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A_∞-Algebra from Supermanifolds

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Abstract

Inspired by the analogy between different types of differential forms on supermanifolds and string fields in superstring theory, we construct new multilinear non-associative products of forms which yield, for a single fermionic dimension, an A_∞-algebra as in string field theory. For multiple fermionic directions, we give the rules for constructing non-associative products, which are the basis for a full A_∞-algebra structure to be yet discussed.

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1 Introduction

Quite recently the discovery of some new algebraic and geometric structures in physics and mathematics has led to a renewed interest in the study of supermanifolds and their peculiar geometry [1–14]. One of the main motivation comes from superstring theory. As is well-know there are two ways to construct the supersymmetric sigma model representing the perturbative Lagrangian of superstrings: 1) the Ramond-Nevue-Schwarz (RNS) sigma model, with world-sheet anti-commuting spinors and world-sheet supersymmetry and 2) the Green-Schwarz/Pure Spinor (GS/PS) sigma model, with anti-commuting target space spinors and target space supersymmetry.

The RNS formulation has several interesting pros, but difficulties appear when one computes higher genus amplitudes. The quantization procedure by gauge-fixing the world-sheet supergravity leads to ghost insertions in the conformal field theory correlation functions for anomaly cancellation. Nonetheless, some of those insertions are rather delicate and they lead to inconsistencies at higher loops. These insertions are the so-called *Picture Changing Operators* (Z or X and Y) introduced in [15, 16]. Recently, string perturbation theory has been revised by Witten [17], pointing out that some of the inconsistencies can be by-passed by a suitable integration theory of forms on supermanifolds and, in particular, on super Riemann surfaces. As reviewed in [2, 18], the integration theory of forms on supermanifolds takes into account the complex of *integral forms*. The latter are distributional-like forms, usually written as $\delta(d\theta)$, which serve to control the integration over the $d\theta$'s. The need of integral forms in the context of superstrings and super Riemann surfaces was already pointed out by Belopolski [19] and discussed, from a strictly mathematical point of view, by Manin [20] and other authors. In particular, in [19] a practical way to handle forms was discussed and described.

On the other hand, for GS/PS sigma model, it was discovered [21–23] that amplitude computations (at tree level and at higher orders) also require the insertion of Picture Changing Operators to cancel the anomalies and to make the amplitudes meaningful. In this context, the geometry related to those operators was less clear, so that they were built in analogy with RNS superstrings. Nonetheless, eventually, they appear to be integral forms for the target superspace (we recall that in GS/PS sigma model, the quantum fields are maps from a given Riemann surface to a target supermanifold) and for them the usual rule of Cartan differential calculus can be used.

Both the case of PCO's for RNS and GS/PS superstrings can be understood from a pure geometrical point of view by insisting on having a meaningful integration theory for supermanifolds (in the RSN case, integration on super Riemann surfaces with given punctures and boundaries, while in the case of GS/PS for target space supermanifolds with a given supermetric), which in turn requires to understand at a deeper level the peculiarities arising whenever part of the geometry is anticommuting. In particular, it turns out that a new number - beside the ordinary form degree - is needed to describe forms on a supermanifold [2]: this is called *picture number* and, essentially, it counts the number of delta functions of the differential 1-forms, namely the $\delta(d\theta_i)$. As discussed in several papers (see for example [1]), two Dirac delta forms anti-commute and that implies an upper bound to the number of delta forms that can appear in a given form. In particular, the picture number can range from zero (in which case we denote those forms as *superforms*) to the maximum value which coincides with the fermionic dimension of the supermanifold. In that case we refer to the related complex as to the *integral forms* complex (see also [2]): in particular, working on a supermanifold of dimension $n|p$, forms of degree n and picture p are actually sections of the Berezinian sheaf, and they can be integrated over. If the fermionic dimension of the superamanifold is greater than one, then between superforms - having picture number equal to zero - and integral forms - having picture number equal to the fermionic dimension of the supermanifold -, we can have forms having a middle-dimensional picture, that are not superforms, nor integral forms, namely they have some Dirac delta functions, but less than the maximum possible number. These are called *pseudoforms*. So far, for the sake of exposition only an algebraic characterization of superforms, integral forms and pseudoforms has been hinted. Notheless, remarkably, a sheaf theoretical description, disclosing interesting relations and dualities, might be given [10].

Once that the "zoo" of forms is established, differential operators relating these forms can be defined. Besides the usual differential operators d, it emerges a new differential operator denoted by η which anticommutes with d and it is nilpotent. This is physically motivated by the embedding of the $N = 1$ RNS superstring into a $N = 4$ supersymmetric sigma model as shown in [24]. In the language of $N = 4$ superconformal symmetry the two operators d and η are the two anticommuting supercharges of the superconformal algebra. Furthermore, η is crucial for a useful characterization of the superstring Hilbert space. Indeed, that Hilbert space contains those states generated by quantizing the superghosts (Small Hilbert Space (SHS) [15, 25]) The same Hilbert space can be represented in terms of a different set of quantized fields and two descriptions are identical by excluding the zero mode of one of these quantum fields (to be precise the superghosts β and γ are reparametrized by two fermionic degrees of freedom and one bosonic degree of freedom). On the other hand, including a zero mode, the Hilbert space gets doubled, leading to the so-called Large Hilbert Space (LHS). The original SHS lives in the kernel of η, but in LHS additional structures emerge leading to an explicit solution of the constraints [24, 26].

Translating this set up in geometric terms on supermanifolds, we found that the LHS corresponds to an enlarged set of forms which contains also *inverse forms* (see [10]) on which the corresponding operator η can be built. Similarly as above, excluding inverse forms is achieved by imposing η-closure on these extended complexes. Again, drawing from string theory experience, we can construct two additional operators known as PCO Z and Y. They are built in terms of Dirac delta function integral representation and they act on the entire space of forms [10, 19].

The graded (supersymmetric) wedge product makes forms on a supermanifold in an algebra. In particular given two forms on a supermanifold, the wedge product acts by adding their form degrees - as it is usual -, and also their picture numbers. It follows that, in general, it maps two forms into a forms having a greater (or equal, in a limiting case) picture number. The operators d and η act as derivations on the exterior algebra of forms, while the PCO's are *not* derivation with respect to the wedge product.
String theory, in particular its second quantized version, *superstring field theory*, has actually yet another construction to hint [26]. Indeed, over the years, there have been several proposed actions reproducing the full fledged superstring spectrum where the insertion of PCO's is crucial (see for instance [27, 28] and again [26] for further examples). Nonetheless, none of them turned out to be fully consistent. One of the main problem is due to the location of PCO's insertions. For first quantized amplitudes, the position of the PCO's is harmless, since on-shell the insertion turns out to be position-independent. On the contrary, for an off-shell second quantized action that it is pivotal. The position of the PCO's breaks the gauge invariance of the theory, leading to inconsistent results. To avoid this problem, new multilinear operations forming a *non-associative algebra* known as A_∞-algebra have been proposed recently by Erler,

Konopka and Sachs in [26].

Again, mimicking what has been done for string field theory, but using now ingredients that arise from the geometry of a supermanifold, we can construct multilinear products of forms. Some of them have precisely the same form of those coming from superstring field theory - constructed in terms of wedge products and PCO's insertions -, but on the other hand the richness that emerges from the geometry of forms on supermanifolds leads to new products, turning the exterior algebra of forms into a non-associative algebra which might be lead to a generalization the above-mentioned A_∞-algebra construction [26]. This completes the construction of new products of forms endowed with new algebraic properties. Applications of these new structures are still premature, but can be foreseen in several directions.

The paper is organized as follows: in sec. 2, we review the basic ingredients in the theory of forms on supermanifolds. In sec. 3, we discuss the differential operators d and η. In sec. 4, we review a useful construction of the PCO Z and we show some computations as illustrative examples. In sec. 5, we introduce and discuss the A_∞-algebra of forms on a supermanifold for the case of one fermionic directions and we provide the definition of multiproducts for higher dimesions. In sec. 5.1 we compute the M_3 product in terms of M_2 products. In sec. 5.2 we give some explicit examples. Finally, in Appendix A we explain the mathematical foundations of A_∞-algebra via cotensor algebra and coderivations and in Appendix B we provide some useful computations.

2 Forms on Supermanifolds and their Local Representation

In a supermanifold $\mathcal{SM}^{(n|m)}$, locally described by the coordinates (x^a, θ^α), with $a = 1, \ldots, n$ and $\alpha = 1, \ldots, m$, we consider the spaces of forms $\Omega^{(p|r)}$ [1]. A given $(p|r)$-form ω can be expressed in terms of local generators as a formal sum as follows

$$\begin{aligned} \omega &= \sum_{l,h,r} \omega_{[a_1 \ldots a_l](\alpha_1 \ldots \alpha_h)[\beta_1 \ldots \beta_r]}(x, \theta) \times \\ &\times \; dx^{a_1} \ldots dx^{a_l} (d\theta^{\alpha_1})^{u(\alpha_1)} \ldots (d\theta^{\alpha_h})^{u(\alpha_h)} \delta^{(g(\beta_1))}(d\theta^{\beta_1}) \ldots \delta^{(g(\beta_r))}(d\theta^{\beta_r}) \end{aligned} \tag{2.1}$$

where $u(\alpha) \geq 0$ is the power of the monomial $d\theta^\alpha$ and where $g(\alpha)$ denotes the differentiation degree of the Dirac delta form with respect to the 1-form $d\theta^\alpha$. Namely, $\delta^{(g(1))}(d\theta^1)$ is the

$g(1)$-derivative of the Dirac delta with respect to the variable $d\theta^1$.[1] The total form degree of $\omega^{(p|r)}$ is

$$l+\sum_{j=1}^{h}u(\alpha_j)-\sum_{k=1}^{r}g(\beta_k)=p\in\mathbb{Z}\,,\qquad \{\alpha_1,\dots,\alpha_r\}\neq\{\beta_1,\dots,\beta_r\}\quad\forall i=1,\dots,h. \tag{2.2}$$

Note that each α_l in the above summation must be different from any β_k, otherwise the degree of the differentiation of the Dirac delta function could be reduced and the corresponding 1-form $d\theta^{\alpha_k}$ could be removed from the basis. The picture number r corresponds to the number of Dirac delta forms. The components $\omega_{[i_1\dots i_l](\alpha_1\dots\alpha_m)[\beta_1\dots\beta_r]}(x,\theta)$ of ω are superfields[2], *i.e.* local sections of the structure sheaf of the supermanifold $\mathcal{SM}$.

The graded wedge product is defined as usual

$$\wedge:\Omega^{(p|r)}(\mathcal{SM})\otimes\Omega^{(q|s)}(\mathcal{SM})\longrightarrow\Omega^{(p+q|r+s)}(\mathcal{SM})\,. \tag{2.3}$$

where $0\leq p,q\leq n$ and $0\leq r,s\leq m$. Due to the anticommuting properties of the Dirac delta forms $\delta(d\theta^\alpha)\delta(d\theta^\beta)=-\delta(d\theta^\beta)\delta(d\theta^\alpha)$ this product can be set equal to zero, if two delta forms has the same $d\theta$ as argument.[3]

Actually, supergeometry allows for an even richer scenario: introducing the *inverse forms* as in [10], the complex made of the spaces $\Omega^{(p|q)}$ gets extended as follows

1. For picture $q=0$, there are new superforms in $\Omega^{(p|0)}$ that can also carry a *negative* form degree $p<0$. Locally, for a supermanifold of dimension $n|m$ we will have expressions of this kind

$$\omega^{(p|0)}=\sum_{l=0}^{n}\sum_{r=0}^{m}\sum_{a_i=1}^{n}\sum_{\alpha_j=1}^{m}\omega_{[a_1\dots a_l](\alpha_1\dots\alpha_r)}(x,\theta)dx^{a_1}\dots dx^{a_l}(d\theta^{\alpha_1})^{u(\alpha_1)}\dots(d\theta^{\alpha_r})^{u(\alpha_r)} \tag{2.4}$$

   together with the constrain $p=l+\sum_r u(\alpha_r)$ and where $u(\alpha_j)\in\mathbb{Z}$ is the power of the monomial $d\theta^{\alpha_r}$, that can now take also negative values. For example, on $\mathbb{C}^{1|1}$, one might consider forms of degree -1, having the following form

$$\omega^{(-1|0)}_{\mathbb{C}^{1|1}}=\omega_0(x,\theta)\frac{1}{d\theta}+\omega_1(x,\theta)\frac{dx}{d\theta^2} \tag{2.5}$$

[1] We recall that $d\theta^1\delta^{(g(1))}(d\theta^1)=-g(1)\delta^{(g(1)-1)}(d\theta^1)$.

[2] Symmetrization and (anti)-symmetrization correspond to the parity of the generators involed.

[3] In addition, it follows $\delta(d\theta^\alpha)\delta'(d\theta^\alpha)=0$, and consequently $\delta^{(p)}(d\theta^\alpha)\delta^{(q)}(d\theta^\alpha))=0$ for any derivative p,q of the Dirac delta forms.

2. Notice that, in general, whenever the supermanifold has fermionic dimension greater that 1, each space $\Omega^{(p|0)}$ for $p \in \mathbb{Z}$ has an *infinite* number of generators - even for $p \geq 0$. Consider for example the case of $\mathbb{C}^{1|2}$: allowing for inverse forms, beside 1, the space of $\Omega^{(0|0)}_{\mathbb{C}^{1|2}}$ is generated by all of the expressions of the kind $d\theta_1^{p_1} d\theta_2^{p_2}$ with $p_1 = -p_2$ and $dx d\theta_1^{p_1} d\theta_2^{p_2}$ with $p_1 + 1 = -p_2$ where $p_1, p_2 \in \mathbb{Z}$.

The spaces $\Omega^{(p|r)}$ with intermediate picture, namely when $0 < r < m$, are infinitely generated:

1. There might appear derivatives of Dirac delta forms of any order $\delta^{g(\alpha_l)}(d\theta^{\alpha_l})$ which reduce the form degree.

2. There might be any powers of $(d\theta^\alpha)^{u(\alpha)}$ different from those contained into the Dirac delta's, namely $\prod_{r=0}^{l} (d\theta^{\alpha_r})^{u(\alpha_r)} \prod_{s=l+1}^{m} \delta^{g(s)}(d\theta^{\alpha_s})$ where $\alpha_i \neq \alpha_j$ with $i = 1, \ldots, l$ and $j = l+1, \ldots, m$. The powers $u(\alpha)$ can be positive or negative, while the $g(s)$ are non-negative.

Finally, the complex $\Omega^{(p|m)}$ is bounded from above, since there are no other form above the top form $\Omega^{(n|m)}$ and, at a given form degree, each space $\Omega^{(p|m)}$ is finitely-generated.

The *odd* differential operator d maps forms of the type $\Omega^{(p|r)}$ into forms of the type $\Omega^{(p+1|r)}$ increasing the form number without changing the picture. The action of the differential operator d on the Dirac delta functions is by chain rule, namely $\delta(f(d\theta))) = \delta'(f(d\theta)) df(d\theta)$, so that, in particular, $d\delta(d\theta) = 0$.

We now focus on a supermanifold of dimension $(1|2)$ for simplicity and we consider the form spaces $\Omega^{(p|q)}$ with $0 \leq q \leq 2$. A simple but non-trivial example of supermanifold of dimension $(1|2)$ is the projective superspace $\mathbb{P}^{1|2}$ over the complex numbers. For a detailed discussion about the geometry of projective superspaces see for example [8]. It is defined starting with two patches U_0 and U_1 and the mapping of the coordinates z_0, θ_0^α to the coordinates z_1, θ_1^α is given by the homolorphic transition functions

$$z_0 \longmapsto z_1 = \frac{1}{z_0}\,, \qquad \theta_0^\alpha \longmapsto \theta_1^\alpha = \frac{\theta_0^\alpha}{z_0} \quad \alpha = 1, 2. \tag{2.6}$$

Its Berezinian bundle is generated by the section $\omega^{(1|2)} = dz \delta(d\theta^1) \delta(d\theta^2)$ which is globally defined, and indeed the supermanifold $\mathbb{P}^{1|2}$ is an example of Calabi-Yau supermanifold [7].

For $q=0$ and $p \in \mathbb{Z}$, we call the space $\Omega^{(p|0)}_{\mathbb{P}^{1|2}}$, the space of *superforms* . For $q=1$ and $p \in \mathbb{Z}$, we call $\Omega^{(p|1)}_{\mathbb{P}^{1|2}}$ the space of *pseudoforms* and finally, for $q=2$ and $p \leq 1$, we call $\Omega^{(p|2)}_{\mathbb{P}^{1|2}}$ the space of *integral* forms.

3 The Differential Operators d and η

We now work over the supermanifold $\mathbb{P}^{1|2}$. There are two differential operators acting on the complex of forms: the obvious one is the usual odd differential d

$$d : \Omega^{(p|q)}_{\mathbb{P}^{1|2}} \longrightarrow \Omega^{(p+1|q)}_{\mathbb{P}^{1|2}} \tag{3.1}$$

As already stressed, it increases the form number, but it does not change the picture. We now introduce another differential operator that will be used in what follow, but first we need some auxiliary material.

Let D be a vector field in the tangent bundle of the supermanifold $\mathcal{T}_{\mathbb{P}^{1|2}}$. In local coordinates is expressed as

$$D = D^z(z,\theta)\frac{\partial}{\partial z} + D^\alpha(z,\theta)\frac{\partial}{\partial \theta^\alpha}. \tag{3.2}$$

Then, for a *constant odd vector field* one has

$$D = D^1\frac{\partial}{\partial \theta^1} + D^2\frac{\partial}{\partial \theta^2} \tag{3.3}$$

with $D^1, D^2 \in \mathbb{C}$. Clearly, two odd vector fields D and D', are linearly independent if $\det(D,D') = D^1 D^{2'} - D^{1'} D^2 \neq 0$.

In general, given a vector field D, one can define the inner product ι_D which acts as

$$\begin{aligned} \iota_D : \Omega^{(p|q)}_{\mathbb{P}^{1|2}} &\longrightarrow \Omega^{(p-1|q)}_{\mathbb{P}^{1|2}} \,, \\ \omega &\longmapsto \iota_D(\omega) \end{aligned} \tag{3.4}$$

where $\iota_D(\omega)(X_1,\ldots,X_{p-1}) := \omega(D, X_1, \ldots, X_{p-1})$. For $D := D^\alpha \partial_{\theta^\alpha}$, an odd constant vector field as requested above one has, in particular that

$$\iota_D(d\theta^\alpha) = d\theta^\alpha(D) = D^\alpha. \tag{3.5}$$

Also, notice that it satisfies the usual Cartan algebra

$$\mathcal{L}_D = [d, \iota_D]\,, \qquad [\mathcal{L}_D, \iota_{D'}] = \iota_{\{D,D'\}}\,, \qquad \{\iota_D, \iota_{D'}\} = 0\,, \tag{3.6}$$

where $\mathcal{L}_D$ is the Lie derivative along D. We stress that in the first identity the commutator $[\cdot,\cdot]$ replaces the anticommutator $\{\cdot,\cdot\}$ since the differential operator ι_D has parity opposed to that of D - so that if D is odd, one has $|\iota_D| = 0$ - and the differential d is odd. For constant D and D', $\{D, D'\} = 0$ and $\iota_D^2 \neq 0$.

As we learnt from string theory (see [24]) there is another interesting odd differential operator which can be defined from ι_D (again for D an odd constant vector field) upon using the Euler representation of the sine: [4]

$$\eta = -2\Pi \lim_{\epsilon\to 0} \sin(i\epsilon\iota_D) : \Omega^{(p|q)}_{\mathbb{P}^{1|2}} \to \Omega^{(p+1|q+1)}_{\mathbb{P}^{1|2}} \tag{3.7}$$

where Π is the parity-change functor (see [10]) that simply changes the parity of the expression to which is applied, without affecting any other property. Acting with η on the inverse forms $1/d\theta^\alpha$, we have

$$\eta\left(\frac{1}{d\theta^\alpha}\right) = \delta(d\theta^\alpha)\,, \tag{3.8}$$

where $1/d\theta^\alpha$ is even and $\delta(d\theta^\alpha)$ is odd according to the axioms defining these distributions. The differential operator η acts as follows

$$\begin{cases} \eta\Big(\frac{1}{(d\theta^\alpha)^p}\Big) = \frac{(-1)^{p-1}}{(p-1)!}\delta^{(p-1)}(d\theta^\alpha) & p > 1, \\ \eta\Big((d\theta^\alpha)^p\Big) = 0 & p \geq 0, \\ \eta\Big(\delta^{(p)}(d\theta^\alpha)\Big) = 0 & p \geq 0, \end{cases} \tag{3.9}$$

whilst it does not act on the differentials of even coordinates. It is easy to verify that

$$\eta^2 = 0\,, \qquad \{d, \eta\} = 0, \tag{3.10}$$

as in string theory [24]. In addition, η is a graded-derivation with respect to the exterior algebra

$$\eta(\omega_A \wedge \omega_B) = \eta(\omega_A) \wedge \omega_B + (-1)^{|\omega_A|}\omega_A \wedge \eta(\omega_B)\,. \tag{3.11}$$

where ω_A and ω_B are forms of the complex $\Omega^{p|q}$.

[4] We use the normalization such that $\delta(x) = \int_{-\infty}^{\infty} e^{itx} dt$ and $\Theta(x) = -i \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{e^{itx}}{t+i\epsilon} dt$, and $\Theta'(x) = \delta(x)$. In this way, in order to match the correct assignments we need the factor -2 in the definition of η in (3.7) .

The operator η has been introduced in string theory to select the Small Hibert Space (SHS) inside the Large Hilbert space (LHS). As discussed in [10], the LHS for a supermanifold is constructed by adding the inverse forms (which are still distribution-like forms) to $\Omega^{(p|q)}$ with $q < m$. Thus, the equation

$$\eta(\omega^{(p|q)}) = 0 \tag{3.12}$$

selects the forms that are in the SHS. See the paragraph toward the end of page 17 for more on this point, which will prove crucial in what follows.

4 The PCO Z_D

When computing amplitudes in string theory, one introduces the *picture changing operators* (PCO) in order to change the picture of the vertex operators as to saturate the superghost charges according to the anomaly cancellation. In string theory the PCO are independent of the position of their insertion into the amplitude since the string fields are on-shell. On the other hand, in the string field theory action, when the string fields are off-shell, the position of the PCO really matters. The consequence of a "wrong" choice is the loss of the gauge invariance of the theory.

In the present supergeometric framework, operators analogous to the PCO of string theory can be defined as acting on the complex of forms $\Omega^{(p|q)}$, moving from one picture to another and leaving the form number unchanged. In addition, it can be shown that they are isomorphisms in de Rham cohomology [1].

With these preliminary remarks, we can define the PCO Z_D (see for example [3]) and ancillary operators as follows

$$Z_D := \{d, -i\Theta(\iota_D)\}\,, \qquad \Theta(\iota_D) := -i \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt \frac{e^{it\iota_D}}{t+i\epsilon}\,, \qquad \delta(\iota_D) := \int_{-\infty}^{\infty} dt e^{it\iota_D}. \tag{4.1}$$

The latter two act as follows on a certain polynomial functions f of the $d\theta$'s:

$$\begin{aligned} \Theta(\iota_D) f(d\theta^\alpha) &= -i \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt \frac{f(d\theta^\alpha + itD^\alpha)}{t+i\epsilon}, \\ \delta(\iota_D) f(d\theta^\alpha) &= \int_{-\infty}^{\infty} dt f(d\theta^\alpha + itD^\alpha). \end{aligned} \tag{4.2}$$

We add the subscript D to the PCO to recall that it depends on the choice of the odd vector D. Note that the operator Z_D is *formally* d-exact, indeed $Z_D = \{d, -i\Theta(\iota_D)\}$, but the operator $\Theta(\iota_D)$, acting on $\Omega^{(p|q)}$, brings from the SHS to the LHS. However, by computing the variation of Z_D by a change of D (that is $D \to D + \delta D$), Z_D transform ad $\delta Z_D = \{d, \delta(\iota_D)\delta D^\alpha \iota_\alpha\}$ which is exact and $\delta(\iota_D)\delta D^\alpha \iota_\alpha$ acts on $\Omega^{(p|q)}$ staying into the SHS.

The two operators $\Theta(\iota_D)$ and $\delta(\iota_D)$ act as follows

$$\Theta(\iota_D) : \Omega^{(p|q)} \longrightarrow \Omega^{(p-1|q-1)} , \qquad \delta(\iota_D) : \Omega^{(p|q)} \longrightarrow \Omega^{(p|q-1)} . \tag{4.3}$$

Both reduce the picture, but the first one reduces also the form degree. This is consistent with the usual relation between the Heaviside Θ function and the Dirac delta δ. In the case of $q = 0$, acting on zero-picture forms, they vanish. Notice also that, differently from η, they are *not* derivations of the exterior algebra of forms. In appendix A, we will give some explicit computations in order to clarify the action of $\Theta(\iota_D)$ and $\delta(\iota_D)$ operators.

Note that if $m > 1$, we have m linear independent odd vectors D_i, therefore we can define a PCO corresponding to each of them. In addition, we have a map between integral forms and superforms as the product of all independent PCO's

$$Z_{max} = \prod_{i=1}^{m} \Big\{ d, -i\Theta(\iota_{D_i}) \Big\} : \Omega^{(p|m)} \longrightarrow \Omega^{(p|0)} . \tag{4.4}$$

Due to the anticommutative properties of $\Theta(\iota_{D_i})$, it is easy to prove that Z_{max} does not depend on the choice of the odd vector fields D_i.

The situation is very similar to N=2 string framework [29, 30] where two different PCO's have been introduced corresponding to the two sectors of superghosts. In [29, 30], it has been pointed out that they are non-locally invertible and some care has to be used to define the BRST cohomology. In contrast to N=2 framework, we have the advantage that the PCO associated to different vector fields D_i do commute $[Z_D, Z'_D] = 0$; notice that in general their commutation relations are automatically d-exact, but in our case they are trivially zero as explicitly proven in app. A, see eqs (B.10)-(B.12).

In general, for $\omega \in \Omega^{(p|q)}$ when $q = 0, 1$, the absence of inverse forms is guaranteed if

$$\eta(\omega) = 0 \qquad \omega \in \Omega^{(p|q)},\ q = 0, 1. \tag{4.5}$$

For $q = 2$, η acts trivially since there is no room for inverse forms. For $q = 0, 1$, we observe

$$
\begin{aligned}
\eta(Z_D(\omega)) &= \eta\left(\{d, -i\Theta(\iota_D)\}(\omega)\right) = \eta\Big(- id\Theta(\iota_D)\omega - i\Theta(\iota_D)d\omega\Big) \\
&= -i(-d\eta\Theta(\iota_D)\omega + \{\eta, \Theta(\iota_D)\}d\omega - \Theta(\iota_D)\eta(d\omega)) \\
&= -i(-d\{\eta, \Theta(\iota_D)\}\omega + d\Theta(\iota_D)\eta(\omega) + \{\eta, \Theta(\iota_D)\}d\omega - \Theta(\iota_D)\eta(d\omega)) \\
&= Z_D(\eta(\omega)) = 0\,. \qquad (4.6)
\end{aligned}
$$

where we have used $\{\eta, -i\Theta(\iota_D)\} = 1$ for any D^α and $\{d, \eta\} = 0$. Therefore, the PCO-transformed ω, namely $Z_D(\omega)$, is independent of inverse forms if ω is independent.

5 Multiproducts of Superforms

In the complex of forms there is a natural bilinear map represented by the usual exterior product $\wedge : \Omega^{(p|q)} \otimes \Omega^{(p'|q')} \to \Omega^{(p+p'|q+q')}$. Note that, in general, the wedge product changes the form degree *and* the picture according to the previous formula. In addition, for what concerns our setting, when working over $\mathbb{P}^{1|2}$, for $q = 1, q' = 2$ or $q = 2, q' = 1$ or $q = q' = 2$, the products are trivial. In general, as observed in the previous sections, the differential d, η are derivations of the exterior algebra. Also, the exterior product can be extended to consider the inverse forms letting $p < 0$ for $q = 0$.

In this section we will show how to construct new bilinear maps that change the picture according to a different prescription. For example, in string field theory, the role of a $(1|1)$ form is played by a string vertex operator with ghost number 1 and picture number 1. The authors in [26] constructed a new bilinear map which take the product of these two vertices into a new vertex with quantun numbers $(2|1)$, namely ghost number two and the same picture. This new product is not associative and it leads to a structure of A_∞-algebra: we will show how this structure arises geometrically from superforms defined on a supermanifold.

We star briefly recalling the definition of A_∞-algebra and then we give an explicit realization of it in terms of the complex of forms on a supermanifold. For the sake of readability and self-consistency of this paper, the interested reader could find a detailed construction of the A_∞-algebra via *cotensor algebra* and *coderivations* on it in Appendix A. This is an algebraic inclined approach to A_∞-algebra which - restricted to our purposes - has the merit to allow the use of commutators between coderivations and the related identities, thus simplifying computations and yielding more compact and readable expressions. This approach is used by Erler and collaborators in [26] and further details can be found in the more mathematically inclined papers [31–33]. For the original approach to A_∞-algebras via homotopy theory due to Stashef in [34], we suggest the interested reader to look for example at [35] and at the last chapter of the book [36], where an extensive treatment is provided.

An A_∞-algebra is a graded vector space $V := \bigoplus_{p\in\mathbb{Z}} V^{(p)}$ with $p \in \mathbb{Z}$, endowed with graded maps (homogeneous and linear) $M_n : V^{\otimes n} \to V$, for $n \geq 1$ and of degree 1 satisfying the relations:

1. $M_1 M_1 = 0$, *i.e.* the pair (A, M_1) is a differential complex;
2. $M_1 M_2 + M_2 (M_1 \otimes 1 + 1 \otimes M_1) = 0$, *i.e.* M_1 is derivation with respect to the multiplication M_2.
3. $M_2 (1 \otimes M_2 + M_2 \otimes 1) + M_1 M_3 + M_3 (M_1 \otimes 1 \otimes 1 + 1 \otimes M_1 \otimes 1 + 1 \otimes 1 \otimes M_1) = 0$
4. for $n \geq 1$ we have $\sum (-1)^{r+st} M_u (1^{\otimes r} \otimes M_s \otimes 1^{\otimes t}) = 0$. The sum is over the decompositions $n = r + s + t$ and $u = 1 + r + t$.

Note that, as said above, we have employed the convention of [26] and we take all of the multilinear maps of degree 1, so that acting on elements $a_1, a_2 \in V$ the second relations yields

$$M_1 M_2 (a_1, b_1) + M_2 \Big(M_1(a_1) \otimes a_2 + (-1)^{|a_1|} a_1 \otimes M_1(a_2) \Big) = 0, \tag{5.1}$$

where the sign is due to Koszul sign rule, since we have commuted $a_1 \in V$ with M_1 and $\deg(M_1) = 1$. To make contact with the notation adopted in the appendix, we stress that the map M_n above are denoted with m_n in the appendix, as to match the mathematical literature. In the present case, we consider a bi-graded vector space $V = \bigoplus_{p,q} A^{(p|q)} = \bigoplus_q V^{(\bullet|q)}$ where the first number denote the form degree and the second the picture number, namely we take $V^{(\bullet|q)}$ to be $\Omega^{(\bullet|q)}$, the complex of forms at fixed picture number q (note that, strictly speaking this is a complex of *sheaves* of vector spaces). The form number is an integer for $q < m$, where m is the maximum number of fermion dimensions. For $q = m$, the form number is bounded by n, the maximum number of bosonic dimension.

We define the graded multilinear maps $M_n^{(-\ell)} : V^{\otimes n} \to V$, with $n \geq 1$ as follows

$$\begin{aligned} M_n^{(-\ell)} : \Omega^{(p_1|q_1)} \otimes \cdots \otimes \Omega^{(p_n|q_n)} &\longrightarrow \Omega^{p'|q'} \\ \omega_1 \otimes \ldots \otimes \omega_n &\longmapsto M_n^{(-\ell)}(\omega_1, \ldots, \omega_n), \end{aligned} \tag{5.2}$$

where $p' = \sum_{i=1}^n p_i$ and $q' = \sum_{i=1}^n q_i - \ell$. In other words, the map $M_n^{(-\ell)}$ is defined as to lower the sum of the pictures by ℓ, *i.e.* if $\omega_1, \ldots, \omega_n$ have picture $q_1, \ldots, q_n$ respectively, the form $M_n^{(-l)}(\omega_1, \ldots, \omega_n)$ is of picture $q' = \sum_{i=1}^n q_i - \ell$. Again, here the maps $M_n^{(-\ell)}$ play the role of the multilinear maps m_n of the Appendix A.

For $n = 1$ there are the two representatives, d and η. We set:

$$M_1^{(0)} := d\,, \qquad M_1^{(1)} := \eta\,. \tag{5.3}$$

In the simple case of two fermionic dimensions, namely for example working in $\mathbb{P}^{1|2}$ or, even more simply, over a superpoint $\mathbb{A}^{0|2}$ (where $q \leq 2$), we define the 2-products as

$$\begin{aligned} M_2^{(0)} &:= \wedge \\ M_2^{(-1)} &:= \frac{1}{3}\left[Z_D M_2^{(0)} + M_2^{(0)}(Z_D \otimes 1 + 1 \otimes Z_D)\right] \\ M_2'^{(-1)} &:= \frac{1}{3}\left[Z_{D'} M_2^{(0)} + M_2^{(0)}(Z_{D'} \otimes 1 + 1 \otimes Z_{D'})\right] \\ M_2^{(-2)} &:= \frac{1}{3^2}\left[Z_{D'} M_2^{(-1)} + M_2^{(-1)}(Z_{D'} \otimes 1 + 1 \otimes Z_{D'})\right] \\ &= \frac{1}{3^2}\Big[Z_{D'} Z_D M_2^{(0)} + Z_{D'} M_2^{(0)}(Z_D \otimes 1 + 1 \otimes Z_D) + Z_D M_2^{(0)}(Z_{D'} \otimes 1 + 1 \otimes Z_{D'}) \\ &\qquad + M_2^{(0)}\left(Z_{D'} Z_D \otimes 1 + Z_{D'} \otimes Z_D + Z_D \otimes Z_{D'} + 1 \otimes Z_{D'} Z_D\right)\Big]\,. \end{aligned} \tag{5.4}$$

They are built starting from the (graded supersymmetric) wedge product $M_2^{(0)} = \wedge$ by inserting the PCOs into the product in a symmetric way respecting the tensor structure. Note that there are two 2-products reducing the picture by one, *i.e.* $M_2^{(-1)}$ and $M_2'^{(-1)}$. This is due to the presence of two independent fermionic directions. Also, note that we do not have to choose the order of Z_D and Z'_D since they commute (see section 4 of this paper).

Notice that in the case $q = q' = 1$ the product $M_2^{(-1)}$ maps elements in $\Omega^{(p|1)} \otimes \Omega^{(p'|1)}$ to $\Omega^{(p+p'|1)}$ and in the case $q = q' = 2$, the product $M_2^{(-2)}$ maps elements in $\Omega^{(p|2)} \otimes \Omega^{(p'|2)}$ to $\Omega^{(p+p'|2)}$. These products preserve the picture and are bilinear maps from $\Omega^{(\bullet|q)} \otimes \Omega^{(\bullet|q)}$ to $\Omega^{(\bullet|q)}$.

It is convenient to rewrite these maps using the notion of *coderivations*, in terms of which we can use the graded commutators, denoted here and in the following as $[\,\cdot\,,\,\cdot\,]$, simplifying the computations. The interested reader could find in the Appendix a more detailed treatment of cotensor algebras and coderivations on them in relation to A_∞-algebras. This formalism was first introduced in string field theory in [37].

Given a multilinear map $\Delta_n : A^{\otimes n} \to A$ of the graded vector space A, we define the *associated coderivation* $\mathbf{\Delta}_{N,n} : A^{\otimes N} \to A^{\otimes(N-n+1)}$ for any $N \geq n$ as follows:[5]

$$\mathbf{\Delta}_{N,n} := \sum_{k=0}^{N-n} 1^{\otimes(N-k-n)} \otimes \Delta_n \otimes 1^{\otimes k} \tag{5.5}$$

acting on the spaces $A^{\otimes N \geq n}$. Note that if $N = n$, one has that $\mathbf{\Delta}_{N,n} = \Delta_n$, so in this particular context one can see that a coderivation "extends" a certain multilinear map Δ_n to higher tensor powers of A by suitably tensoring it with the identity map. From a more rigorous point of view, given a multilinear map $\Delta_n : A^{\otimes n} \to A$, the map $\mathbf{\Delta}_{N,n}$ has to be seen as the associated map in the cotensor algebra of A (see again the Appendix): from this point of view the presence of the subscript N is superfluous - and possibly misleading - as the cotensor algebra of a graded vector space V contains all its tensor powers and therefore indeed all of the maps $\mathbf{\Delta}_{N,n}$ for any possible $N \geq n$.
Let us for example consider a multilinear map $\Delta_2 : A^{\otimes 2} \to A$ and let us take $N = 3$, in the above formula, so that one has that the associated coderivation $\mathbf{\Delta}_{3,2} : A^{\otimes 3} \to A^{\otimes 2}$ is given by

$$\mathbf{\Delta}_{3,2} = \sum_{k=0}^{1} 1^{\otimes(1-k)} \otimes \Delta_2 \otimes 1^{\otimes k} = 1 \otimes \Delta_2 + \Delta_2 \otimes 1. \tag{5.6}$$

It can be seen that the commutator of two coderivations associated to the multilinear maps $\Delta_m : A^{\otimes m} \to A$ and $\Delta'_n : A^{\otimes n} \to A$ respectively, is the coderivation associated with the commutator of the maps Δ_m and Δ'_n, where the commutator acts as

$$[\Delta_m, \Delta'_n] : A^{\otimes(n+m-1)} \longrightarrow A \tag{5.7}$$

[5] In the following the coderivation associated to a map will be noted by same character, but in boldface style. Also note that, strictly speaking, defining the $\mathbf{\Delta}_{N,n}$ as *coderivations* is slight abuse of notation. See the Appendix for details.

and it is defined as

$$[\Delta_m, \Delta'_n] := \tag{5.8}$$
$$:= \Delta_m \left[\sum_{k=0}^{m-1} 1^{\otimes(m-k-1)} \otimes \Delta'_n \otimes 1^{\otimes k} \right] - (-1)^{\deg \Delta_m \cdot \deg \Delta'_n} \Delta'_n \left[\sum_{k=0}^{n-1} 1^{\otimes(n-k-1)} \otimes \Delta_m \otimes 1^{\otimes k} \right],$$

Looking for example at the first bit of the commutator in the previous equation, one has that $\sum_{k=0}^{m-1} 1^{\otimes(m-k-1)} \otimes \Delta'_n \otimes 1^{\otimes k} : A^{\otimes(m+n-1)} \to A^{\otimes m}$ and hence it maps tensors in the domain of the multilinear map Δ_m, as it should. Similar story goes on for the second bit. Actually, at a closer look it might be observed that the previous equation (5.8) is not really a definition but it is just the *projection* on the first factor V of the cotensor algebra of V of the commutator of two coderivations, as it is explained in the Appendix A. Again, let us see by means of an example that the coderivation associated to the commutator as defined above, is the commutator of the coderivations: let us consider the multilinear maps $\Delta_2 : A^{\otimes 2} \to A$ and $\Delta'_1 : A \to A$, both of degree 1, then their commutator is given by

$$[\Delta_2, \Delta'_1] = \Delta_2(1 \otimes \Delta'_1) + \Delta_2(\Delta'_1 \otimes 1) + \Delta'_1 \Delta_2. \tag{5.9}$$

In turn, let us set $N = 3$ and let us compute the coderivation associated to this commutator, which is then a map $F([\Delta_2, \Delta'_1]) : A^{\otimes 3} \to A^{\otimes 2}$

$$\begin{aligned} F([\Delta_2, \Delta'_1]) &= \sum_{k=0}^{1} 1^{\otimes 1-k} \otimes [\Delta_2, \Delta'_1] \otimes 1^{\otimes k} \\ &= 1 \otimes \Delta_2(1 \otimes \Delta'_1) + 1 \otimes \Delta_2(\Delta'_1 \otimes 1) + 1 \otimes \Delta'_1 \Delta_2 + \\ &\quad + \Delta_2(1 \otimes \Delta'_1) \otimes 1 + \Delta_2(\Delta'_1 \otimes 1) \otimes 1 + \Delta'_1 \Delta_2 \otimes 1. \end{aligned} \tag{5.10}$$

On the other hand, one has that

$$F(\Delta_2) = 1 \otimes \Delta_2 + \Delta_2 \otimes 1 \qquad F(\Delta'_1) = 1^{\otimes 2} \otimes \Delta'_1 + 1 \otimes \Delta'_1 \otimes 1 + \Delta'_1 \otimes 1^{\otimes 2}. \tag{5.11}$$

Computing the commutator, $[F(\Delta_2), F(\Delta'_1)] = F(\Delta_2)F(\Delta'_1) - (-1)^{\deg \Delta_2 \cdot \deg \Delta'_1} F(\Delta_2)F(\Delta_1)$, yields

$$
\begin{aligned}
[F(\Delta_2), F(\Delta_1')] = & (1 \otimes \Delta_2 + \Delta_2 \otimes 1)(1^{\otimes 2} \otimes \Delta_1' + 1 \otimes \Delta_1' \otimes 1 + \Delta_1' \otimes 1^{\otimes 2}) + \\
& + (1 \otimes \Delta_1' + \Delta_1' \otimes 1)(1 \otimes \Delta_2 + \Delta_2 \otimes 1) \\
= & 1 \otimes \Delta_2(1 \otimes \Delta_1') + 1 \otimes \Delta_2(\Delta_1' \otimes 1) - \Delta_1' \otimes \Delta_2 + \\
& + \Delta_2 \otimes \Delta_1' + \Delta_2(1 \otimes \Delta_1') \otimes 1 + \Delta_2(\Delta_1' \otimes 1) \otimes 1 + \\
& + 1 \otimes \Delta_1'\Delta_2 - \Delta_2 \otimes \Delta_1' + \Delta_1' \otimes \Delta_2 + \Delta_1'\Delta_2 \otimes 1 \\
= & 1 \otimes \Delta_2(1 \otimes \Delta_1') + 1 \otimes \Delta_2(\Delta_1' \otimes 1) + 1 \otimes \Delta_1'\Delta_2 \\
& + \Delta_2(1 \otimes \Delta_1') \otimes 1 + \Delta_2(\Delta_1' \otimes 1) \otimes 1 + \Delta_1'\Delta_2 \otimes 1,
\end{aligned} \tag{5.12}
$$

matching the (5.10). Notice the presence of the minus signs, due to the degree of the maps and their commutations.

Let $\mathbf{M}_n^{(-\ell)}$ be the coderivation constructed in terms of the multilinear map $M_n^{(-\ell)}$ above - that is, to make contact with the notation of Appendix A, $\mathbf{M}_n^{(-\ell)}$ will play the role of $\mathfrak{m}_n$. We thus denote by $\mathbf{M}_2^{(0)}$ the coderivation constructed in terms of $M_2^{(0)}$, *i.e.* we consider the case $\ell = 0$: then one has an *associative* DG-algebra, where $\mathbf{M}_1^{(0)}$ is simply the coderivation constructed in terms of the differential d. We have the relations

$$
[\mathbf{M}_1^{(0)}, \mathbf{M}_1^{(0)}] = 0 \,, \qquad [\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(0)}] = 0 \,, \qquad [\mathbf{M}_2^{(0)}, \mathbf{M}_2^{(0)}] = 0 \,, \tag{5.13}
$$

with $\mathbf{M}_n^{(0)} = 0$ for $n > 2$. An explicit realization of this on a manifold or on a supermanifold is just the ordinary de Rham complex of forms or superforms, where $M_1^{(0)} := d$ and $M_2^{(0)} := \wedge$, which is an example of *associative* DG-algebra.

Let us consider now the case $\ell = 1$. Using the (anti)commutators, we can rewrite the second coderivation $\mathbf{M}_2^{(-1)}$ of (5.4) as follows

$$
\mathbf{M}_2^{(-1)} = \frac{1}{3}\Big\{\mathbf{Z}_D, \mathbf{M}_2^{(0)}\Big\}, \tag{5.14}
$$

where $\mathbf{Z}_D$ is the coderivation associated to Z_D.

It is easy to verify the first A_∞-relation

$$
\begin{aligned}
[\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(-1)}] &= \Big[\mathbf{M}_1^{(0)}, \frac{1}{3}\Big\{\mathbf{Z}_D, \mathbf{M}_2^{(0)}\Big\}\Big] \\
&= -\frac{1}{3}\Big\{[\mathbf{Z}_D, [\mathbf{M}_2^{(0)}, \mathbf{M}_1^{(0)}]\Big\} + \frac{1}{3}\Big\{\mathbf{M}_2^{(0)}, [\mathbf{M}_1^{(0)}, \mathbf{Z}_D]\Big\} = 0
\end{aligned} \tag{5.15}
$$

since $[\mathbf{M}_2^{(0)}, \mathbf{M}_1^{(0)}] = 0$ by (5.13) (corresponding to the Leibniz rule) and $[\mathbf{M}_1^{(0)}, \mathbf{Z}_D] = 0$ since the PCO is d-closed, $[d, Z_D] = 0$.
On the other hand, by an explicit computation (see sec. (6.2)), one can see that the associativity of $\mathbf{M}_2^{(-1)}$ is violated

$$[\mathbf{M}_2^{(-1)}, \mathbf{M}_2^{(-1)}] = \boldsymbol{\Delta}_3^{(-2)} \neq 0\,, \tag{5.16}$$

where the violation comes from a 3-product $\Delta_3^{(-2)}$. Using again the relations (5.13) and (5.14), we have

$$[\mathbf{M}_1^{(0)}, \boldsymbol{\Delta}_3^{(-2)}] = [\mathbf{M}_1^{(0)}, [\mathbf{M}_2^{(-1)}, \mathbf{M}_2^{(-1)}]] = -2[\mathbf{M}_2^{(-1)}, [\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(-1)}]] = 0 \tag{5.17}$$

which implies that $\boldsymbol{\Delta}_3^{(-2)}$ is d-closed. If $\boldsymbol{\Delta}_3^{(-2)}$ were formally exact, namely if an $\mathbf{M}_3^{(-2)}$ exists such that

$$\boldsymbol{\Delta}_3^{(-2)} = -\frac{1}{2}[\mathbf{M}_1^{(0)}, \mathbf{M}_3^{(-2)}]\,, \tag{5.18}$$

then it would follow

$$\frac{1}{2}[\mathbf{M}_2^{(-1)}, \mathbf{M}_2^{(-1)}] + [\mathbf{M}_1^{(0)}, \mathbf{M}_3^{(-2)}] = 0, \tag{5.19}$$

which might be used as the starting point to build a A_∞-algebra.
The proof that $\boldsymbol{\Delta}_3^{(-2)}$ is indeed exact is deferred to the next subsection where the operator $\Theta(\iota_D)$ is used to compute $\mathbf{M}_3^{(-2)}$. Repeating this analysis, one finds a series of coderivations $\mathbf{M}_n^{(1-n)}$ satisfying the A_∞-relations.

As stated at the beginning of the section, there is another differential operator $M_1^{(1)} = \eta$, acting on the space of forms as a graded-derivation of the exterior algebra, namely with respect to $M_2^{(0)}$. Before proceeding, we underline that the need of this new differential operator is essential in the present construction. Indeed, the explicit construction of the A_∞-algebra is based on the existence of $\Theta(\iota_D)$ which is not a Dirac delta function distribution (compared to Dirac delta function, the latter has no compact support) and it maps outside of the forms as discussed in the introduction. Nonetheless, after the computations are performed, we have to check whether the multiproducts are indeed in the correct functional space of forms. That can be done by checking the vanishing of the commutator of the multiproducts with $M_1^{(1)} = \eta$.

This is to be looked at as in close relation with the procedure used in [26] to introduce the A_∞-algebra. The authors of [26], indeed, produce an A_∞-interacting theory via *string field redefinition*: the crucial ingredient of the construction is the condition that, even if the field redefinition is in the LHS, the multiproducts have to be in the SHS, *i.e.* they have to commute with η.

In the following we will denote the coderivation associated with $M_1^{(1)} = \eta$ simply by $\boldsymbol{\eta}$, as to underline the similarity with the procedure in [26]: indeed $\boldsymbol{\eta}$ will be somewhat "ancillary" with respect to the mutiproducts $M_n^{(\ell)}$ and their associated coderivations $\mathbf{M}_n^{(\ell)}$, as it only controls that at the end of the day everything stays in the SHS.

Getting back to the coderivation notation, this can be expressed as

$$[\boldsymbol{\eta}, \mathbf{M}_2^{(0)}] = 0\,. \tag{5.20}$$

We can check that $\boldsymbol{\eta}$ is also a graded-derivation of $\mathbf{M}_2^{(-1)}$ by observing that

$$\begin{aligned}
[\boldsymbol{\eta}, \mathbf{M}_2^{(-1)}] &= \Big[\boldsymbol{\eta}, \frac{1}{3}\Big\{\mathbf{Z}_D, \mathbf{M}_2^{(0)}\Big\}\Big] \\
&= \frac{1}{3}\Big\{\mathbf{Z}_D, \Big[\mathbf{M}_2^{(0)}, \boldsymbol{\eta}\Big]\Big\} + \frac{1}{3}\Big\{\mathbf{M}_2^{(0)}, \Big[\boldsymbol{\eta}, \mathbf{Z}_D\Big]\Big\} = 0.
\end{aligned} \tag{5.21}$$

The right hand side vanishes since $\boldsymbol{\eta}$ is a derivation of the wedge product by (5.20), and the PCO Z_D commutes with η as proven in (4.6). This implies that the result of the product $\mathbf{M}_2^{(-1)}$ is still in the SHS if both the forms on which it acts are in the SHS, in other words the action of $\mathbf{M}_2^{(-1)}$ preserves the SHS. In the same way, using the complete set of A_∞-relations, one has to prove that all coderivations $\mathbf{M}_n^{(1-n)}$ commute with $\boldsymbol{\eta}$. Finally, note that the set of coderivations $\mathbf{M}_n^{(1-n)}$ are associated to the multiproducts

$$M_n^{(1-n)} : (\Omega^{(\bullet|1)})^{\otimes n} \longrightarrow \Omega^{(\bullet|1)} \tag{5.22}$$

since the wedge product of n forms in $\Omega^{(\bullet|1)}$ has picture n, and the product $\mathbf{M}_n^{(1-n)}$ lower the picture back to 1 .

Let us now consider the last product $M_2^{(-2)}$ in eq. (5.4). As discussed in the beginning, there are two possible products $M_2^{(-1)}$ and $M_2'^{(-1)}$ which can be constructed. They are associated to the two independent odd vectors D and D' that determine the two independent directions in the space of $d\theta$'s (dually). However, in picture 2, we find that there is only one possible product. Using again the coderivation notation we have

$$\mathbf{M}_2^{(-2)} = \frac{1}{3}\Big\{\mathbf{Z}_{D'}, \mathbf{M}_2^{(-1)}\Big\} = \frac{1}{3^2}\Big\{\mathbf{Z}_{D'}, \Big\{\mathbf{Z}_D, \mathbf{M}_2^{(0)}\Big\}\Big\} = \frac{1}{3^2}\Big\{\mathbf{Z}_D, \Big\{\mathbf{Z}_{D'}, \mathbf{M}_2^{(0)}\Big\}\Big\} \tag{5.23}$$

where the last equality is due to the Jacobi identity $(\{A, \{B, C\}\} = \{B, \{A, C\}\} + [[A, B], C])$ and the vanishing of the commutator between Z_D and Z'_D, then it satisfies

$$[\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(-2)}] = 0\,, \tag{5.24}$$

indeed, upon using again the Jacobi identity

$$\Big[\mathbf{M}_1^{(0)}, \frac{1}{3}\Big\{\mathbf{Z}_{D'}, \mathbf{M}_2^{(-1)}\Big\}\Big] = -\frac{1}{3}\Big\{\mathbf{Z}_{D'}, \Big[\mathbf{M}_2^{(-1)}, \mathbf{M}_1^{(0)}\Big]\Big\} + \frac{1}{3}\Big\{\mathbf{M}_2^{(-1)}, \Big[\mathbf{M}_1^{(0)}, \mathbf{Z}_{D'}\Big]\Big\} \tag{5.25}$$

and the right hand side vanishes by the equation (5.15) and by the fact that $[d, Z_{D'}] = 0$.
A simple computation shows that the associativity of $\mathbf{M}_2^{(-2)}$ is violated by a coderivation $\mathbf{M}_3^{(-4)}$ as follows

$$\frac{1}{2}[\mathbf{M}_2^{(-2)}, \mathbf{M}_2^{(-2)}] + [\mathbf{M}_1^{(0)}, \mathbf{M}_3^{(-4)}] = 0\,. \tag{5.26}$$

Again, as above, one can compute the first relations between these products and the commutation relation with $\boldsymbol{\eta}$. The final result gives the coderivations (and the corresponding multiproducts) $\mathbf{M}_n^{(2-2n)}$. Note that these multiproducts act as follows

$$M_n^{(2-2n)} : (\Omega^{(\bullet|2)})^{\otimes n} \to \Omega^{(\bullet|2)}\,. \tag{5.27}$$

We can generalize the set of products to multiplications of forms with *different* pictures. Indeed $\mathbf{M}_2^{(-l)}$ acts on any type of forms regardless their picture, mapping them into a different complex, according to

$$M_2^{(-l)} : \Omega^{(\bullet|p)} \otimes \Omega^{(\bullet|p')} \to \Omega^{(\bullet|p+p'-l)}. \tag{5.28}$$

This leads to study the commutator

$$[\mathbf{M}_2^{(-l)}, \mathbf{M}_2^{(-h)}] = \Delta_3^{-(l+h)}. \tag{5.29}$$

Since any $\mathbf{M}_2^{(-l)}$ satisfies $[\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(-l)}] = 0$ (*i.e.* the first A_∞-relation), we have that $\Delta_3^{(-(l+h))} = [\mathbf{M}_1^{(0)}, \mathbf{M}_3^{(-(l+h))}]$ defining the new co-derivation $\mathbf{M}_3^{(-(l+h))}$. It follows that the multiplicative structure can be extended to $\Omega^{(\bullet|\bullet)}$.

5.1 The Construction of $M_3^{(2-2h)}$

In this subsection we construct explicitly the 3-product $M_3^{(2-2h)}$ for $h \geq 2$, following the suggestions in [26]. We first review the construction of $M_3^{(-2)}$ and then we derive the formula for the 3-product which lowers the total picture changing by 4.

We first define the following coderivation

$$\widetilde{\mathbf{M}}_2^{(-1)} := \frac{1}{3}\Big\{ -i\,\boldsymbol{\Theta}(\iota_D), \mathbf{M}_2^{(0)}\Big\} \tag{5.30}$$

where $\boldsymbol{\Theta}(\iota_D)$ is the coderivation associated to the map $\Theta(\iota_D)$ introduced above. This has the following properties

$$\mathbf{M}_2^{(-1)} = [\mathbf{M}_1^{(0)}, \widetilde{\mathbf{M}}_2^{(-1)}]\,, \qquad \mathbf{M}_2^{(0)} = [\boldsymbol{\eta}\,, \widetilde{\mathbf{M}}_2^{(-1)}]. \tag{5.31}$$

The first equation means that $\mathbf{M}_2^{(-1)}$ is formally exact and it follows from the Jacobi identity, indeed

$$\begin{aligned}\Big[\mathbf{M}_1^{(0)}, \frac{1}{3}\Big\{ -i\,\boldsymbol{\Theta}(\iota_D), \mathbf{M}_2^{(0)}\Big\}\Big] &= -\frac{1}{3}\Big\{ -i\boldsymbol{\Theta}(\iota_D), \Big[\mathbf{M}_2^{(0)}, \mathbf{M}_1^{(0)}\Big]\Big\} + \frac{1}{3}\Big\{\mathbf{M}_2^{(0)}, \Big[\mathbf{M}_1^{(0)}, -i\boldsymbol{\Theta}(\iota_D)\Big]\Big\} \\ &= \frac{1}{3}\Big\{\mathbf{Z}_D, \mathbf{M}_2^{(0)}\Big\},\end{aligned} \tag{5.32}$$

where we have used that $[\mathbf{M}_2^{(0)}, \mathbf{M}_1^{(0)}] = 0$. Noticing that by definition, $\mathbf{M}_2^{(-1)} = \frac{1}{3}\{\mathbf{Z}_D, \mathbf{M}_2^{(0)}\}$, as seen in the previous section, concludes the proof. The second property follows again from the Jacobi identity, since

$$\begin{aligned}\Big[\boldsymbol{\eta}, \frac{1}{3}\Big\{ -i\boldsymbol{\Theta}(\iota_D), \mathbf{M}_2^{(0)}\Big\}\Big] &= -\frac{1}{3}\Big\{ -i\boldsymbol{\Theta}(\iota_D), [\mathbf{M}_2^{(0)}, \boldsymbol{\eta}]\Big\} + \frac{1}{3}\Big\{[\mathbf{M}_2^{(0)}, [\boldsymbol{\eta}, -i\boldsymbol{\Theta}(\iota_D)]\Big\} \\ &= \frac{1}{3}\Big\{\mathbf{M}_2^{(0)}, [\boldsymbol{\eta}, -i\boldsymbol{\Theta}(\iota_D)]\Big\} = \mathbf{M}_2^{(0)},\end{aligned} \tag{5.33}$$

where we have used that $[\eta, -i\Theta(\iota_D)] = 1$. Observing that $\{M_2^{(0)}, 1\} = M_2^{(0)}(1 \otimes 1 + 1 \otimes 1) + M_2^{(0)} = 3\, M_2^{(0)}$ one concludes the proof.

Now, inserting the first equation (5.30) into eq. (5.16), we find that

$$\boldsymbol{\Delta}_3^{(-2)} = [\mathbf{M}_2^{(-1)}, [\mathbf{M}_1^{(0)}, \widetilde{\mathbf{M}}_2^{(-1)}]], \tag{5.34}$$

which give us an explicit formula for the associator $\boldsymbol{\Delta}_3^{(-2)}$ in terms of the coderivations $\mathbf{M}_2^{(-1)}$ and $\widetilde{\mathbf{M}}_2^{(-1)}$. Using the Jacobi identity and the relation $[\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(-1)}] = 0$ in (5.13), the right-hand side of (5.34) can be re-written as

$$\boldsymbol{\Delta}_3^{(-2)} = [\mathbf{M}_1^{(0)}, [\mathbf{M}_2^{(-1)}, \widetilde{\mathbf{M}}_2^{(-1)}]]\,. \tag{5.35}$$

This concludes the proof that the associator $\mathbf{\Delta}_3^{(-2)}$ is formally d-exact. Finally, using this relation in (5.18) we have

$$-\frac{1}{2}[\mathbf{M}_1^{(0)},[\mathbf{M}_2^{(-1)},\widetilde{\mathbf{M}}_2^{(-1)}]]+[\mathbf{M}_1^{(0)},\mathbf{M}_3^{(-2)}]=0\,. \tag{5.36}$$

It follows that

$$\Big[\mathbf{M}_1^{(0)},\Big(-\frac{1}{2}[\mathbf{M}_2^{(-1)},\widetilde{\mathbf{M}}_2^{(-1)}]+\mathbf{M}_3^{(-2)}\Big)\Big]=0, \tag{5.37}$$

from which we deduce

$$\mathbf{M}_3^{(-2)}=\frac{1}{2}[\mathbf{M}_1^{(0)},\widetilde{\mathbf{M}}_3^{(-2)}]+\frac{1}{2}[\mathbf{M}_2^{(-1)},\widetilde{\mathbf{M}}_2^{(-1)}], \tag{5.38}$$

where the first term with $\widetilde{\mathbf{M}}_3^{(-2)}$ is added, being a trivial solution to the above equation. Here $\widetilde{\mathbf{M}}_3^{(-2)}$ is an arbitrary trilinear map, but it is needed in order that the trilinear map $\mathbf{M}_3^{(-2)}$ is in the SHS, namely $[\boldsymbol{\eta},\mathbf{M}_3^{(-2)}]=0$. The calculation is done in the same way as in [26]. This concludes the explicit computation of $\mathbf{M}_3^{(-2)}$.

Let us now move to the 2-product $\mathbf{M}_2^{(-2)}$. This changes the picture by two. We have

$$\mathbf{M}_2^{(-2)}=\frac{1}{3^2}\Big\{\mathbf{Z}_{D'},\Big\{\mathbf{Z}_D,\mathbf{M}_2^{(0)}\Big\}\Big\}=\frac{1}{3}\Big\{\mathbf{Z}_{D'},\frac{1}{3}\Big\{[\mathbf{M}_1^{(0)},-i\,\mathbf{\Theta}(\iota_D)],\mathbf{M}_2^{(0)}\Big\}\Big\}, \tag{5.39}$$

where we have used the coderivation associated to the definition of the PCO $Z_D=[d,-i\Theta(\iota_D)]$. Now, by the Jacobi identity and using the Leibniz rule $[\mathbf{M}_1^{(0)},\mathbf{M}_2^{(0)}]=0$, we get

$$\mathbf{M}_2^{(-2)}=\frac{1}{3}\Big\{\mathbf{Z}_{D'},\Big[\mathbf{M}_1^{(0)},\frac{1}{3}\Big\{-i\,\mathbf{\Theta}(\iota_D),\mathbf{M}_2^{(0)}\Big\}\Big]\Big\}=\frac{1}{3}\Big\{\mathbf{Z}_{D'},\Big[\mathbf{M}_1^{(0)},\widetilde{\mathbf{M}}_{2,D}^{(-1)}\Big]\Big\}, \tag{5.40}$$

where $\widetilde{\mathbf{M}}_{2,D}^{(-1)}$ is defined as in (5.30), but we add the subscript D to recall that it depends upon the choice of D. Using again the Jacobi identity and that the PCO is closed, *i.e.* $[\mathbf{M}_1^{(0)},\mathbf{Z}_{D'}]=0$, this yields

$$\mathbf{M}_2^{(-2)}=\Big[\mathbf{M}_1^{(0)},\frac{1}{3}\Big\{\mathbf{Z}_{D'},\widetilde{\mathbf{M}}_{2,D}^{(-1)}\Big\}\Big]. \tag{5.41}$$

Thus, we can define

$$\widetilde{\mathbf{M}}_2^{(-2)}=\frac{1}{3}\Big\{\mathbf{Z}_{D'},\widetilde{\mathbf{M}}_{2,D}^{(-1)}\Big\}\,. \tag{5.42}$$

Notice that acting the $\boldsymbol{\eta}$ (the coderivation associated to η), we have

$$[\boldsymbol{\eta},\widetilde{\mathbf{M}}_2^{(-2)}]=\frac{1}{3}\Big\{\mathbf{Z}_{D'},[\boldsymbol{\eta},\widetilde{\mathbf{M}}_{2,D}^{(-1)}]\Big\}=\frac{1}{3}\Big\{\mathbf{Z}_{D'},\mathbf{M}_2^{(0)}\Big\}=\mathbf{M}_{2,D'}^{(-1)}\,, \tag{5.43}$$

and consequently, we have

$$[\boldsymbol{\eta}, \mathbf{M}_2^{(-2)}] = \frac{1}{3}\Big\{\mathbf{Z}_{D'}, [\boldsymbol{\eta}, \widetilde{\mathbf{M}}_2^{(-2)}]\Big\} = [\mathbf{M}_1^{(0)}, \mathbf{M}_{2,D'}^{(-1)}] = -\frac{1}{3}\Big\{\mathbf{Z}_{D'}, [\mathbf{M}_1^{(0)}, \mathbf{M}_2^{(0)}\Big\} = 0 \tag{5.44}$$

where the last equation follows from the Leibniz rule. This implies that $\mathbf{M}_2^{(-2)}$ is in the SHS as it should, consistently with the construction. In addition, from (5.42), we have

$$\widetilde{\mathbf{M}}_2^{(-2)} = \frac{1}{3}\Big\{[\mathbf{M}_1^{(0)}, -i\,\boldsymbol{\Theta}(\iota_{D'})], \widetilde{\mathbf{M}}_{2,D}^{(-1)}\Big\} = \Big[\mathbf{M}_1^{(0)}, \widetilde{\widetilde{\mathbf{M}}}_2^{(-2)}\Big] - \frac{1}{3}\Big\{ -i\,\boldsymbol{\Theta}(\iota_{D'}), \mathbf{M}_{2,D}^{(-1)}\Big\}\,, \tag{5.45}$$

using again the Jacobi identities and where we put

$$\widetilde{\widetilde{\mathbf{M}}}_2^{(-2)} = \frac{1}{3}\Big\{ -i\,\boldsymbol{\Theta}(\iota_{D'}), \widetilde{\mathbf{M}}_{2,D}^{(-1)}\Big\}. \tag{5.46}$$

As a consistency check, we have that

$$\Big[\mathbf{M}_1^{(0)}, \widetilde{\mathbf{M}}_2^{(-2)}\Big] = \Big[\mathbf{M}_1^{(0)}, \frac{1}{3}\Big\{ -i\,\boldsymbol{\Theta}(\iota_{D'}), \mathbf{M}_{2,D}^{(-1)}\Big\}\Big] = \frac{1}{3}\Big\{\mathbf{Z}_{D'}, \mathbf{M}_{2,D}^{(-1)}\Big\} = \mathbf{M}_2^{(-2)}\,. \tag{5.47}$$

which reproduces (5.30) for the picture 2. As a final check, we observe that

$$[\boldsymbol{\eta}, \widetilde{\mathbf{M}}_{2,D}^{(-1)}] = \mathbf{M}_2^{(0)}\,, \qquad [\boldsymbol{\eta}, \widetilde{\mathbf{M}}_{2,D'}^{(-1)}] = \mathbf{M}_2^{(0)}\,, \tag{5.48}$$

and therefore

$$\Big[\boldsymbol{\eta}, \widetilde{\widetilde{\mathbf{M}}}_2^{(-2)}\Big] = \widetilde{\mathbf{M}}_{2,D'}^{(-1)} - \widetilde{\mathbf{M}}_{2,D}^{(-1)} \tag{5.49}$$

and for consistency

$$\Big[\boldsymbol{\eta}, \Big[\boldsymbol{\eta}, \widetilde{\widetilde{\mathbf{M}}}_2^{(-2)}\Big]\Big] = [\boldsymbol{\eta}, \widetilde{\mathbf{M}}_{2,D'}^{(-1)}] - [\boldsymbol{\eta}, \widetilde{\mathbf{M}}_{2,D}^{(-1)}] = 0\,, \tag{5.50}$$

since $[\boldsymbol{\eta}, \boldsymbol{\eta}] = 0$.

Inserting (5.41) into the following defining equation for $\mathbf{M}_3^{(-4)}$

$$\frac{1}{2}[\mathbf{M}_2^{(-2)}, \mathbf{M}_2^{(-2)}] + [\mathbf{M}_1^{(0)}, \mathbf{M}_3^{(-4)}] = 0 \tag{5.51}$$

we finally get

$$\mathbf{M}_3^{(-4)} = [\mathbf{M}_1^{(0)}, \widetilde{\mathbf{M}}_3^{(-4)}] + \frac{1}{2}[\mathbf{M}_2^{(-2)}, \widetilde{\mathbf{M}}_2^{(-2)}]\,, \tag{5.52}$$

where $\widetilde{\mathbf{M}}_3^{(-4)}$ is again the trivial term, as above needed to ensure that the 3-coderivation $\mathbf{M}_3^{(-4)}$ being in the SHS. To check this fact, we act with $\boldsymbol{\eta}$ on $\mathbf{M}_3^{(-4)}$ and we perform the computation as for $\mathbf{M}_3^{(-2)}$. Then, it follows

$$[\boldsymbol{\eta}, \widetilde{\mathbf{M}}_3^{(-4)}] = [\widetilde{\mathbf{M}}_2^{(-2)}, \mathbf{M}_{2,D'}^{(-1)}] + [\mathbf{M}_1^{(0)}, \widehat{\mathbf{M}}_3^{(-3)}] \,. \tag{5.53}$$

The last term is added to take into account the non-associativity of $\mathbf{M}_2^{(-1)}$ products as discussed in the previous section. In fact, by consistency, acting with $\boldsymbol{\eta}$ on the l.h.s., we end up with

$$\begin{aligned} &\Big[[\boldsymbol{\eta}, \widetilde{\mathbf{M}}_2^{(-2)}], \mathbf{M}_{2D'}^{(-1)}] + [\mathbf{M}_1^{(0)}, [\boldsymbol{\eta}, \widehat{\mathbf{M}}_3^{(-3)}]] \\ &= \Big[\mathbf{M}_{2,D'}^{(-1)}, \mathbf{M}_{2D'}^{(-1)}] + [\mathbf{M}_1^{(0)}, [\boldsymbol{\eta}, \widehat{\mathbf{M}}_3^{(-3)}]] = 0 \end{aligned} \tag{5.54}$$

which can be finally solved due to A_∞-algebra for the products $\mathbf{M}_{2D'}^{(-1)}$ in terms of the $\mathbf{M}_3^{(-2)}$ and a new possible d-exact correction term. This implies that the new 3-product $M_3^{(-4)}$ associated to the coderivation $\mathbf{M}_3^{(-4)}$ is indeed in the SHS as in the case of the (-2) picture. A complete recursive procedure must be arranged to verify that all higher products and their relations can be constructed along the lines of [26], with the additional features due to multiple directions D_i for constructing different type of products lowering the pictures. The richness of this structure emerging from this algebraic framework has never been explored extending the beautiful construction of [26]. The complete proof is deferred to further publications.

5.2 Some Explicit Examples of Computations

In order to illustrate and make more clear the constructions in the previous section, we discuss specific examples of products of forms. We take into consideration the 2-products $M_2^{(-l)}$ with $l = 0, 1, 2$ and for them we consider a collections of 0-, 1- and 2-forms of the following types

$$\begin{aligned} &\omega_A^{(0|0)} = A(x,\theta) \,, \qquad \omega_B^{(0|1)} = B(x,\theta)\delta(d\theta^1) \,, \\ &\omega_{B'}^{(0|1)} = B'(x,\theta)\delta(d\theta^2) \,, \qquad \omega_C^{(0|2)} = C(x,\theta)\delta(d\theta^1)\delta(d\theta^2) \,, \end{aligned} \tag{5.55}$$

where A, B, B' and C are supefields.[6] First of all, in the table 1 the picture numbers of the resulting forms are listed.

[6]In the following, we write as an upperscript only the picture number $\omega^{(0|a)} \to \omega^{(a)}$ with $a = 0, 1, 2$.

	0×0	0×1	1×1	0×2	1×2	2×2
$M_2^{(0)}$	0	1	2	2	/	/
$M_2^{(-1)}$	/	0	1	1	2	/
$M_2^{(-2)}$	/	/	0	0	1	2

Table 1: In the table, we compute $M_2^{(-l)}(\omega^{(a)}, \omega^{(b)})$ where the pictures a, b are listed in the first row as $a \times b$. The slanted line / denotes a trivial result, while the numbers in the other boxes denote the picture of the resulting form.

Since $M_2^{(0)}$ is the usual wedge product we have

$$\begin{aligned}
M_2^{(0)}(\omega_A^{(0)}, \omega_{A'}^{(0)}) &= AA' \,, \\
M_2^{(0)}(\omega_A^{(0)}, \omega_{B'}^{(1)}) &= AB'\delta(d\theta^2) \,, \\
M_2^{(0)}(\omega_B^{(1)}, \omega_{B'}^{(1)}) &= BB'\delta(d\theta^1)\delta(d\theta^2) \,, \\
M_2^{(0)}(\omega_A^{(0)}, \omega_C^{(2)}) &= AC\delta(d\theta^1)\delta(d\theta^2) \,, \\
M_2^{(0)}(\omega_B^{(1)}, \omega_C^{(2)}) &= 0 \,, \\
M_2^{(0)}(\omega_C^{(2)}, \omega_{C'}^{(2)}) &= 0 \,,
\end{aligned} \tag{5.56}$$

The last two expressions vanish since there is no picture 3 or 4 in our case. Let us compute now $M_2^{(-1)}$

$$\begin{aligned}
M_2^{(-1)}(\omega_A^{(0)}, \omega_{A'}^{(0)}) &= 0 \,, \\
M_2^{(-1)}(\omega_A^{(0)}, \omega_{B'}^{(1)}) &= \frac{1}{3}\Big[Z_D\Big(AB'\delta(d\theta^2) \Big) + AZ_D\Big(B'\delta(d\theta^2) \Big) \Big] \,, \\
M_2^{(-1)}(\omega_B^{(1)}, \omega_{B'}^{(1)}) &= \frac{1}{3}\Big[Z_D\Big(BB'\delta(d\theta^1)\delta(d\theta^2) \Big) + Z_D\Big(B\delta(d\theta^1) \Big) B'\delta(d\theta^2) + B\delta(d\theta^1) Z_D\Big(B'\delta(d\theta^2) \Big) \Big] \,, \\
M_2^{(-1)}(\omega_A^{(0)}, \omega_C^{(2)}) &= \frac{1}{3}\Big[Z_D\Big(AC\delta(d\theta^1)\delta(d\theta^2) \Big) + AZ_D\Big(C\delta(d\theta^1)\delta(d\theta^2) \Big) \Big] \,, \\
M_2^{(-1)}(\omega_B^{(1)}, \omega_C^{(2)}) &= \frac{1}{3}\Big[Z_D\Big(B\delta(d\theta^1) \Big) C\delta(d\theta^1)\delta(d\theta^2) + B\delta(d\theta^1) Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2) \Big) \Big] \,, \\
M_2^{(-1)}(\omega_C^{(2)}, \omega_{C'}^{(2)}) &= 0 \,,
\end{aligned} \tag{5.57}$$

Explicitly, we have

$$
\begin{aligned}
Z_D\Big(AB'\delta(d\theta^2)\Big) &= d\Big[-i\Theta(\iota_D)\Big(AB'\delta(d\theta^2)\Big)\Big] - i\Theta(\iota_D)d\Big(AB'\delta(d\theta^2)\Big) \\
&= \partial_2(AB')\,, \\
Z_D\Big(B'\delta(d\theta^2)\Big) &= d\Big[-i\Theta(\iota_D)\Big(B'\delta(d\theta^2)\Big)\Big] - i\Theta(\iota_D)d\Big(B'\delta(d\theta^2)\Big) \\
&= \partial_2 B'\,, \\
Z_D\Big(B\delta(d\theta^1)\Big) &= d\Big[-i\Theta(\iota_D)\Big(B\delta(d\theta^1)\Big)\Big] - i\Theta(\iota_D)d\Big(B\delta(d\theta^1)\Big) \\
&= \partial_1 B\,, \\
Z_D\Big(AC\delta(d\theta^1)\delta(d\theta^2)\Big) &= d\Big[-i\Theta(\iota_D)\Big(AC\delta(d\theta^1)\delta(d\theta^2)\Big)\Big] - i\Theta(\iota_D)d\Big(AC\delta(d\theta^1)\delta(d\theta^2)\Big) \\
&= -2D^\alpha\partial_\alpha(AC)\delta(D\cdot d\theta)\,,
\end{aligned}
$$

$$
\begin{aligned}
Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big) &= d\Big[-i\Theta(\iota_D)\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)\Big] - i\Theta(\iota_D)d\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big) \\
&= -2D^\alpha\partial_\alpha C\delta(D\cdot d\theta)\,.
\end{aligned} \tag{5.58}
$$

Finally

$$
\begin{aligned}
M_2^{(-1)}(\omega_A^{(0)},\omega_{A'}^{(0)}) &= 0\,, \\
M_2^{(-1)}(\omega_A^{(0)},\omega_{B'}^{(1)}) &= \frac{1}{3}\Big(\partial_2(AB') + A\partial_2 B'\Big)\,, \\
M_2^{(-1)}(\omega_B^{(1)},\omega_{B'}^{(1)}) &= \frac{1}{3}\Big[-2D^\alpha\partial_\alpha(BB')\delta(D\cdot d\theta) + \partial_1 BB'\delta(d\theta^2) - B\partial_2 B'\delta(d\theta^1)\Big]\,, \\
M_2^{(-1)}(\omega_A^{(0)},\omega_C^{(2)}) &= \frac{1}{3}\Big[-2D^\alpha\partial_\alpha(AC)\delta(D\cdot d\theta) - 2AD^\alpha\partial_\alpha C\delta(D\cdot d\theta)\Big] \\
&= -\frac{2}{3}\Big[D^\alpha\partial_\alpha(AC) + AD^\alpha\partial_\alpha C\Big]\delta(D\cdot d\theta)\,, \\
M_2^{(-1)}(\omega_B^{(1)},\omega_C^{(2)}) &= \frac{1}{3}\Big[\partial_1 BC\delta(d\theta^1)\delta(d\theta^2) + 2BD^\alpha\partial_\alpha C\delta(d\theta^1)\delta(D\cdot d\theta)\Big] \\
&= \frac{1}{3}\Big[\partial_1 BC + \frac{2}{D^1}BD^\alpha\partial_\alpha C\Big]\delta(d\theta^1)\delta(d\theta^2)\,, \\
M_2^{(-1)}(\omega_C^{(2)},\omega_{C'}^{(2)}) &= 0\,,
\end{aligned} \tag{5.59}
$$

The resulting products have the correct picture assignment as depicted in the table 1. We notice that the result depends upon the choice of the PCO and therefore they depend upon the odd vector field D.

Now, we consider the last case: $M_2^{(-2)}$. We have

$$
\begin{aligned}
M_2^{(-2)}(\omega_A^{(0)},\omega_{A'}^{(0)}) &= 0\,,\\
M_2^{(-2)}(\omega_A^{(0)},\omega_{B'}^{(1)}) &= 0\,,\\
M_2^{(-2)}(\omega_B^{(1)},\omega_{B'}^{(1)}) &= \frac{1}{9}\Big[Z_{D'}Z_D\Big(BB'\delta(d\theta^1)\delta(d\theta^2)\Big)+Z_{D'}\Big(M_2^{(-1)}(\omega_B^{(1)},\omega_{B'}^{(1)})\Big)+Z_D\Big(M_2'^{(-1)}(\omega_B^{(1)},\omega_{B'}^{(1)})\Big)\\
&+ Z_{D'}\Big(B\delta(d\theta^1)\Big)Z_D\Big(B'\delta(d\theta^2)\Big)+Z_D\Big(B\delta(d\theta^1)\Big)Z_{D'}\Big(B'\delta(d\theta^2)\Big)\Big]\,,\\
M_2^{(-2)}(\omega_A^{(0)},\omega_C^{(2)}) &= \frac{1}{9}\Big[Z_{D'}Z_D\Big(AC\delta(d\theta^1)\delta(d\theta^2)\Big)+Z_{D'}M_2^{(-1)}(\omega_A^{(0)},\omega_C^{(2)})+Z_DM_2'^{(-1)}(\omega_A^{(0)},\omega_C^{(2)})\\
&+ AZ_{D'}Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)\Big]\,,\\
M_2^{(-2)}(\omega_B^{(1)},\omega_C^{(2)}) &= \frac{1}{9}\Big[Z_{D'}M_2^{(-1)}(\omega_B^{(1)},\omega_C^{(2)})+Z_DM_2'^{(-1)}(\omega_B^{(1)},\omega_C^{(2)})\\
&+ Z_{D'}\Big(B\delta(d\theta^1)\Big)Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)+Z_D\Big(B\delta(d\theta^1)\Big)Z_{D'}\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)\\
&+ B\delta(d\theta^1)Z_{D'}Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)\Big]\,,\\
M_2^{(-2)}(\omega_C^{(2)},\omega_{C'}^{(2)}) &= \frac{1}{9}\Big[Z_{D'}Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)\Big(C'\delta(d\theta^1)\delta(d\theta^2)\Big)\\
&+ Z_{D'}\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)Z_D\Big(C'\delta(d\theta^1)\delta(d\theta^2)\Big)\\
&+ Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)Z_{D'}\Big(C'\delta(d\theta^1)\delta(d\theta^2)\Big)\\
&+ \Big(C\delta(d\theta^1)\delta(d\theta^2)\Big)Z_{D'}Z_D\Big(C'\delta(d\theta^1)\delta(d\theta^2)\Big)\Big]\,.
\end{aligned}
\tag{5.60}
$$

where we have denoted $M_2'^{(-1)}$ the 2-product with respect to the odd vector D'. Each single piece is computed as follows

$$
\begin{aligned}
Z_{D'}\Big(M_2^{(-1)}(\omega_B^{(1)},\omega_{B'}^{(1)})\Big) &= -\frac{2}{3}\epsilon^{\beta\alpha}\partial_\beta\partial_\alpha(BB')+\frac{1}{3D'^1}D'^\beta\partial_\beta(\partial_1BB')-\frac{1}{3D'^2}D'^\beta\partial_\beta(B\partial_2B')\,,\\
Z_D\Big(M_2'^{(-1)}(\omega_B^{(1)},\omega_{B'}^{(1)})\Big) &= -\frac{2}{3}\epsilon^{\beta\alpha}\partial_\beta\partial_\alpha(BB')+\frac{1}{3D^1}D^\beta\partial_\beta(\partial_1BB')-\frac{1}{3D^2}D^\beta\partial_\beta(B\partial_2B')\,,\\
Z_D\Big(B\delta(d\theta^1)\Big) &= \partial_1B\,,\\
Z_{D'}\Big(B\delta(d\theta^1)\Big) &= \partial_1B\,,\\
Z_D\Big(B'\delta(d\theta^2)\Big) &= \partial_2B'\,,\\
Z_{D'}\Big(B'\delta(d\theta^2)\Big) &= \partial_2B'\,,\\
Z_{D'}\Big(M_2^{(-1)}(\omega_B^{(1)},\omega_C^{(2)})\Big) &= -\frac{2}{3}D'^\alpha\partial_\alpha\Big[\partial_1BC+\frac{2}{D^1}BD^\alpha\partial_\alpha C\Big]\delta(D'\cdot d\theta)\,,
\end{aligned}
$$

$$
\begin{aligned}
Z_D\Big(M_2'^{(-1)}(\omega_B^{(1)},\omega_C^{(2)})\Big) &= -\frac{2}{3}D^\alpha\partial_\alpha\Big[\partial_1 BC+\frac{2}{D'^1}BD'^\alpha\partial_\alpha C\Big]\delta(D\cdot d\theta)\,,\\
Z_D\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big) &= -2D^\alpha\partial_\alpha C\delta(D\cdot d\theta)\,,\\
Z_{D'}\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big) &= -2D'^\alpha\partial_\alpha C\delta(D'\cdot d\theta)\,,\\
Z_{D'}Z_D\Big(BB'\delta(d\theta^1)\delta(d\theta^2)\Big) &= 2\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta(BB')\,,\\
Z_DZ_{D'}\Big(C\delta(d\theta^1)\delta(d\theta^2)\Big) &= 2\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta C\,.
\end{aligned}
\tag{5.61}
$$

Now, it is a simple matter to replace each single pieces into the definitions given in the previous equations. For example, we get

$$
\begin{aligned}
M_2^{(-2)}(\omega_C^{(2)},\omega_{C'}^{(2)}) &= \frac{1}{9}\Big[(\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta C)C'\delta(d\theta^1)\delta(d\theta^2)+4D'^\alpha\partial_\alpha C\delta(D'\cdot d\theta)D^\alpha\partial_\alpha C'\delta(D\cdot d\theta)\\
&+ 4D^\alpha\partial_\alpha C\delta(D\cdot d\theta)D'^\alpha\partial_\alpha C'\delta(D'\cdot d\theta)+C\delta(d\theta^1)\delta(d\theta^2)(\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta C')\Big]\\
&= \frac{1}{9}\Big[(\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta C)C'+2\epsilon^{\alpha\beta}\partial_\alpha C\partial_\beta C'+C(\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta C')\Big]\delta(d\theta^1)\delta(d\theta^2)\\
&= \frac{1}{9}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta(CC')\delta(d\theta^1)\delta(d\theta^2)\,.
\end{aligned}
\tag{5.62}
$$

which is independent of D and of D'. Since $M_2^{(-2)}$ maps two 2-picture forms into a 2-picture form it also preserves the invariance under $SL(2)$ isometries and therefore the result can be written in manifestly invariant way.

In the same way, one can compute the other expressions. Finally, we can check the non-associativity for the last expression, namely we can check that

$$
M_2^{(-2)}\Big(\omega_C^{(2)},M_2^{(-2)}(\omega_{C'}^{(2)},\omega_{C''}^{(2)})\Big)+M_2^{(-2)}\Big(M_2^{(-2)}(\omega_C^{(2)},\omega_{C'}^{(2)}),\omega_{C''}^{(2)}\Big)\neq 0\,.
\tag{5.63}
$$

leading to a 3-product $M_3^{(-4)}$ source of the A_∞-algebra.

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Appendix A: A Nod to Coderivations and A_∞-Algebras

For the sake of readability of the paper and for future reference, we now briefly introduce some basic elements in the theory of A_∞-algebras.

Generally speaking, A_∞-algebras are examples of non-associative algebras, first introduced by Stashef, see [34] in the context of homotopy theory. In what follow, by the way, we will give a different treatment - somewhat more abstract - compared to the original one, based on the notion of *cotensor algebra* and *coderivations* (see [33] for an extended and in depth discussion). We start recalling that, over a field or a ring k, a $\mathbb{Z}$-graded *coassociative coalgebra* is a pair (C, Δ) where $C \coloneqq \bigoplus_{i\in\mathbb{Z}} C^{(i)}$ is $\mathbb{Z}$-graded k-module and $\Delta : C \to C \otimes C$ is a *coassociative coproduct*, that is it satisfies $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

So far we have described objects in the category $\mathsf{CoAlg}_k(C)$: to complete the categorial description we need to introduce morphisms between the objects of the category. Given two coalgebras (C, Δ_C) and (D, Δ_D), we call a *cohomomorphisms* degree 0 maps $\mathfrak{F} : C \to D$ satisfying $\Delta_D\mathfrak{F} = (\mathfrak{F} \otimes \mathfrak{F})\Delta_C$. Notice that here the degree is a $\mathbb{Z}$-degree and it refers to the $\mathbb{Z}$-grading of the k-modules C and D, that is for a cohomomorphism $\mathfrak{F} : C \to D$, if c is a homogeneous element of degree i in C, *i.e.* if $c \in C^{(i)} \subset C$, then $\mathfrak{F}(c)$ is a homogeneous element of degree i in D, *i.e.* $\mathfrak{F}(c) \in D^{(i)}$.

A *coderivation* $\mathfrak{D} : C \to C$ on C is a degree 1 map that satisfies the *coLeibniz rule*, $\Delta\mathfrak{D} = (1 \otimes \mathfrak{D} + \mathfrak{D} \otimes 1)\Delta$. In particular, one defines a *differentially graded* (coassociative) *coalgebra* a triple $(C, \Delta, \mathfrak{D})$ where the pair (C, Δ) is a coassociative coalgebra and $\mathfrak{D}$ is a coderivation satisfying $\mathfrak{D}^2 = 0$.

More in general, given a coalgebra (C, Δ), one can allow cohomomorphisms and coderivations of any degree, each satisfying the defining properties. In this case one can introduce the $\mathbb{Z}$-graded k-module $CoEnd_k^\bullet(C) \coloneqq \bigoplus_{n\in\mathbb{Z}} CoEnd_k^n(C)$, where $CoEnd_k^n(C) \coloneqq \{\mathfrak{F} : C \to C : \deg(\mathfrak{F}) = n\}$, and its sub-module $CoDer_k^\bullet(C) \coloneqq \bigoplus_{n\in\mathbb{Z}} CoDer_k^n(C)$, where $CoDer_k^n(C) \coloneqq \{\mathfrak{D} \in CoEnd_k^n(C) : \Delta\mathfrak{D} = (\mathfrak{D} \otimes 1 + 1 \otimes \mathfrak{D})\Delta\}$. Defining the graded commutator as

$$[\mathfrak{F}, \mathfrak{G}] = \mathfrak{F} \circ \mathfrak{G} - (-1)^{|\mathfrak{F}||\mathfrak{G}|}\mathfrak{G} \circ \mathfrak{F} \tag{A.1}$$

for $\deg \mathfrak{F} = |\mathfrak{F}|$ and $\deg(\mathfrak{G}) = |\mathfrak{G}|$, one can observe the following fundamental fact: $(CoDer_k^\bullet(C), [\cdot, \cdot])$ is a *Lie subalgebra* of $CoEnd_k^\bullet(C)$. The proof is straighforward: it is enough to check that the commutator closes in $CoDer_k^\bullet(C)$. This is a very useful result, which will be constantly exploited in the following, since it allows one to use the various operations in the Lie algebras, *e.g.* Jacobi identity, when dealing with coderivations. Notice, in any case, that it is *not* true that the composition of two coderivation yields a coderivation: this mirrors the fact that the composition of two derivative (vector fields) does not yields a derivative (vector field), but a suitable commutator of them does.

Possibly the most important example of coalgebra is the *cotensor algebra* of a $\mathbb{Z}$-graded k-module V. The cotensor algebra of V is the pair $(\mathsf{T}(V), \Delta)$, where $\mathsf{T}(V) \coloneqq \bigoplus_{n=0}^{\infty} V^{\otimes n}$ and the coassociative multiplication Δ is defined as

$$\begin{aligned} \Delta : \mathsf{T}(V) &\longrightarrow \mathsf{T}(V) \otimes \mathsf{T}(V) \\ v_1 \otimes \ldots \otimes v_n &\longmapsto \textstyle\sum_{k=0}^{n} (v_1 \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes v_n) \end{aligned} \tag{A.2}$$

For example, one has that for the tensor $v_1 \otimes v_2 \otimes_3 \in V^{\otimes 3} \subset \mathsf{T}(V)$

$$\Delta(v_1 \otimes v_2 \otimes v_3) = 1 \otimes (v_1 \otimes v_2 \otimes v_3) + v_1 \otimes (v_2 \otimes v_3) + (v_1 \otimes v_3) \otimes v_3 + (v_1 \otimes v_2 \otimes v_3) \otimes 1$$

where the first summand belongs to $k \otimes V^{\otimes 3} \subset \mathsf{T}(V) \otimes \mathsf{T}(V)$, the second to $V \otimes V^{\otimes 2} \subset \mathsf{T}(V) \otimes \mathsf{T}(V)$ and so on.

We are interested into the coderivations corresponding to this coproduct: these can be characterized by dualizing the construction for the ordinary tensor algebra. In particular, there is a map from the coderivations on $\mathsf{T}(V)$ to the multilinear maps $\mathsf{T}(V) \to V$, which is just given by the composition of $\mathfrak{D} : \mathsf{T}(V) \to \mathsf{T}(V)$ with the projection $\mathfrak{p}_1 : \mathsf{T}(V) \to V$. Its inverse is constructed via the following steps: first, one takes a collection of multilinear maps $\{m_k : V^{\otimes k} \to V\}_{k \geq 0}$ such that $\deg(m_k) = 1$ for any $k \geq 0$ and with $v_1 \otimes \ldots \otimes v_k \mapsto m_k(v_1, \ldots, v_k)$ for any $v_1, \ldots, v_k \in V$ of homogeneous degree $\deg(v_k) \coloneqq |v_k|$ so that $\deg(m_k(v_1, \ldots, v_k)) \coloneqq \sum_k |v_k| + 1$ and where we define $m_0 : k \to V$ is so that $m_0(1) \in V$ has degree 1. Then the maps are extended to the whole $\mathsf{T}(V)$ as follows $m_k \mapsto \mathfrak{m}_k : \mathsf{T}(V) \to \mathsf{T}(V)$ with

$$\mathfrak{m}_k(v_1, \ldots, v_n) \coloneqq \sum_{\ell=1}^{n-k} (-1)^{\sum_{i=1}^{\ell-1} |v_i|} v_1 \otimes \ldots v_{\ell-1} \otimes m_k(v_\ell, \ldots, v_{\ell+k-1}) \otimes v_{\ell+k} \otimes \ldots \otimes v_n, \tag{A.3}$$

where we notice that the sign is there because of the Koszul rule of commutation of the degree 1 map m_k with the homogeneous tensors $v_1, \ldots, v_{\ell-1} \in V$, with $\deg(v_k) = |v_k|$ and that, in particular, $\mathfrak{m}_k : V^{\otimes n} \subset \mathsf{T}(V) \to V^{\otimes n-k+1} \subset \mathsf{T}(V)$. Finally, the coderivation $\mathfrak{m} : \mathsf{T}(V) \to \mathsf{T}(V)$ is given summing over all of the maps $\mathfrak{m}_k$, as follows

$$\mathfrak{m}(v_1, \ldots, v_n) \coloneqq \sum_{k \geq 0} \mathfrak{m}_k(v_1, \ldots, v_n). \tag{A.4}$$

Once a coderivation $\mathfrak{m} = \sum_k \mathfrak{m}_k$ on $\mathsf{T}(V)$ has been constructed it is natural to ask whenever it is is a codifferential, that is whenever it is such that $\mathfrak{m}^2 = 0$: the definition is related to this question. Let $(\mathsf{T}(V), \Delta)$ be the cotensor algebra of a $\mathbb{Z}$-graded vector space and let $\mathfrak{m} = \sum_k \mathfrak{m}_k$ be the coderivation on $\mathsf{T}(V)$. Then we call a *weak* A_∞-algebra the differentially graded coalgebra $(\mathsf{T}(V), \Delta, \mathfrak{m})$, that is for $(\mathsf{T}(V), \Delta, \mathfrak{m})$ to be a weak A_∞-algebra, the coderivation $\mathfrak{m}$ is actually a codifferential, satisfying $\mathfrak{m}^2 = 0$. In particular, a weak A_∞-algebra is an A_∞-algebra if $m_0 : k \to V$ is the zero map.
We now look forward to unravel the condition $\mathfrak{m}^2 = 0$ in the definition of an A_∞-algebra in order to see which sort of condition it gives on the multilinear maps m_k. With an eye to the previous section, first of all let us observe the following fact: $\mathfrak{m}^2 : \mathsf{T}(V) \to \mathsf{T}(V)$ is a (degree 2) map having image into $V \oplus V^{\otimes 2} \oplus \ldots \subset \mathsf{T}(V)$ (recall that m_0= 0), therefore requiring $\mathfrak{m}^2 = 0$ is equivalent to write down the corresponding conditions in all of the summand $V^{\otimes k \geq 1}$ of the image separately. Definining $\mathfrak{p}_k : \mathsf{T}(V) \to V^{\otimes k}$ the projection map onto the i-th component of $\mathsf{T}(V)$, the condition $\mathfrak{m}^2 = 0$ is equivalent to $\mathfrak{p}_k \circ \mathfrak{m}^2 = 0$ for any $k \geq 0$. What is crucial, though, is that due to the anticommutativity of the $\mathfrak{m}_i$'s, one has that $\mathfrak{p}_1 \mathfrak{m}^2 = 0$ is a sufficient condition for $\mathfrak{m}^2 = 0$: therefore the only thing we will be concerned will be the projection onto the first factor $V \subset \mathsf{T}(V)$.
The condition $\mathfrak{p}_1 \mathfrak{m}^2 = 0$ can be rewritten in terms of the multilinear maps $m_k : V^{\otimes k} \to V$ making up the codifferential in a very compact and elegant fashion. It reads:

$$\sum_{\kappa+\ell=n+1} \sum_{i=0}^{\kappa-1} (-1)^{\sum_{j=1}^{i} |v_j|} m_\kappa \left(v_1, \ldots, v_i, m_\ell(v_{i+1}, \ldots, v_{i+\ell}), v_{i+\ell+1}, \ldots, v_n\right) = 0, \tag{A.5}$$

where any tensor $v_j \in V$ is understood to be homogeneous of degree $|v_j|$. Notice that the map is well-defined, indeed m_ℓ acts on a ℓ-tensor $v_i \otimes \ldots \otimes v_{i+\ell} \in V^{\otimes \ell}$ and m_κ acts on a $i+1+(n-i-\ell) = n+1-\ell = \kappa$-tensor $v_1 \otimes \ldots, v_i \otimes m_\ell(v_{i+1}, \ldots, v_{i+\ell}) \otimes v_{i+\ell+1} \otimes \ldots \otimes v_n \in V^{\otimes \kappa}$.
We are now in the position to write the first relations coming from the previous equation (A.5): notice that since $m_0 = 0$, the first non-trivial relation is given by the choice $\kappa + \ell = 2$ with $\kappa = \ell = 1$ and it reads

$$m_1^2(v_1) = 0, \tag{A.6}$$

which says that, recalling that m_1 has degree 1, the pair (V, m_1) is a *complex* of $\mathbb{Z}$-graded k-modules, having $m_1 : V^{(n)} \to V^{(n+1)}$ as the differential of the complex. The second relation comes from the choice $\kappa+\ell=3$ and it yields the condition

$$m_1(m_2(v_1,v_2))+m_2(m_1(v_1),v_2)+(-1)^{|v_1|}m_2(v_1,m_1(v_2))=0, \tag{A.7}$$

which is the Leibniz rule for the differential m_1 with respect to the product $m_2 : V^{\otimes 2}\to V$. The third relations - possibily the most characterizing one for an A_∞-algebra - comes from choosing $\kappa+\ell=4$, so that one has

$$\begin{aligned} m_2(m_2(v_1,v_2),v_3)+(-1)^{|v_1|}m_2(v_1,m_2(v_2,v_3))+&\\ +m_1(m_3(v_1,v_2,v_3))+m_3(m_1(v_1),v_2,v_3)+(-1)^{|v_1|}m_3(v_1,m_1(v_2),v_3)+&\\ +(-1)^{|v_1|+|v_2|}m_3(v_1,v_2,m_1(v_3))=0.& \end{aligned} \tag{A.8}$$

This condition means that the associativity for the product m_2 is broken by the terms containing the 3-product $m_3 : V^{\otimes 3}\to V$: one says that m_2 is associative *up to homotopy* in m_3. Keep going up, one sees that the 3-associativity for m_3 is broken by a term in m_4 and so on. In this context the Lie algebra structure on the coderivations offers a very compact and useful environment to reproduce the above relations, defining an A_∞-algebra. In general, given a coderivation as in (A.4) one has to compute

$$[\mathfrak{m},\mathfrak{m}]=\sum_{k,l}[\mathfrak{m}_k,\mathfrak{m}_l]\,. \tag{A.9}$$

The right hand side has to be considered carefully. First of all we note that we have taken $\deg(m_k)=1$ for any k, likewise we define $\deg\mathfrak{m}_k=\deg(m_k)=1$, so the commutator above is indeed an *anti*commutator for any k and l, that is

$$[\mathfrak{m}_k,\mathfrak{m}_l]=\mathfrak{m}_k\mathfrak{m}_l+\mathfrak{m}_l\mathfrak{m}_k. \tag{A.10}$$

Again, it is useful to divide the various cases by letting $k+l=n+1$ for $n\geq 1$ as above: then we have that in general $[\mathfrak{m}_k,\mathfrak{m}_l] : V^{\otimes\geq(k+l-1)}\to V^{\otimes\geq 1}$. To make contact with A_∞-relations, we restrict our attention to the case the image of the commutator is just V and we look at the first instances. For $n=1$ we have

$$[\mathfrak{m}_1,\mathfrak{m}_1](v_1)=2m_1(v_1). \tag{A.11}$$

For $n=2$ we have

$$[\mathfrak{m}_1,\mathfrak{m}_2](v_1,v_2)=m_1(m_2(v_1,v_2))+m_2(m_1(v_1),v_2)+(-1)^{|v_1|}m_2(v_1\otimes m_1(v_2)). \tag{A.12}$$

Notice that for $n = 3$ we should start considering more than one commutator, indeed we find $[\mathfrak{m}_1, \mathfrak{m}_3]$ and $[\mathfrak{m}_2, \mathfrak{m}_2]$. Clearly, as n grows there will be more and more commutators to take into account. Now, the A_∞-algebra relations can be written in a very compact way using these commutators, for example the first relations reads

$$[\mathfrak{m}_1, \mathfrak{m}_1] = 0, \qquad [\mathfrak{m}_1, \mathfrak{m}_2] = 0, \qquad [\mathfrak{m}_1, \mathfrak{m}_3] + \frac{1}{2}[\mathfrak{m}_2, \mathfrak{m}_2] = 0, \tag{A.13}$$

where the projection on V is understood.

Appendix B: How to compute with $\Theta(\iota_D)$ and $\delta(\iota_D)$

In order to clarify the action of $\Theta(\iota_D), \delta(\iota_D)$ and Z_D, we present some detailed calculations. Let us compute the action of $\Theta(\iota_D)$ on $\delta(d\theta^\alpha)$ with $\alpha = 1, 2$.

$$\begin{aligned} \Theta(\iota_D)\delta(d\theta^\alpha) &= -i \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{e^{it\iota_D}}{t + i\epsilon} \delta(d\theta^\alpha) = -i \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{\delta(d\theta^\alpha + iD^\alpha t)}{t + i\epsilon} \\ &= -\frac{1}{D^\alpha} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{\delta(t - i\frac{d\theta^\alpha}{D^\alpha})}{t + i\epsilon} = \frac{i}{d\theta^\alpha} \in \Omega^{(-1|0)}_{\mathbb{P}^{1|2}} \end{aligned} \tag{B.1}$$

where the coefficient D^α drops out from the computation (but it must be different from zero in order to have a meaningful computation). In the same way, we have

$$\delta(\iota_D)\delta(d\theta^\alpha) = \int_{-\infty}^{\infty} dt e^{it\iota_D} \delta(d\theta^\alpha) = \int_{-\infty}^{\infty} dt \delta(d\theta^\alpha + iD^\alpha t) = -\frac{i}{D^\alpha} \in \Omega^{(0|0)}_{\mathbb{P}^{1|2}}, \tag{B.2}$$

using the distributional properties. Again the requirement that D^α is different from zero is crucial.

Let us compute the action of $\Theta(\iota_D)$ on the product $d\theta^\beta \delta(d\theta^\alpha)$. We assume that $\alpha \neq \beta$, otherwise it vanishes. Applying the same rules we have

$$\begin{aligned} \Theta(\iota_D)\Big(d\theta^\beta \delta(d\theta^\alpha)\Big) &= -i \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{e^{it\iota_D}}{t + i\epsilon} \Big(d\theta^\beta \delta(d\theta^\alpha)\Big) \\ &= -i \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{(d\theta_\beta + iD_\beta t)\delta(d\theta^\alpha + iD^\alpha t)}{t + i\epsilon} \\ &= \frac{-i}{iD^\alpha} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{(d\theta^\beta + iD^\beta t)}{t + i\epsilon} \delta\Big(t - \frac{id\theta_\alpha}{D^\alpha}\Big) \\ &= -\frac{1}{D^\alpha}\Big(d\theta_\beta + iD^\beta \frac{id\theta^\alpha}{D_\alpha}\Big) \frac{D^\alpha}{id\theta^\alpha} \\ &= i\Big(\frac{d\theta^\beta}{d\theta^\alpha} - \frac{D^\beta}{D^\alpha}\Big) \in \Omega^{(0|0)}_{\mathbb{P}^{1|2}} \end{aligned} \tag{B.3}$$

from which it immediately follows that if $\alpha = \beta$, then both members vanish. Analogously, we have

$$\begin{aligned}
\delta(\iota_D)\Big(d\theta^\beta \delta(d\theta^\alpha)\Big) &= \int_{-\infty}^{\infty} dt e^{it\iota_D}\Big(d\theta^\beta \delta(d\theta^\alpha)\Big) = \int_{-\infty}^{\infty} dt(d\theta_\beta + iD_\beta t)\delta(d\theta^\alpha + iD^\alpha t) \\
&= \frac{1}{iD^\alpha}\int_{-\infty}^{\infty} dt(d\theta^\beta + iD^\beta t)\delta\Big(t - \frac{id\theta_\alpha}{D^\alpha}\Big) \\
&= \frac{1}{iD^\alpha}\Big(d\theta^\beta - \frac{D^\alpha}{D^\beta}d\theta^\alpha\Big) \in \Omega^{(1|0)}_{\mathbb{P}^{1|2}} \qquad \text{(B.4)}
\end{aligned}$$

which also vanishes if $\alpha = \beta$.

Let us also consider the following expressions

$$\begin{aligned}
\Theta(\iota_D)\Big(\frac{1}{d\theta_\beta}\delta(d\theta_\alpha)\Big) &= -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\frac{\delta(d\theta_\alpha + iD_\alpha t)}{(d\theta_\beta + iD_\beta t)(t+i\epsilon)} \\
&= \frac{-i}{iD_\alpha}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\frac{1}{(d\theta_\beta + iD_\beta t)(t+i\epsilon)}\delta\Big(t - \frac{id\theta_\alpha}{D_\alpha}\Big) \\
&= -\frac{1}{D_\alpha}\frac{1}{\Big(d\theta_\beta + iD_\beta\frac{id\theta_\alpha}{D_\alpha}\Big)}\frac{D_\alpha}{id\theta_\alpha} \\
&= i\frac{1}{\Big(\frac{d\theta_\beta}{d\theta_\alpha} - \frac{D_\beta}{D_\alpha}\Big)}\frac{1}{d\theta_\alpha^2} \in \Omega^{(-2|0)}_{\mathbb{P}^{1|2}} \qquad \text{(B.5)}
\end{aligned}$$

which is an inverse form. Notice that if $\alpha = \beta$, the product $\Big(\frac{1}{d\theta^\beta}\delta(d\theta^\alpha)\Big)$ is ill-defined, and this is consistent with the fact that also the right-hand side is divergent.

Let us now compute the action of $\Theta(\iota_D)$ on $\Omega^{(0|2)}_{\mathbb{P}^{1|2}}$. This is done as follows

$$\begin{aligned}
\Theta(\iota_D)\Big(\delta(d\theta_1)\delta(d\theta_2)\Big) &= -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\frac{e^{it\iota_D}}{t+i\epsilon}\delta(d\theta_1)\delta(d\theta_2) \\
&= -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\frac{\delta(d\theta_1 + itD_1)\delta(d\theta_2 + itD_2)}{t+i\epsilon} \\
&= \frac{i}{D_1D_2}\left(D_1\frac{\delta\Big(d\theta_2 - \frac{D_2}{D_1}d\theta_1\Big)}{d\theta_1} - D_2\frac{\delta\Big(d\theta_1 - \frac{D_1}{D_2}d\theta_2\Big)}{d\theta_2}\right) \\
&= -\frac{i}{D_1D_2}\Big(\frac{D_1}{d\theta_1} + \frac{D_2}{d\theta_2}\Big)\delta\Big(\frac{d\theta_1}{D_1} - \frac{d\theta_2}{D_2}\Big) \\
&= = -i\Big(\frac{D_1}{d\theta_1} + \frac{D_2}{d\theta_2}\Big)\delta(D\cdot d\theta) \in \Omega^{(-1|1)}_{\mathbb{P}^{1|2}}\,. \qquad \text{(B.6)}
\end{aligned}$$

where $(D\cdot d\theta) = D_\alpha\epsilon^{\alpha\beta}d\theta_\beta$.

Notice that the linear combination of $d\theta_1$ and $d\theta_2$ appearing in the first factor is linearly independent from the linear combination appearing in the Dirac delta argument. Notice also that the sign between the two Dirac delta's in the second line is due to the fermionic nature of dt and of the Dirac delta form. This sign is crucial for the left-hand side and the right-hand side of eq. (B.6) be consistent. Indeed, if we interchange $\delta(d\theta_1)$ with $\delta(d\theta_2)$ in the left-hand side we get an overall minus sign; on the other hand, on the right-hand side of the equation, by exchanging $d\theta_1$ and $d\theta_2$ in the Dirac delta argument again a sign emerges.

Finally, we can consider another independent odd vector field D' and the corresponding operator $\Theta(\iota_{D'})$. Acting on (B.6) it yields

$$\Theta(\iota_{D'})\Theta(\iota_D)\Big(\delta(d\theta_1)\delta(d\theta_2)\Big) = \frac{\det(D', D)}{(D' \cdot d\theta)(D \cdot d\theta)} \in \Omega^{(-2|0)}_{\mathbb{P}^{1|2}} \tag{B.7}$$

where $(D \cdot d\theta) = D_\alpha \epsilon^{\alpha\beta} d\theta_\beta$ and $\det(D', D) = D'_\alpha \epsilon^{\alpha\beta} D_\beta = D' \cdot D$. Notice that in this case, by interchanging $\delta(d\theta_1)$ with $\delta(d\theta_2)$, we get again an overall minus sign. This is obtained also by exchanging the coefficients of the vectors D and D', and in this way we get a minus sign from the determinant $\det(D', D)$.

Let us also consider the action of $\delta(\iota_D)$ on the product of $\delta(d\theta^1)\delta(d\theta^2)$. We have

$$\delta(\iota_D)(\delta(d\theta^1)\delta(d\theta^2)) = -i\delta(D \cdot d\theta) \in \Omega^{(0|1)}_{\mathbb{P}^{1|2}}, \tag{B.8}$$

and finally

$$\delta(\iota_{D'})\delta(\iota_D)(\delta(d\theta^1)\delta(d\theta^2)) = \det(D', D) \in \Omega^{(0|0)}_{\mathbb{P}^{1|2}}, \tag{B.9}$$

which also follows from (B.7) by the identity $d\theta^\alpha \Theta(\iota_D) = \delta(d\theta^\alpha)$.

The action of a second PCO decreases the picture number as to bring elements of $\Omega^{p|2}_{\mathbb{P}^{1|2}}$ into superforms having picture number equal to zero. Note that since the PCO's Z is formally exact as stress above, it maps cohomology classes into cohomology classes, $H^{(p|2)}_{dR} \to H^{(p|0)}_{dR}$, therefore it is natural to expect that it can only properly acts on cohomology classes, and indeed, acting on representatives of $H^{(p|2)}_{dR}$ one never gets inverse forms. Nonetheless, it can be shown that acting on generic elements of the space $\Omega^{(p|2)}_{\mathbb{P}^{1|2}}$, not necessarily closed, one never produces inverse forms. Let us show this first in a very simple example.
Consider a generic integral form in $\Omega^{1|2}_{\mathbb{P}^{1|2}} \cong \mathcal{B}er(\mathbb{P}^{1|2})$

$$\omega^{(1|2)} = A(z, \theta) dz \delta(d\theta^1)\delta(d\theta^2) \tag{B.10}$$

where $A(z,\theta^\alpha)$ is a superfield in the local coordinates of $\mathbb{P}^{1|2}$. Being (the analog of) a top-form it is naturally closed. Acting with Z_D one gets

$$
\begin{aligned}
Z_D(\omega^{(1|2)}) &= d\left[-i\Theta(\iota_D)Adz\delta(d\theta^1)\delta(d\theta^2)\right] - i\Theta(\iota_D)\left[d\left(Adz\delta(d\theta^1)\delta(d\theta^2)\right)\right] \\
&= d\Big[A\left(\frac{D^1}{d\theta^1}+\frac{D^2}{d\theta^2}\right)dz\,\delta(D\cdot d\theta)\Big] \\
&= 2\Big((D^1\partial_1 A + D^2\partial_2 A)\,dz\,\delta(D\cdot d\theta)\Big) \\
&= 2D^\alpha\partial_\alpha A\,dz\,\delta(D\cdot d\theta)\Big) \in \Omega^{(1|1)}_{\mathbb{P}^{1|2}},
\end{aligned}
\tag{B.11}
$$

where $\partial_\alpha A$ are the derivatives with respect to θ^α of the superfield A. The result is in $\Omega^{(1|1)}_{\mathbb{P}^{1|2}}$, it is closed and no inverse form is required. However, the form (B.11) is not the most general (1|1)-pseudoform.

Let us act with a second PCO :

$$
Z_{D'}\Big[2D^\alpha\partial_\alpha A\,dz\,\delta(D\cdot d\theta)\Big)\Big] \;=\; 2\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta A\,dz \in \Omega^{(1|0)}_{\mathbb{P}^{1|2}}
\tag{B.12}
$$

which is a superform in $\Omega^{(1|0)}_{\mathbb{P}^{1|2}}$, it does not contain any inverse form and it is independent of the odd vector fields D, D'. Note that this particular expression is closed, since $\partial_1^2 = \partial_2^2 = \{\partial_1,\partial_2\} = 0$. No inverse form is needed in the present case.

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