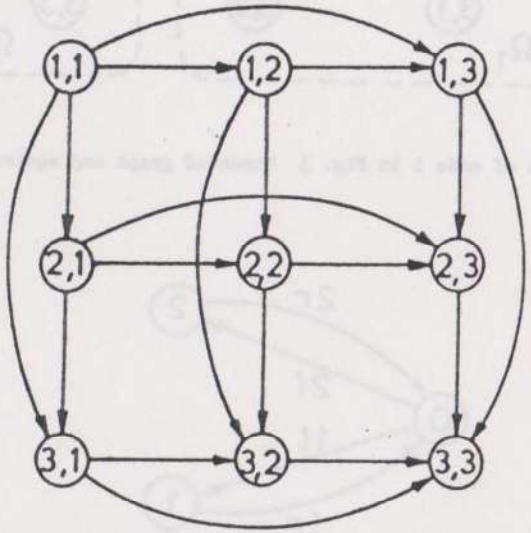

Fig. 6 – Expanded graph of Fig. 3 with both activities distributed as a canonical PH of order two.

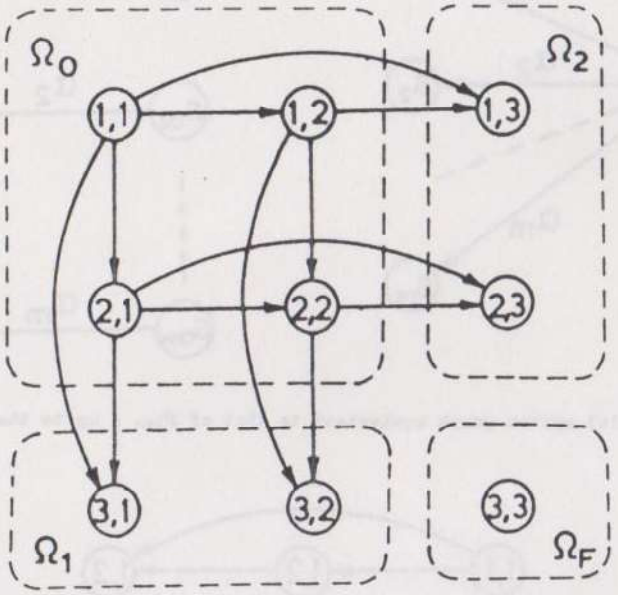

Fig. 7 – Expansion of node 1 in Fig. 3 Expanded graph and equivalent sets of states.

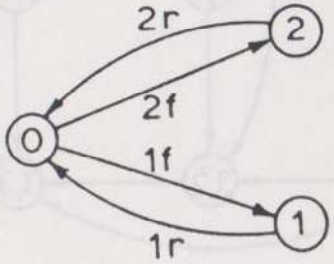

Fig. 8 – Graph of a two-component series system with maintenance. f = component failure; = component repair.

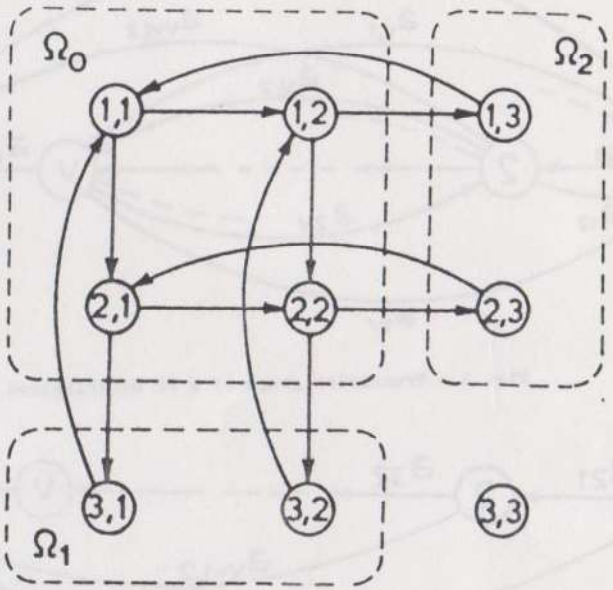

Fig. 9 – Complete expansion of Fig. 8 with 1f and 2f Erlang (2), and 1r, 2r exponential.