

Topological effective field theories for Dirac fermions from index theorem

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Dirac fermions have a central role in high energy physics but it is well known that they emerge also as quasiparticles in several condensed matter systems supporting topological order. We present a general method for deriving the topological effective actions of (3+1) massless Dirac fermions living on general backgrounds and coupled with vector and axial–vector gauge fields. The first step of our strategy is standard (in the Hermitian case) and consists in connecting the determinants of Dirac operators with the corresponding analytical indices through the zeta–function regularization. Then, we introduce a suitable splitting of the heat kernel that naturally selects the purely topological part of the determinant (*i.e.* the topological effective action). This topological effective action is expressed in terms of gauge fields using the Atiyah–Singer index theorem which computes the analytical index in topological terms. The main new result of this paper is to provide a consistent extension of this method to the non Hermitian case where a well–defined determinant does not exist. Quantum systems supporting relativistic fermions can thus be topologically classified on the basis of their response to the presence of (external or emergent) gauge fields through the corresponding topological effective field theories.

PACS numbers: 71.10.-w, 73.43.-f; 11.15.-q

INTRODUCTION

The issue of connections between topological quantum field theories and condensed matter systems has been intensively investigated over the years in a variety of different contexts [1–3]. For example, in two space dimensions the occurrence of topological order in ground states of Hamiltonians [4] and the existence of quasiparticle excitations associated with fractional statistics [5] are crucial features of a variety of many–body microscopic systems. The existence of such kind of phases of matter represents a challenge also for theoretical physics because they cannot be described by Landau theory of spontaneous symmetry breaking and thus require more sophisticated mathematical tools.

Besides the case of quantum Hall states which are described by Chern–Simons–type topological effective field theories [6, 7], topological phases of matter have been recently recognized or conjectured to occur in different physical substrata, such as topological insulators, topological superconductors, graphene and cold atoms in optical lattices (see *e.g.* [8–11]).

In several cases, Dirac fermions emerge as quasiparticles evolving in the bulk. It is worth recalling that before the discoveries of graphene [12] and topological insulators [13], it was not generally thought that relativistic quantum field theory could be so relevant in solid–state physics. However, it is by now generally accepted that the bulk of topological insulators and superconductors can be described by gapped Dirac Hamiltonians [14] and that the appearance of relativistic massless particles in the bulk is a hallmark of certain gapless topological

phases [15–17].

The topological properties together with the relativistic dynamics of fermions provide strong evidence that the low energy effective theories can be identified with topological quantum field theories such as Chern–Simons and BF theories [18], the latter being defined in both (2+1) and (3+1)–dimensions. Indeed, actions of topological effective field theories (TEFT) describe universal global properties of the physical states and the coefficient of the topological terms can be identified with the topological order parameter of the system.

It is therefore of crucial importance to single out the specific bosonic field theories corresponding to different microscopic fermionic systems. TEFTs emerge through the path integral quantization prescription by integrating out the fermionic degrees of freedom in the Dirac actions with fermions coupled to suitably chosen gauge fields.

In this article we present a general method, firstly outlined in [19], for deriving the topological part of the effective actions of (3+1) massless Dirac fermions living on general (flat or curved) background manifolds and coupled with vector and axial–vector gauge fields. When an axial–vector field is coupled with fermions, the corresponding Dirac operators are non–Hermitian and we propose a consistent way of defining also in this case a corresponding TEFT. Although the total effective action, defined as usual as a regularized determinant, might be affected by anomalies, our definition of its topological part (see Eqs. (27)–(35) below) is gauge and axial invariant.

Our method suggests also a sort of topological classification of these fermionic systems depending on their response to the presence of gauge fields, considered as external or emergent fields [20]. Such an approach follows the same philosophy proposed in [21], where topological insulators and superconductors have been classified on the basis of quantum anomalies [22].

The first step of our strategy is standard (in the Hermitian case) and consists in connecting the determinants of Dirac operators with the corresponding analytical indices through the zeta–function regularization. Then, we introduce a suitable splitting of the heat kernel that naturally selects the purely topological part of the determinant (*i.e.* the topological effective action). This topological effective action is expressed in terms of gauge fields using the Atiyah–Singer index theorem which computes the analytical index in topological terms [23]. The main new result of this paper is to provide a consistent extension of this method to the non Hermitian case (where a well–defined determinant does not exist).

It is worth to remark that various types or versions of index theorems have already found direct applications in many condensed matter systems, see *e.g.* [24–26]. However, we are not going to address here a systematic analysis of specific microscopic models which might sustain the proposed topological actions. Rather, in the final section we illustrate a couple of applications. The first one is about three–dimensional topological insulators where massless Dirac fermions emerge in the bulk at quantum critical points. In this regime our results might help in analyzing the response of such kind of substrata in the presence of both an electromagnetic and an Abelian axial–vector gauge fields. In the second final remark it is argued that there might be established a 2D–3D duality for topological insulators based on a common characterization of their effective actions in terms of BF theories.

TOPOLOGICAL EFFECTIVE ACTIONS AND INDEX THEOREM

Effective field theories are fully consistent quantum field theories valid in a limited energy range described by appropriate degrees of freedom. In a quantum system composed by fermions and bosons with relativistic dynamics and described by a microscopic action S , the corresponding effective action S_{eff} , is usually obtained by integrating out the fermionic fields

$$e^{-S_{\text{eff}}} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S}. \quad (1)$$

With this formal prescription in mind, we start considering massless Dirac fermions in a (3+1)–dimensional flat spacetime coupled to a generic vector gauge field

$A_\mu = A_\mu^\alpha T_\alpha$, where $\mu=1,2,3,4$ and T_α are the generators of the Lie algebra \mathfrak{g} of a compact Lie group G , $\alpha = 1, 2, \dots, r$ ($r = \text{rank}(\mathfrak{g})$). The action is given by

$$S = \int d^4x \bar{\psi} \mathcal{D} \psi = \int d^4x \bar{\psi} \gamma^\mu (\partial_\mu + A_\mu) \psi, \quad (2)$$

where γ^μ are the 4×4 Euclidean Dirac matrices in the chiral basis and \mathcal{D} is the Hermitian Dirac operator which has the following matrix form

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}, \quad (3)$$

where $\mathcal{D}_- = \mathcal{D}_+^\dagger$ and $\mathcal{D}_+ = \mathcal{D}_-^\dagger$ are the chiral components of the operator (here the slash complies with the Feynman slash–notation taken with respect to the Pauli matrices).

Formally, integrating out the fermionic fields, we get

$$S_{\text{eff}} = -\log \det \mathcal{D} := -\frac{1}{2} \log \det \mathcal{D}^2, \quad (4)$$

where $\det \mathcal{D}$ is the determinant of the Dirac operator and the projectors are given by

$$\left(\frac{1+\gamma_5}{2}\right) \mathcal{D}^2 = \mathcal{D}_+^\dagger \mathcal{D}_+; \quad \left(\frac{1-\gamma_5}{2}\right) \mathcal{D}^2 = \mathcal{D}_+ \mathcal{D}_+^\dagger, \quad (5)$$

with $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$.

All the physical properties of the quantum system (at one loop) are captured by the fermionic determinant that needs to be regularized. In particular, the zeta–function regularization of such kind of determinants can be briefly summarized as follows [27–29]. Consider a positive–definite Hermitian second order elliptic operator L . The zeta function associated to L is defined as

$$\zeta(s; L) = \sum_\lambda \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{tr} e^{-Lt}, \quad (6)$$

where λ are the positive eigenvalues of L , $\Gamma(s)$ is the gamma function, and the zeta function is expressed in terms of the corresponding heat kernel. When the operator has also zero eigenvalues, it is usual to deform L into $L' = L + \epsilon$, where ϵ is a positive number, so that this new operator has only positive eigenvalues [30]. Its determinant is defined in terms of its zeta function as

$$\log \det L' = -\frac{d}{ds} \zeta(0; L'), \quad (7)$$

which is related to the Ray–Singer torsion [31]. Divergences in the above expression are regularized by using

the s -regularized determinant, related to the zeta function as follows

$$(\log \det L')_s = -\mu^{2s} \Gamma(s) \zeta(s; L'), \quad (8)$$

where μ is a positive constant of dimension of a mass and $Re(s) > 2$.

In the case under examination $L' = \mathcal{D}^2 + \epsilon$, and the limit $\epsilon \rightarrow 0$ is taken after the regularization to capture the effective action, namely

$$\begin{aligned} S_{\text{eff}} &= \lim_{\epsilon \rightarrow 0} -\frac{1}{2} (\log \det \mathcal{D}^2 + \epsilon)_s \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2} \mu^{2s} \int_0^\infty dt t^{s-1} \text{tr} e^{-(\mathcal{D}^2 + \epsilon)t}. \end{aligned} \quad (9)$$

Upon manipulating the heat-kernel (by taking into account the formal properties of exponentials and projection operators) one gets

$$\begin{aligned} &\text{tr} e^{-(\mathcal{D}^2 + \epsilon)t} \\ &= \text{tr} \left(\frac{1+\gamma_5}{2} \right) e^{-(\mathcal{D}^2 + \epsilon)t} - \text{tr} \left(\frac{1-\gamma_5}{2} \right) e^{-(\mathcal{D}^2 + \epsilon)t} \\ &+ 2 \text{tr} \left(\frac{1-\gamma_5}{2} \right) e^{-(\mathcal{D}^2 + \epsilon)t} = K_1 + K_2, \end{aligned} \quad (10)$$

where, due to relations (5),

$$K_1 = \text{tr} e^{-(\mathcal{D}_+^\dagger \mathcal{D}_+ + \epsilon)t} - \text{tr} e^{-(\mathcal{D}_+ \mathcal{D}_+^\dagger + \epsilon)t} \quad (11)$$

and

$$K_2 = 2 \text{tr} e^{-(\mathcal{D}_+ \mathcal{D}_+^\dagger + \epsilon)t}. \quad (12)$$

K_1 is the difference of two traces and turns out to be connected with the zero-modes of the Dirac operator. For this reason it includes necessarily the contributions due to topological terms while K_2 includes the non-topological ones. Taking into account only K_1 as given in (11), the topological part of the effective action (9) can be defined as follows

$$\begin{aligned} S_{\text{eff}}^{\text{top}}(s) &:= \lim_{\epsilon \rightarrow 0} \frac{1}{2} \mu^{2s} \Gamma(s) [\zeta(s; \mathcal{D}_+^\dagger \mathcal{D}_+ + \epsilon) - \zeta(s; \mathcal{D}_+ \mathcal{D}_+^\dagger + \epsilon)]. \end{aligned} \quad (13)$$

Recall that a local formula for the analytical index of an elliptic operator Q is given by [32, 33]

$$\text{ind } Q = \epsilon^s [\zeta(s; Q^\dagger Q + \epsilon) - \zeta(s; Q Q^\dagger + \epsilon)]. \quad (14)$$

The topological part of the regularized effective action can thus be expressed in terms of the analytic index as

$$S_{\text{eff}}^{\text{top}}(s) = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \mu^{2s} \Gamma(s) \epsilon^{-s} \text{ind } \mathcal{D}_+. \quad (15)$$

Choosing the normalization $\mu = \sqrt{c\epsilon}$, with c and s , $Re(s) > 2$, arbitrary parameters such that $c^s \Gamma(s) = -2\pi i$, the topological part of the effective action can be defined as

$$S_{\text{eff}}^{\text{top}} := -i\pi \text{ind } \mathcal{D}_+. \quad (16)$$

Note that such normalization is compatible with the results found on employing different regularization schemes in the case of massive fermions [34]. Indeed, taking a finite value of ϵ and considering it as a square mass term ($\epsilon = m^2$), (15) remains unchanged and the TEFTs obtained from this relation are in complete agreement with respect to the effective actions in three space dimensions established in [21] by resorting to quantum anomalies. (Actually this is not surprising since it is well known that there exists a deep connection between index theorem and quantum anomalies [22].) The arbitrariness in the choice of the normalization coefficient is simply due to the fact that the original path integral (1) is unnormalized.

Consider now a generalization of the fermionic action (2) where the fermions are coupled to both a vector A_μ and an axial-vector B_μ gauge fields with value in \mathfrak{g} (Lie-algebra indices $\alpha = 1, 2, \dots, r$ are hidden)

$$\begin{aligned} S &= \int d^4x \bar{\psi} \mathcal{D} \psi \\ &= \int d^4x \bar{\psi} \gamma^\mu (\partial_\mu + A_\mu + \gamma_5 B_\mu) \psi. \end{aligned} \quad (17)$$

The Dirac operator is not Hermitian in Euclidean signature, namely

$$\mathcal{D}^\dagger = \mathcal{D} + \mathcal{A} - \gamma_5 \mathcal{B} \neq \mathcal{D}. \quad (18)$$

The corresponding chiral operators are given by

$$\mathcal{D}_+ = \mathcal{D} + \mathcal{A} + \mathcal{B}; \quad \mathcal{D}_- = \mathcal{D} + \mathcal{A} - \mathcal{B}. \quad (19)$$

The operators \mathcal{D}^2 and $(\mathcal{D}^\dagger)^2$ do not have a well-defined determinant. However, the operators $\mathcal{D}^\dagger \mathcal{D}$ and $\mathcal{D} \mathcal{D}^\dagger$ do have a determinant because they are Hermitian.

With such remarks in mind, associated effective actions might be hard to define. However, as we are here interested only in topological contributions, one can use the fact that reversing the orientation of the background

manifold is equivalent to pass from \mathcal{D} to \mathcal{D}^\dagger . Denoting by S_{eff}^\dagger and $S_{\text{eff}}^\downarrow$ respectively the ‘‘effective actions’’ defined on the background manifold M (with a fixed orientation) and on \overline{M} (with the opposite one), the following definition can be given

$$\begin{aligned} S_{\text{eff}}^\dagger + S_{\text{eff}}^\downarrow &= -\log \det \mathcal{D} - \log \det \mathcal{D}^\dagger \\ &:= -\frac{1}{2} \log \det \mathcal{D}^\dagger \mathcal{D} - \frac{1}{2} \log \det \mathcal{D} \mathcal{D}^\dagger. \end{aligned} \quad (20)$$

Note that, although the first equality is purely formal, the expression in the second row is well defined and necessarily contains (being invariant under a change of orientation) the topological part of the effective action to be evaluated.

The projectors act as follow

$$\begin{aligned} \left(\frac{1+\gamma_5}{2}\right) \mathcal{D}^\dagger \mathcal{D} &= \mathcal{D}_+^\dagger \mathcal{D}_+; \\ \left(\frac{1-\gamma_5}{2}\right) \mathcal{D}^\dagger \mathcal{D} &= \mathcal{D}_-^\dagger \mathcal{D}_-; \\ \left(\frac{1+\gamma_5}{2}\right) \mathcal{D} \mathcal{D}^\dagger &= \mathcal{D}_- \mathcal{D}_-^\dagger; \\ \left(\frac{1-\gamma_5}{2}\right) \mathcal{D} \mathcal{D}^\dagger &= \mathcal{D}_+ \mathcal{D}_+^\dagger. \end{aligned} \quad (21)$$

Also in this case we regularize using the zeta–function prescription, namely

$$\begin{aligned} &-\lim_{\epsilon \rightarrow 0} \frac{1}{2} (\log \det \mathcal{D}^\dagger \mathcal{D} + \epsilon)_s - \lim_{\epsilon \rightarrow 0} \frac{1}{2} (\log \det \mathcal{D} \mathcal{D}^\dagger + \epsilon)_s \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2} \mu^{2s} \int_0^\infty dt t^{s-1} \text{tr} \left[e^{-(\mathcal{D}^\dagger \mathcal{D} + \epsilon)t} + e^{-(\mathcal{D} \mathcal{D}^\dagger + \epsilon)t} \right] \end{aligned} \quad (22)$$

and

$$\begin{aligned} &\text{tr} \left[e^{-(\mathcal{D}^\dagger \mathcal{D} + \epsilon)t} + e^{-(\mathcal{D} \mathcal{D}^\dagger + \epsilon)t} \right] \\ &= \text{tr} \left(\frac{1+\gamma_5}{2} \right) \left[e^{-(\mathcal{D}^\dagger \mathcal{D} + \epsilon)t} + e^{-(\mathcal{D} \mathcal{D}^\dagger + \epsilon)t} \right] \\ &\quad - \text{tr} \left(\frac{1-\gamma_5}{2} \right) \left[e^{-(\mathcal{D}^\dagger \mathcal{D} + \epsilon)t} + e^{-(\mathcal{D} \mathcal{D}^\dagger + \epsilon)t} \right] \\ &\quad + 2 \text{tr} \left(\frac{1-\gamma_5}{2} \right) \left[e^{-(\mathcal{D}^\dagger \mathcal{D} + \epsilon)t} + e^{-(\mathcal{D} \mathcal{D}^\dagger + \epsilon)t} \right] \\ &= K_1 + K_2, \end{aligned} \quad (23)$$

where, thanks to relations (21), we have that

$$\begin{aligned} K_1 &= \text{tr} e^{-(\mathcal{D}_+^\dagger \mathcal{D}_+ + \epsilon)t} - \text{tr} e^{-(\mathcal{D}_+ \mathcal{D}_+^\dagger + \epsilon)t} \\ &\quad - \text{tr} e^{-(\mathcal{D}_-^\dagger \mathcal{D}_- + \epsilon)t} + \text{tr} e^{-(\mathcal{D}_- \mathcal{D}_-^\dagger + \epsilon)t}; \end{aligned} \quad (24)$$

$$K_2 = 2 \text{tr} e^{-(\mathcal{D}_+ \mathcal{D}_+^\dagger + \epsilon)t} + 2 \text{tr} e^{-(\mathcal{D}_- \mathcal{D}_-^\dagger + \epsilon)t}. \quad (25)$$

K_1 includes the topological terms. On the basis of (13) and (14), the topological effective action can be defined as

$$\begin{aligned} S_{\text{eff}}^{\text{top}}(s) &:= \lim_{\epsilon \rightarrow 0} \frac{1}{2} \mu^{2s} \Gamma(s) [\zeta(s; \mathcal{D}_+^\dagger \mathcal{D}_+ + \epsilon) - \zeta(s; \mathcal{D}_+ \mathcal{D}_+^\dagger + \epsilon) \\ &\quad - \zeta(s; \mathcal{D}_-^\dagger \mathcal{D}_- + \epsilon) + \zeta(s; \mathcal{D}_- \mathcal{D}_-^\dagger + \epsilon)] \end{aligned} \quad (26)$$

and finally we get

$$\begin{aligned} S_{\text{eff}}^{\text{top}} &= \lim_{\epsilon \rightarrow 0} \frac{1}{2} \mu^{2s} \Gamma(s) \epsilon^{-s} (\text{ind } \mathcal{D}_+ - \text{ind } \mathcal{D}_-) \\ &= -i\pi (\text{ind } \mathcal{D}_+ - \text{ind } \mathcal{D}_-), \end{aligned} \quad (27)$$

where we have used the same normalization adopted in expression (16).

Thus we have found that for a non–Hermitian Dirac operator a topological effective action can be defined as the difference of the analytic indices of \mathcal{D}_+ and \mathcal{D}_- .

CLASSIFICATION OF EFFECTIVE FIELD THEORIES

For an elliptic differential operator on a compact smooth manifold in even dimension, the Atiyah–Singer index theorem states that the analytical index is equal to the topological index [23]. The latter is a topological invariant depending only on the fibre bundles (in the present cases spinor bundles) living on the manifold. Most significant situations in applications require the explicit expression of the topological index of Dirac operators in terms of fields on either closed manifolds or compact manifolds with (suitable) boundary components. Recall that the topological index for a compact manifold with a non–empty boundary is different from the index for a manifold without boundary. The Atiyah–Patodi–Singer index theorem indeed generalizes the Atiyah–Singer index theorem by adding the so–called η invariants to the other topological indices [32, 35]. However, being η a geometric invariant characterizing the boundary, it is not involved in the following analysis since we are going to focus only onto the topological (bulk) sector of the effective actions.

Combining the index theorem with the results of the previous section, we give below the explicit forms of a few topological effective field theories emerging from microscopic quantum systems in (3+1)–dimensional (flat or curved) spacetimes, where massless fermions are coupled with non–Abelian gauge fields (the Abelian analogues being derivable in a straightforward way). In

the following M is a four-dimensional smooth orientable compact Riemannian manifold (with or without boundaries) or a (generalized) oriented cylinder, namely $M = \mathcal{M}^3 \times [0, 1]$, with \mathcal{M}^3 a three-dimensional Riemannian compact orientable smooth manifold (in turns with or without boundaries; moreover M is endowed with the natural product 4-metric). Of course suitable boundary conditions should be imposed on the fields according to the specific theories and/or applications of interest. A complete field-theoretic treatment of such issue in the case of (Abelian) TEFT of the BF-type can be found in [36].

Fermions coupled with a vector gauge field on a flat spacetime

The microscopic action is

$$S = \int d^4x \bar{\psi} \gamma^\mu (\partial_\mu + A_\mu) \psi. \quad (28)$$

The topological effective action turns out to be

$$S_{\text{eff}}^{\text{top}} = -i\pi \text{ind } \mathcal{D}_+ = \frac{i}{8\pi} \int_M \text{tr } F_A \wedge F_A, \quad (29)$$

where $\mathcal{D}_+ = \partial + A$, the trace is taken over the gauge indices and $F_A = dA + A \wedge A$ is the curvature form [37].

Fermions on a curved spacetime

$$S = \int d^4x \det(e) \bar{\psi} \gamma^\alpha e_\alpha^\mu (\partial_\mu + \omega_\mu) \psi, \quad (30)$$

where e_α^μ are the tetrads and ω_μ the spin connection [23].

$$S_{\text{eff}}^{\text{top}} = -i\pi \text{ind } \mathcal{D}_+ = -\frac{i}{96\pi} \int_M \text{tr } R_\omega \wedge R_\omega, \quad (31)$$

where $\mathcal{D}_+ = \partial + \omega$ and $R_\omega = d\omega + \omega \wedge \omega$ is the Riemann two-form.

Fermions coupled with a vector gauge field on a curved spacetime

$$S = \int d^4x \det(e) \bar{\psi} \gamma^\alpha e_\alpha^\mu (\partial_\mu + A_\mu + \omega_\mu) \psi. \quad (32)$$

$$\begin{aligned} S_{\text{eff}}^{\text{top}} &= -i\pi \text{ind } \mathcal{D}_+ \\ &= \frac{i}{8\pi} \int_M \text{tr } F_A \wedge F_A - \frac{i(\dim \rho)}{192\pi} \int_M \text{tr } R_\omega \wedge R_\omega, \end{aligned} \quad (33)$$

where $\mathcal{D}_+ = \partial + A + \omega$ and $\dim \rho$ is the dimension of the gauge group representation.

Fermions coupled with vector and axial gauge fields on a flat spacetime

$$S = \int d^4x \bar{\psi} \gamma^\mu (\partial_\mu + A_\mu + \gamma_5 B_\mu) \psi. \quad (34)$$

$$\begin{aligned} S_{\text{eff}}^{\text{top}} &= -i\pi (\text{ind } \mathcal{D}_+ - \text{ind } \mathcal{D}_-) \\ &= \frac{i}{8\pi} \int_M \text{tr } F_{A+B} \wedge F_{A+B} - \frac{i}{8\pi} \int_M \text{tr } F_{A-B} \wedge F_{A-B}, \end{aligned} \quad (35)$$

where $\mathcal{D}_+ = \partial + A + B$, $\mathcal{D}_- = \partial + A - B$.

Fermions coupled with vector and axial gauge fields on a curved spacetime

$$S = \int d^4x \det(e) \bar{\psi} \gamma^\alpha e_\alpha^\mu (\partial_\mu + A_\mu + \gamma_5 B_\mu + \omega_\mu) \psi. \quad (36)$$

$$\begin{aligned} S_{\text{eff}}^{\text{top}} &= -i\pi (\text{ind } \mathcal{D}_+ - \text{ind } \mathcal{D}_-) \\ &= \frac{i}{8\pi} \int_M \text{tr } F_{A+B} \wedge F_{A+B} - \frac{i}{8\pi} \int_M \text{tr } F_{A-B} \wedge F_{A-B}, \end{aligned} \quad (37)$$

where $\mathcal{D}_+ = \partial + A + B + \omega$, $\mathcal{D}_- = \partial + A - B + \omega$. (Note that here the contributions of the ω -field cancels out.)

In the last two cases the definition of the field strength F reads

$$F_{A \pm B} = d(A \pm B) + (A \pm B) \wedge (A \pm B). \quad (38)$$

APPLICATIONS

In this section we start focusing on three-dimensional strong topological insulators such as $Bi_{1-x}Sb_x$ (bismuth antimony) which have a gapped bulk and gapless surface

states protected by time-reversal symmetry [38]. This means that they are robust against non-magnetic impurities/disorder which cannot destroy the conducting states. These surface states consist of an odd number of massless Dirac fermions [39].

We can characterize such topological insulators through their response properties to an electromagnetic field denoted a_μ . Indeed in the bulk this response can be associated with a topological θ -term which (in Lorentzian signature) reads

$$S_\theta = \frac{\theta}{32\pi^2} \int d^4x \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau} = \frac{\theta}{8\pi^2} \int F_a \wedge F_a \quad (39)$$

and is commonly referred to as axion electrodynamics [8, 9, 40]. For a generic value of θ the axion term breaks time-reversal symmetry T as well as parity P . These discrete symmetries are preserved only for $\theta = \pi$ and $\theta = 0 \bmod 2\pi$, which characterize topological insulators and standard insulators respectively. Since the topological bulk action can be formulated in terms of a Chern-Simons action on the boundary, $\theta = \pi$ implies a one-half quantum Hall effect on the surface [41]

$$S_{CS} = \frac{\sigma_{xy}}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \quad (40)$$

where $\sigma_{xy} = \pm 1/2$ (in unit of e^2/h) is the Hall conductivity. Time reversal symmetry is broken on the surface also if it is preserved in the bulk, and thus the boundary surface becomes insulating, *i.e.* Dirac fermions acquire mass. Conversely, we can say also that a topological axion term describes the electromagnetic response of topological insulators when T is broken on the boundary surface.

Three-dimensional massless Dirac points are predicted to exist at the phase transition between a topological and a normal insulator [38, 42]. In the alloy $Bi_{1-x}Sb_x$ which possesses a rhombohedral crystal structure, the evolution of its band structure has been experimentally studied. Changing the Sb -concentration x , $Bi_{1-x}Sb_x$ passes from a topological insulating state to a normal insulating one. The quantum critical point is at $x \approx 0.04$, where the gap in the bulk becomes zero and massless three-dimensional Dirac point is realized.

We argue that also at this critical point the material possesses a topological behavior thanks to the fact that the electromagnetic response of the bulk gives rise again to the axion term. Indeed, nearby the quantum critical point, the tight binding Hamiltonian can be described in terms of its continuum limit. The effective topological action for massless Dirac fermions coupled with an electromagnetic field, *i.e.* the Abelian case of (29) in the previous section, is given exactly by the action (39) with $\theta = \pi$ (with Lorentzian signature the imaginary unit in front of (29) drops out). We argue that in this

massless regime a new topological response might be made manifest.

It was pointed out in [43] the relevant role of BF theory in 2 and 3D topological insulators. Recall that in two-dimensional topological insulators quantum spin Hall phase can be realized by means of a superposition of two quantum Hall systems with the up- and down-spins having opposite (effective) magnetic field. A double (2+1) Abelian Chern-Simons theory, equivalent to an Abelian BF theory, describes consistently the realization of this quantum phase [44].

On the basis of this remark, we argue that a (3+1) Abelian BF effective term could be associated to the bulk at quantum critical point in three-dimensional topological insulators when an Abelian axial-vector field b_μ together with the electromagnetic field a_μ are introduced. It turns then out that the response of the system is described by the corresponding Abelian version of the effective action (34) (in Lorentzian signature), namely

$$\begin{aligned} S_{BF} &= \frac{1}{8\pi} \int (F_{a+b} \wedge F_{a+b} - F_{a-b} \wedge F_{a-b}) \\ &= \frac{1}{2\pi} \int F_b \wedge F_a \end{aligned} \quad (41)$$

which represents a BF term because F_b takes exactly the role of the B-field (a similar result for Dirac materials has been found recently [45]). Moreover, the above action induces surface contributions since it can be expressed as a (2+1) Abelian BF theory on the boundary and, by resorting to Stokes' theorem, one gets

$$\begin{aligned} S_{BF} &= \frac{1}{2\pi} \int_M F_b \wedge F_a = \frac{1}{2\pi} \int_M db \wedge da \\ &= \frac{1}{2\pi} \int_M d(b \wedge da) = \frac{1}{2\pi} \int_{\partial M} b \wedge da, \end{aligned} \quad (42)$$

where PT invariance is manifest. This effective action and the unbroken time-reversal symmetry are compatible with the presence of quantum spin Hall states on the surface of the system if we identify b_μ with an external (Zeeman) field coupled to the (edge) spin currents [46, 47]. Note that such bulk-boundary correspondence can be interpreted as a sort of 2D-3D duality for topological insulators.

Further applications in condensed matter involving the effective actions on curved backgrounds of the previous section should include other microscopic quantum systems such as topological superconductors in connection with thermal Hall effect [21, 48, 49], Dirac semi-metals and 3D optical lattices.

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