SUPERSTRING FIELD THEORY, SUPERFORMS AND SUPERGEOMETRY

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Abstract. Inspired by superstring field theory, we study differential, integral, and inverse forms and their mutual relations on a supermanifold from a sheaf-theoretical point of view. In particular, the formal distributional properties of integral forms are recovered in this scenario in a geometrical way. Further, we show how inverse forms "extend" the ordinary de Rham complex on a supermanifold, thus providing a mathematical foundation of the Large Hilbert Space used in superstrings. Last, we briefly discuss how the Hodge diamond of a supermanifold looks like, and we explicitly compute it for super Riemann surfaces.

Contents

1. Introduction

Supergeometry is a fascinating branch of mathematics that prompted from the physical motivation of describing fermionic degrees of freedom. As is well known since the first years of quantum mechanics, identical particles can appear in two types: bosons and fermions. They have different properties, but essentially their wave functions, describing the states of those particles, have to be either fully symmetrized under the exchange of two identical particles in the case of bosons, or fully anti-symmetrized in the case of fermions. Such a requirement is easily implemented by representing the fermions in terms of anticommuting variables, also said *Grassmann variables* belonging to a superalgebra. This original motivation stemming from physics has given a strong impulse to the study of supergeometry, a context in which commuting and anticommuting variables can be treated on the same footing and described in a unified fashion. Nonetheless, further important developments were motivated by string theory and string field theory.

In string theory, in order to include the fermionic physical degrees of freedom and also protecting the theory from unwanted tachyonic fields and stabilising the vacuum, one needs fermionic coordinates (either in the vector representation of Lorentz group, RNS formulation, or in the spinor representation, GS/pure spinor formulation). In this respect, the spacetime is enriched by these additional coordinates and the supergeometry starts playing a fundamental role. On one side, string theory needs the supergeometry formulation to define vertex operators, correlation functions and amplitudes, on the other side the geometry emerging from that embodies those anticommuting variables in the properties of supermanifolds.

During the last years, several research articles [1, 2] pointed out new important applications of supergeometry in the context of string theory. In particular, it has been observed that the correlation functions of vertex operators, after integrating over conformal fields, are special types of differential forms - known as *integral forms* - on the supermoduli space of super Riemann surfaces. To complete the computation, one needs an integration on that supermoduli space, which proved to be a formidable hard problem as one has to confront with some typical supergeometric subtleties, as recently shown by Donagi and Witten in [3]. By the way, this kind of issues called for the definition of an integration theory on supermanifolds. This has been developed and it revealed new interesting features of differential forms: 1) the differential forms on a supermanifold are characterized by two numbers: the *form degree* and the *picture number*, 2) the complex of superforms must be extended to integral forms. This is obtained by adding to the complex additional lines with fixed picture and variable form degree. 3) In general, picture-zero differential forms have no upper bound to their form degree, whilst integral forms, *i.e.* those forms having maximal picture number, have no lower bound to their form degree (which can also be negative). Finally, differential forms with a generic picture number are unbounded from above and from below. In addition, at a given form number the forms with a non-maximal picture number span an infinite-dimensional space. 4) New differential operators can be defined in order to remove or to add picture to the differential forms.

All these features are easily discussed in the context of conformal field theory where the calculations can be performed. Nevertheless, some of the computations have a geometrical origin and therefore these features can be translated in term of geometrical properties. For that purpose, we use a *sheaf-theoretical* approach to supermanifolds. Nonetheless, to keep our exposition as readable and concrete as possible, we will use as prototypical example for our considerations and constructions the *projective superspaces* $\mathbb{P}^{n|m}$, whose supermanifold structure is non-trivial but easy-enough to allows us for explicit computations in order to identify the sheaves involved and make clear their sheaf-theoretical local-to-global nature. Also, some of the computational properties of integral forms are to be ascribed to their *distributional* nature and therefore it is shown how analytical distributional properties and geometrical aspects fuse into a precise description. This also motivates the introduction of a new type of superforms, called negative-degree superforms or *inverse forms*, which have interesting properties. They play an essential role in the comparison between string theory and supergeometry. Indeed, in the string theory framework it is known how to enlarge the physical spectrum of states (called *Large Hilbert Space*) in order to gain a useful description of the BRST cohomology (vertex operator observables): in this paper we will show how this is achieved from a purely geometrical approach, shedding some light on the supergeometrical origin of concepts underlying string field theory. The Large Hilbert Space has new features that have never been considered in supergeometry revealing new interesting results.

The main motivations of the present work is the translation into a mathematical framework closer to the physics applications of the properties of differential superforms, integral forms and inverse forms via sheaf theory. A classical reference for the theory of forms and integration on supermanifold is Voronov's book [4].

A future goal is to understand if the A_∞-algebra appearing in super string field theory [29, 30, 31] could show up also in the supergeometric context, possibly in a natural fashion. Along this line a preliminary study has been recently published in [41]

The plan of the paper is the following: in sec. 2, we revise some ingredients from physical perspective such as the beta-gamma ghost fields, their fermionization, their vertex operators and their OPE algebras. In addition, we recall some basics facts regarding distributions and how they have to be understood in the present context; finally, picture changing operator are preliminarily discussed here. In sec. 3 and sec. 4, we recall basic facts about supermanifolds and we introduce some of the natural sheaves (namely the tangent, the cotangent and the Berezinian sheaves) that can be defined over a supermanifold and that will enter our description. In sec. 5, we introduce a global definition of the sheaves of integral forms and related complex. In sec. 6, we introduce the new concept of negative-degree superforms (*a.k.a.* inverse forms) and their complex and we discuss the cohomology of Large Hilbert Space in two interesting instances. In sec. 7, some issues in higher odd dimensions are addressed and discussed. Finally, in sec. 8, using mostly *Serre duality*, we briefly study the *Hodge diamond* of a complex supermanifold, by underlying the differences arising in comparison with the ordinary well-understood case: the relevant case of super Riemann surfaces is described in some details.

2. The Large Hilbert Space, PCO's and New Superforms

The ideas of the *Large Hilbert Space* (LHS) and of the *Picture Changing Operators* (PCO) have been introduced in string theory [5], in order to quantize the ghost fields associated to the superdiffeomorphisms on the worldsheet. Nonetheless those ideas can be imported in the geometry of supermanifolds and, as will be shown, lead to new interesting addition to the space of integral forms. In particular, it will be shown that the space of distributions such as the Dirac delta forms (local expressions for integral forms), used so far as *prototypes* is not large enough and it must be augmented to the full set of distributional forms with compact support.

In the quantization of superstring theory (see [6] for a comprehensive and complete review using the notation of the present section), one introduces two sets of conformal fields with conformal weights $(2, -1)$ and $(3/2, -1/2)$ needed to fix the local supersymmetry and worldsheet diffeomorphisms. They are named *ghost* and *superghost fields* and denoted by $(b(z), c(z))$ and $(\beta(z), \gamma(z))$, respectively. The first set is made of anticommuting fields, while the second one by commuting real fields. The quantization of the latter requires some additional care since any function of the zero mode of γ enters in the cohomology. Such a degree of freedom has the same properties of the differential $d\theta$ of the worldsheet anticommuting local coordinate θ of the super Riemann surface in the local coordinate system (z, θ).

A powerful way to deal with the quantization of these fields is by performing a *fermionization* (see [5]) by expressing the set $(\beta(z), \gamma(z))$ in terms of two anticommuting fields $(\xi(z), \eta(z))$ (with conformal weight $(0, 1)$) and one chiral boson $\phi(z)$ as follows

$$\begin{aligned} &\gamma(z) = :\eta(z)\, e^{\phi}(z):, \qquad \beta(z) = :\partial\xi(z)\, e^{-\phi(z)}:, \\ &\delta(\gamma(z)) = :e^{-\phi(z)}:, \qquad \delta(\beta(z)) = :e^{\phi(z)}:, \end{aligned} \tag{2.1}$$

The colon notation, as usual, denotes the normal ordering in the products. In the second line, we have computed the Dirac delta functions of the fields $\gamma(z)$ and $\beta(z)$ and it is not difficult to show that they indeed satisfy the correct properties $\gamma\delta(\gamma) = 0$, $\gamma\delta'(\gamma) = -\delta(\gamma)$ as for the usual Dirac distribution $\delta(x)$. The tools needed are the elementary quantization techniques of conformal field theory, reviewed in classical string theory manuals ([6] and [7]).

The "standard" Hilbert Space (or *Small Hilbert Space*, SHS henceforth) is identified with the Fock space resulting from the quantization of the (η,ξ) and ϕ conformal field theories. In that space the zero mode of the field $\xi(z)$ is absent in the expression (2.1) and any operator built in terms of positive powers of γ,β and derivatives of $\delta(\gamma),\delta(\beta)$ can be easily written without using the zero mode of ξ. For instance, we have

$$\begin{aligned}\gamma^p &= \frac{1}{(p-1)!}\,\eta\partial\eta\cdots\partial^{(p-1)}\eta\, e^{p\phi},\\ \beta^p &= \frac{1}{(p-1)!}\,\partial\xi\partial^2\xi\cdots\partial^p\xi\, e^{-p\phi},\\ \delta^{(p)}(\gamma) &= \partial\xi\cdots\partial^p\xi\, e^{-(p+1)\phi},\\ \delta^{(p)}(\beta) &= \eta\partial\eta\cdots\partial^{p-1}\eta\, e^{(p+1)\phi}.\end{aligned} \tag{2.2}$$

Switching to the usual language of supergeometry in a complex supermanifold of dimension $1|1$, identifying $\gamma\sim d\theta$ and $\delta(\gamma)\sim\delta(d\theta)$ and neglecting at the moment dz, the expression γ^p belongs to $\Omega^{p;0}_{\mathcal{M}}$ (the space of superforms of zero picture), while $\delta^{(p)}(\gamma)$ belongs to $\Omega^{-p;1}_{\mathcal{M}}$ (the space of integral forms see [8]) and sec. 5. In formulae (2.1), there are also the fields β and $\delta(\beta)$: they are translated into the geometric language as $\beta\sim\iota_D$, namely the interior derivative, where $D=\partial_\theta$ and $\delta(\beta)\sim\delta(\iota_D)$ (notice that the interior derivative w.r.t. an odd vector field is an even derivation, therefore the Dirac delta of ι_D is defined). To invert the relation between the fields, we have

$$\begin{aligned}\eta &= \partial\gamma\,\delta(\gamma) = \partial\Theta(\gamma)\\ \xi &= \Theta(\beta),\end{aligned} \tag{2.3}$$

where Θ is the Heaviside function, which can be given an integral representation as

$$\Theta(\mathcal{R}) = \lim_{\epsilon\to 0^+}\Big(-i\int_{-\infty}^{\infty}\frac{dt}{t+i\epsilon}\,\exp(+it\mathcal{R})\Big).$$

for a given operator $\mathcal{R}$. Notice that, to represent completely the field ξ in terms of the original set of fields γ,β, one needs to enlarge the space of distribution by considering also the Heaviside function. Nevertherless, that distribution involves the field β, but apparently we do not require the same enlargement also for γ. However, using conformal field theory techniques, namely by using the OPE's

$$\beta(z)\gamma(w) = \frac{1}{z-w} + (reg),$$

and, consequently, from the definitions of β and γ in eq (2.1) we have the basic OPEs

$$\eta(z)\xi(w) = \frac{1}{z-w} + (reg), \qquad \phi(z)\phi(w) = -\log(z-w).$$

Then, bringing the two quantities $\Theta(\beta(z))$ and $\delta(\gamma(w))$ close to each other on the worldsheet (represented here by the points z and w appearing in the arguments of β and of γ), one can show that,

$$\Theta(\beta(z))\,\delta(\gamma(w)) = \frac{1}{\gamma(w)} + \dots \tag{2.4}$$

where the ellipsis stands for $O(z-w)$, namely those terms which are polynomials in the difference of z and w and vanishing when $w\to z$. This implies that the presence of the zero mode of ξ allows us to consider also the negative powers of γ. This fact has deep consequences in string theory opening the possibility of constructing *open superstring field theory* [9] and it has been used for proving *Sen's conjecture* [10].

We will show that also in the context of supermanifolds, we can consider a *Large Hilbert Space* (LHS), or better an enlarged space of forms enriching the geometrical structures. For that purpose, by identifying $\beta \sim \iota_D$ and $\gamma \sim d\theta$, we can compute the action of the operator $\Theta(\iota_D)$ on $\delta(d\theta)$ as follows

$$\Theta(\iota_D)\delta(d\theta) = \Big(\lim_{\epsilon\to 0^+} -i \int_{-\infty}^{\infty} \frac{dt}{t+i\epsilon}\, e^{+it\iota_D} \Big) \delta(d\theta) = \lim_{\epsilon\to 0^+} -i \int_{-\infty}^{\infty} \frac{dt}{t+i\epsilon} \delta(d\theta - it) = \frac{i}{d\theta} \tag{2.5}$$

This new relation shows that, by allowing for the operator $\Theta(\iota_D)$, we are forced to consider also negative powers of $d\theta$ along the same ideas pursued in string theory. Using the operator $\Theta(\iota_D)$, we are able to map the integral forms complex $\Omega^{p;1}_{\mathcal{M}}$ into the new complex of superforms with negative degree.

The generalisation to derivatives of delta functions is

$$\Theta(\iota_D)\delta^{(n)}(d\theta) = \frac{-i(-1)^{n+1} n!}{(d\theta)^{n+1}}, . \tag{2.6}$$

Therefore, we have a map

$$\Theta(\iota_D) : \Omega^{p;1}_{\mathcal{M}} \longrightarrow \Omega^{p-1;0}_{\mathcal{M}} \tag{2.7}$$

for $p \in \mathbb{Z}$ and $p \leq 1$, and where $\Omega^{p-1;0}_{\mathcal{M}}$ for $p-1 \leq 0$ denotes the space of superforms with inverse powers of $d\theta$. Note that we have to take into account that the derivatives $\delta^{(p)}(d\theta)$ are required to be anticommuting quantities in order to be able to build full-fledged complex of integral forms and the corresponding top forms. In the same way, $\Theta(\iota_D)$ is an odd operator acting on the space of integral forms. Therefore, the action of Θ on $\delta(d\theta)$ yields a commuting quantity, namely $(d\theta)^{-1}$, which is consistent with the algebraic properties.

However from the analytic point of view, we have to clarify an important issue. As is well known, the distributions also emerge by introducing the famous $i\epsilon$-prescription and using the formula (Sokhotski-Plemelj theorem [11])

$$\lim_{\epsilon\to 0^+} \frac{1}{x - x_0 \pm i\epsilon} = \text{p.v.}\Big(\frac{1}{x-x_0}\Big) \mp i\pi\delta(x-x_0) \tag{2.8}$$

where x, x_0 are defined on $\mathbb{R}$, p.v. stands for *principal value* and it is defined as usual as

$$\Big\langle \text{p.v.}\Big(\frac{1}{x}\Big), f(x) \Big\rangle = \lim_{\epsilon\to 0} \Big(\int_{\epsilon}^{\infty} \frac{f(x)dx}{x} + \int_{-\infty}^{-\epsilon} \frac{f(x)dx}{x} \Big) . \tag{2.9}$$

for any test function $f(x)$ with compact support. The integral representation of the Dirac delta function $\delta(x)$ used in the literature contains a $\frac{1}{2\pi}$ factor bringing the factor π in the above expression. If we would like to use the same expression for $x \leftrightarrow d\theta$ and $x_0 \leftrightarrow 0$, taking into account that $\delta(d\theta)$ is an anticommuting operator, we have

$$\lim_{\epsilon\to 0^+} \frac{1}{d\theta \pm i\epsilon} = \text{p.v.}\Big(\frac{1}{d\theta}\Big) \mp \frac{i}{2}\Pi\delta(d\theta) \tag{2.10}$$

where Π is the parity changing functor described in the following section and the inverse power of $d\theta$ is considered as the distribution p.v.$(1/d\theta)$, which is a compact support distribution. Using the Π funtor, we correctly take into account the algebraic properties. The number π disappeared because our integral representation for Dirac delta function is

$$\delta(d\theta) = \int_{-\infty}^{\infty} \exp\Big(it d\theta\Big) dt \tag{2.11}$$

and it does not have the $1/2\pi$ factor in it. The two terms in the r.h.s. of (2.10) are two distributions with different characteristics and different degrees. In particular they belong to $\Omega^{-1;0}_{\mathcal{M}}$ and $\Omega^{0;1}_{\mathcal{M}}$. It is worthwhile noting that the transformation properties of both expressions in the r.h.s. of the equations, under change of patches are exactly the same. This point will be completely elucidated in the forthcoming sections where a coordinated-free definition of the objects considered in this section will be provided.

Still working on a local set of coordinates, we can multiply both sides of (2.10) by θ to get

$$\lim_{\epsilon\to 0^+}\frac{\theta}{d\theta\pm i\epsilon}=\text{p.v.}\Big(\frac{\theta}{d\theta}\Big)\mp\frac{i}{2}\Pi\theta\delta(d\theta)=\alpha^{(-1|0)}\mp\frac{i}{2}\Pi\mathbb{Y}^{(0|1)} \tag{2.12}$$

where we have defined the two quantities

$$\alpha^{(-1|0)}:=\text{p.v.}\Big(\frac{\theta}{d\theta}\Big)\,,\qquad\qquad \mathbb{Y}^{(0|1)}:=\theta\delta(d\theta)\,. \tag{2.13}$$

called *trivializer* and *Picture Changing Operator* (PCO), respectively. The name of the first one is due to its property

$$d\left(\lim_{\epsilon\to 0^+}\frac{\theta}{d\theta\pm i\epsilon}\right)=d\left[\text{p.v.}\Big(\frac{\theta}{d\theta}\Big)\right]\pm i\Pi d\Big(\theta\delta(d\theta)\Big)=1 \tag{2.14}$$

which implies that $d\alpha^{(-1|0)}=1$ and $d\mathbb{Y}^{(0|1)}=0$. Indeed, $\alpha^{(-1|0)}$ is the trivializer of the usual odd differential operator d. The second operator is imported from string theory where it plays a fundamental role for constructing the amplitudes. It is d-closed but it is not exact and, as such, it belongs to the de Rham cohomology group $H^{0;1}_{dR}(\mathcal{M})$ of the supermanifold.

The generalisation to higher powers is:

$$\lim_{\epsilon\to 0^+}\frac{\theta}{d\theta^2\pm i\epsilon}=\text{f.p.}\Big(\frac{\theta}{(d\theta)^2}\Big)\mp\frac{i}{2}\Pi\theta\delta'(d\theta) \tag{2.15}$$

where f.p. is the Hadamard finite part of the integral defined as

$$\Big\langle\text{f.p.}\left(\frac{1}{x^2}\right),f(x)\Big\rangle=\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}\frac{f(x)dx}{x^2}+\int_{\epsilon}^{\infty}\frac{f(x)dx}{x^2}-\frac{2f(0)}{\epsilon}\right] \tag{2.16}$$

for any test function $f(x)$ with compact support.

Let us consider the two elements of $\Omega^{0;1}_{\mathcal{M}}$ given by

$$\mathbb{Y}^{(0|1)}=\theta\delta(d\theta)\,,\qquad \widetilde{\mathbb{Y}}^{(0|1)}=(dz-\theta d\theta)\delta'(d\theta). \tag{2.17}$$

They differ by an exact term, as can be readily checked. We can in the same way define the operators

$$\alpha^{(-1|0)}_{\epsilon}=\frac{\theta}{d\theta\pm i\epsilon}\,,\qquad \widetilde{\alpha}^{(-1|0)}_{\epsilon}=\frac{(dz-\theta d\theta)}{d\theta^2\pm i\epsilon}\,, \tag{2.18}$$

satisfying the equations

$$d\alpha^{(-1|0)}_{\epsilon}=1\,,\quad d\widetilde{\alpha}^{(-1|0)}_{\epsilon}=1\,,\quad \widetilde{\alpha}^{(-1|0)}_{\epsilon}-\alpha^{(-1|0)}_{\epsilon}=d\Omega^{(-2|0)}. \tag{2.19}$$

where $\Omega^{(-2|0)}$ is a negative-degree superform with zero picture. The Large Hibert Space is spanned by the superforms

$$(d\theta)^p,(d\theta\pm i\epsilon)^{-p-1},\qquad p\geq 0\,. \tag{2.20}$$

Since there are two regularizations for the inverses of $d\theta$ associated to the two signs $\pm i\epsilon$, the Hilbert Space is not isomorphic to the original one.

Equivalently, the Large Hilbert Space is spanned by the superforms

$$(d\theta)^p,\ \delta^{(p)}(d\theta),\ \text{f.p.}\left(\frac{1}{d\theta^p}\right),\qquad p\geq 0. \tag{2.21}$$

The Heaviside step operator $\Theta(\iota_D)$ enters the definition of another type of PCO that is given by (see [8]):

$$\mathbb{Z}_D = [d, -i\Theta(\iota_D)] \,. \tag{2.22}$$

which depends on the choice of the vector field D. Note that, being d an odd differential and $\Theta(\iota_D)$ an odd operator, the PCO $\mathbb{Z}_D$ is an even operator.

Acting on $\mathbb{Y}^{(0|1)}$ we get:

$$\Theta(\iota_D)\mathbb{Y}^{(0|1)} = i\frac{\theta}{d\theta}\,, \qquad \mathbb{Z}^{(0|-1)}\mathbb{Y}^{(0|1)} = d\left[-i\Theta(\iota_D)\mathbb{Y}^{(0|1)}\right] = d\left(\frac{\theta}{d\theta}\right) = 1\,. \tag{2.23}$$

The $\mathbb{Z}_D$ operator is in general not invertible but it is possible to find a *non unique* operator $\mathbb{Y}$ such that $\mathbb{Z}\circ\mathbb{Y}$ is an isomorphism in de Rham cohomology *i.e.* the cohomology of the d operator described above. These operators are the called *Picture Raising Operators.* The operators of type $\mathbb{Y}$ are non trivial elements of the de Rham cohomology.

We apply a PCO of type $\mathbb{Y}$ on a given form by taking the graded wedge product: given ω in $\Omega^{p;q}_{\mathcal{M}}$, we have:

$$\omega \xrightarrow{\mathbb{Y}} \omega \wedge \mathbb{Y} \in \Omega^{p;q+1}_{\mathcal{M}} \tag{2.24}$$

If $q = m$, where m is the fermionic dimension of a generic supermanifold of dimension $n|m$, then $\omega \wedge \mathbb{Y} = 0$. In addition, if $d\omega = 0$ then $d(\omega \wedge \mathbb{Y}) = 0$, and if $\omega \neq dK$ then it follows that also $\omega\wedge\mathbb{Y} \neq dU$ where U is a form in $\Omega^{p-1;q+1}_{\mathcal{M}}$. So, given an element of the cohomogy $H^{p;q}_{DR}(\mathcal{M})$, the new form $\omega \wedge \mathbb{Y}$ is an element of $H^{p;q+1}_{dR}(\mathcal{M})$. The $\mathbb{Y}$ and $\mathbb{Z}$ operators give an isomorphism in de Rham cohomology:

$$H^{p;0}_{dR}(\mathcal{M}) \cong H^{p;m}_{dR}(\mathcal{M}) \tag{2.25}$$

Incidentally, this imply that $H^{p;0}_{dR}(\mathcal{M}) = \{0\}$ for $p > n$, where n is the bosonic dimension of the supermanifold, and this means that the de Rham cohomology of superforms cannot capture informations on the odd dimensions [12], [4].

We can build explicitely a *left* inverse for the Θ operator that it is called η_0. From (2.8)

$$\lim_{\epsilon\to 0}\frac{1}{2i}\Big(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\Big) = \lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2} = \delta(x). \tag{2.26}$$

The expression on the left hand side can be rewritten as follows (using the translation operator $e^{i\epsilon\partial_x}$)

$$\lim_{\epsilon\to 0}\frac{1}{2i}\Big(e^{i\epsilon\partial_x} - e^{-i\epsilon\partial_x}\Big)\frac{1}{x} = \lim_{\epsilon\to 0}\sin(\epsilon\partial_x)\frac{1}{x} = \delta(x) \tag{2.27}$$

Let us consider now the formal series $f(x) = \sum_{n=-\infty}^{\infty} c_n x^n$; for each single term with $n > 0$ we have

$$\lim_{\epsilon\to 0}\sin(\epsilon\partial_x)x^n = \lim_{\epsilon\to 0}\frac{1}{2i}\Big((x+i\epsilon)^n - (x-i\epsilon)^n\Big) = 0\,, \tag{2.28}$$

As for the negative powers $n < 0$, we can set $x^{-n} = (-)^n/(n-1)!\,\partial^{(n)}_\alpha (x+\alpha)^{-1}\Big\lfloor_{\alpha=0}$, then we have

$$\lim_{\epsilon\to 0}\sin(\epsilon\partial_x)x^{-n} = \frac{(-)^n}{(n-1)!}\,\partial^{(n)}_\alpha\left(\lim_{\epsilon\to 0}\sin(\epsilon\partial_x)\frac{1}{(x+\alpha)}\right)\Big\lfloor_{\alpha=0} = \frac{(-)^n}{(n-1)!}\delta^{(n)}(x) \tag{2.29}$$

Translating for the even superform $d\theta$ we define,

$$\eta_0 := \frac{i}{2\pi}\Pi\lim_{\epsilon\to 0}\sin(\epsilon\iota_D) \tag{2.30}$$

where again we have used the parity-changing functor Π in order to assign the correct parity to the operator, consistently with the properties of $d\theta$ and of $\delta(d\theta)$. We can check that:

$$\eta_0 \delta(d\theta) = 0\,, \tag{2.31}$$

and

$$\eta_0 \Theta(\iota_D)\delta(d\theta) = \eta_0 \left(\frac{i}{d\theta}\right) = i\delta(\,d\theta)\,,\ \Theta(\iota_D)\eta_0\delta\ (d\theta) = 0\,. \tag{2.32}$$

The operator η_0 is, modulo the multiplicative constant i, the left inverse of the operator Θ acting on pseudoforms, i.e.

$$\eta_0 \Theta\,(i_D) = i\ . \tag{2.33}$$

Also, the operator Θ is, modulo the multiplicative constant i, the left inverse of the operator η_0 on inverse forms of picture degree 0, i.e.

$$\Theta\,(i_D)\,\eta_0 = i\ . \tag{2.34}$$

3. Elements of Supermanifolds

In this section we shall give the most important definitions in the theory of supermanifolds, in order to set some notation and terminology. For a more complete introduction to the theory of supermanifolds via algebraic geometry we suggest the reader to refer to the deep treatment given by Manin in [13], some details of which have been recently spelled out in [14].

As a general setting, we work in the (super) analytic category and we take our ground field to be the complex numbers $\mathbb{C}$.

Our main characters will be *complex supermanifolds.* In general, a complex supermanifold of dimension $n|m$ is a locally ringed space $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$, where $\mathcal{M}$ is a topological space and $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$ is a sheaf of supercommutative algebras over $\mathbb{C}$ on $\mathcal{M}$, called the *structure sheaf* of the supermanifold, such that the following conditions are satisfied:

(1) the pair $(\mathcal{M}, \mathcal{O}_{\mathcal{M}_{red}})$, where $\mathcal{O}_{\mathcal{M}_{red}} := \mathcal{O}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}$, for $\mathcal{J}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M},1} \oplus \mathcal{O}^{\otimes 2}_{\mathcal{M},1}$, is a *complex manifold* of dimension n. The pair $(\mathcal{M}, \mathcal{O}_{\mathcal{M}_{red}})$ is called the *reduced space* of the supermanifold $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$;
(2) the quotient $\mathcal{J}_{\mathcal{M}}/\mathcal{J}^2_{\mathcal{M}}$ is a locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules of rank $0|m$, and it is called the *fermionic sheaf* and denoted by $\mathcal{F}_{\mathcal{M}}$;
(3) the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is *locally* isomorphic to the *exterior algebra* $\bigwedge^{\bullet} \mathcal{F}_{\mathcal{M}}$ over $\mathcal{O}_{\mathcal{M}_{red}}$, seen as a superalgebra.

For the sake of brevity, we will denote a supermanifold by $\mathcal{M}$ and its reduced space by $\mathcal{M}_{red}$. Also, following [13], it is worth noticing that since $\mathcal{F}_{\mathcal{M}}$ is a purely odd sheaf, it would be more appropriate to write $Sym^{\bullet}\mathcal{F}_{\mathcal{M}}$, instead of $\bigwedge^{\bullet} \mathcal{F}_{\mathcal{M}}$, as we will do later on in this paper.

Before we go on we make some comments on this definition. First of all we remark that the first condition mentioned above corresponds, for any supermanifold $\mathcal{M}$, to the existence of a morphism of supermanifolds $\iota : \mathcal{M}_{red} \to \mathcal{M}$ such that ι is a pair $\iota := (\iota, \iota^{\sharp})$, with $\iota : \mathcal{M} \to \mathcal{M}$ the identity on the underlying topological space and $\iota^{\sharp} : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}_{red}}$ is the quotient map by $\mathcal{J}_{\mathcal{M}}$, the *sheaf of ideals* formed by all of the nilpotents. Loosely speaking, this tells that the reduced manifold arises by setting all of the nilpotents in $\mathcal{O}_{\mathcal{M}}$ to zero. More precisely, a more invariant formulation is that to any supermanifold is attached a short exact sequence as follows

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}} \xrightarrow{\iota^{\sharp}} \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0, \tag{3.1}$$

that tells that the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a supermanifold is an *extension* of $\mathcal{O}_{\mathcal{M}_{red}}$ by $\mathcal{J}_{\mathcal{M}}$. In view of this, a very important question in the theory of supermanifolds is whether the defining short exact sequence (3.1) is *split* or not, *i.e.* whether there exists a morphism of supermanifolds

$\pi : \mathcal{M} \to \mathcal{M}_{red}$ given by a pair $(\pi, \pi^\sharp)$ with $\pi : \mathcal{M} \to \mathcal{M}$ being again the identity on the underlying topological space and $\pi^\sharp : \mathcal{O}_{\mathcal{M}_{red}} \to \mathcal{O}_{\mathcal{M}}$ a splitting morphism - called a *projection* - such that $\pi^\sharp \circ \iota^\sharp = id_{\mathcal{O}_{\mathcal{M}}}$. In case the splitting morphism $\pi : \mathcal{M} \to \mathcal{M}_{red}$ exists, then $\mathcal{M}$ is said a *projected* supermanifold, otherwise is said a *non-projected*. In this paper we will not be concerned with the subtleties related to non-projected supermanifolds - which yield complicated problems in the theory of complex supermanifolds that deeply affects the computation of amplitudes in superstring theory [3] -, by the way the interested reader is advised to refer to the recent [15], [16] and [17] to get an idea about the phenomenology related to these kind of supermanifolds. The third condition in the definition of a supermanifold is often briefly referred in short by saying that a complex supermanifold of dimension $n|m$ is locally isomorphic to the superspace $\mathbb{C}^{n|m}$. Actually, more precisely, this third condition is the fundamental request that the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is *locally freely-generated* by linear independent sections, we will denote them by $(z_1, \ldots, z_n, \theta_1, \ldots, \theta_m)$. These are subjected to supercommutativity only: this implies that, locally, every section s in $\mathcal{O}_{\mathcal{M}}$ can be uniquely represented by a power expansion in the theta's, that is

$$s(z, \theta) = s_0(z) + s_i(z)\theta^i + s_{ij}(z)\theta^i\theta^j \ldots, \tag{3.2}$$

in an open set $U \subseteq \mathcal{M}$ and where $s_0, s_i, s_{ij}, \ldots$ are sections in $\mathcal{O}_{\mathcal{M}_{red}}$ over U, *i.e.* holomorphic functions over $\mathcal{U}$. It is crucial to note that since the theta's are nilpotent, this power expansion has a finite number of terms, actually 2^m at most.
A projected supermanifold whose structure sheaf is given itself by an exterior algebra is said to be *split*.

The most important class of split complex supermanifolds is given by (complex) *projective superspaces* $\mathbb{P}^{n|m} := (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n|m}})$, where $\mathcal{O}_{\mathbb{P}^{n|m}} := Sym^\bullet\left(\mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}(-1)\right))$, that is, more extensively,

$$\mathcal{O}_{\mathbb{P}^{n|m}} := \bigoplus_{k\, even} \bigwedge^k \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m} \oplus \bigoplus_{k\, odd} \bigwedge^k \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}. \tag{3.3}$$

When it comes to sheaf-theoretic constructions, projective superspaces $\mathbb{P}^{n|m}$ represent a particularly suitable class of examples as they allow for an immediate and easy *local-to-global* and *global-to-local* correspondence, and as such they will be extensively used throughout the paper. A projective superspace $\mathbb{P}^{n|m}$ has a straightforward local description by patching affine charts. Since the underlying reduced manifold of $\mathbb{P}^{n|m}$ is just $\mathbb{P}^n$, it has a covering $\{U_i\}_{i=0,\ldots,n}$ made by $n+1$ open sets U_i, each characterized by the usual condition on the homogeneous coordinates. These open sets, in turn, make up $n+1$ affine supermanifolds $\tilde{U}_i \cong \mathbb{C}^{n|m}$ with $\tilde{U}_i := (U_i, \mathbb{C}[z_{\ell i}, \theta_{\kappa i}])$, for $i = 0, \ldots, n$ and $\ell \neq i$, $\kappa = 1, \ldots, m$, which cover $\mathbb{P}^{n|m}$.
In the intersections $U_i \cap U_j$ for $0 \leq i < j \leq n+1$ transition functions reads

$$\begin{aligned} \ell \neq i : & \qquad z_{\ell j} = \frac{z_{\ell i}}{z_{ji}}, \\ \ell = i : & \qquad z_{ij} = \frac{1}{z_{ji}}, \\ \kappa = 1, \ldots, m : & \qquad \theta_{\kappa j} = \frac{\theta_{\kappa i}}{z_{ji}}, \end{aligned} \tag{3.4}$$

and this gives an atlas for $\mathbb{P}^{n|m}$.
The interested reader can find a detailed treatment of the supergeometry of projective superspaces in the recent [18].

4. Locally-Free Sheaves on Supermanifolds: Tangent, Cotangent and Berezinian Sheaves

Now that we have introduced what a supermanifold is, let us see what can be defined on it. For our purposes, one of the most important and useful concept is the one of locally-free sheaf, that will completely replace the cumbersome notion of super vector bundle [13], [14], [19].
Given a supermanifold $\mathcal{M}$, a *locally-free sheaf* $\mathcal{G}$ of rank $p|q$ on $\mathcal{M}$ is simply a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules which is locally-isomorphic to $\mathcal{O}_{\mathcal{M}}^{\oplus p} \oplus (\Pi\mathcal{O}_{\mathcal{M}})^{\oplus q}$, where $\Pi\mathcal{O}_{\mathcal{M}}$ is structure sheaf of the supermanifold having reversed parity.
In particular, an *even invertible sheaf* $\mathcal{L}_{ev}$ on $\mathcal{M}$ is a rank $1|0$ locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules, and, likewise, an *odd invertible sheaf* $\mathcal{L}_{odd}$ on $\mathcal{M}$ is a rank $0|1$ locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules. This means that, locally, one has $\mathcal{L}_{ev}\lfloor_U \cong \mathcal{O}_{\mathcal{M}}\lfloor_U$ and $\mathcal{L}_{odd}\lfloor_U \cong \Pi\mathcal{O}_{\mathcal{M}}\lfloor_U$ for U an open set of $\mathcal{M}$.
Note that in general, exactly as in the ordinary context, defining a locally-free sheaf $\mathcal{G}$ of a certain rank on a supermanifold $\mathcal{M}$, amounts to give an open covering of $\mathcal{M}$, call it $\{U_i\}_{i\in I}$, and the transition functions $\{g_{ij}\}_{i,j\in I}$ between two local frames - call them e_{U_i} and e_{U_j} - in the intersections $U_i \cap U_j$ for $i,j \in I$, so that $e_{U_i} = g_{ij} e_{U_j}$. In this fashion, one finds the usual correspondence $\mathcal{G} \leftrightarrow (\{U_i\}_{i\in I}, \{g_{ij}\}_{i,j\in I})$, where if $\mathcal{G}$ has rank $p|q$ then g_{ij} is a $GL_{p|q}$ transformation taking values in $\mathcal{O}_{\mathcal{M}}(U_i \cap U_j))$.
In the case we are considering, say, an even invertible sheaf, this corresponds to transition functions g_{ij} taking values into $(\mathcal{O}^*_{\mathcal{M}})_0 \cong \mathcal{O}^*_{\mathcal{M},0}$, the sheaf of *non-vanishing* sections of the structure sheaf. This is so as the transformation g_{ij} needs to be *invertible* and a *parity-preserving one*: indeed the frames have well-defined parity that get preserved under a change of coordinates. This bears an important consequence: $\mathcal{O}^*_{\mathcal{M},0}$ is a sheaf of *abelian groups*, so that we are allowed to consider its cohomology groups, without confronting the issues related to the definition of non-abelian cohomology (the full sheaf $\mathcal{O}^*_{\mathcal{M}}$ is indeed *not* a sheaf of abelian groups). Notice that in order to define an even invertible sheaf, the transition functions have to be 1-cocycles valued in the sheaf $\mathcal{O}^*_{\mathcal{M},0}$, so that one has the super-analog of the usual correspondence between the *even Picard group* $\mathrm{Pic}_0(\mathcal{M})$ - which is the group of the isomorphism classes of even invertible sheaves on $\mathcal{M}$ - and the cohomology group $H^1(\mathcal{M}, \mathcal{O}^*_{\mathcal{M},0})$, see [18] or [20] for details.
Clearly, the classification and the related moduli problem for higher rank sheaves is much more difficult, and, just like in the ordinary theory, being $GL_{p|q}(\mathcal{O}_{\mathcal{M}})$ non-commutative, the set $H^1(\mathcal{M}, GL_{p|q}(\mathcal{O}_{\mathcal{M}}))$ is not endowed with a group structure, but it is just a pointed-set instead, whose identity is usually taken to be the trivial rank $p|q$ sheaf. By the way we shall not worry about these subtleties in what follows, as we will not be interested into a classification but just into identifying certain sheaves instead, so it will be enough to look at the specific form of the transition functions.

In comparison with the usual commutative geometric context, there is at least one more important operation one can do on a locally-free sheaf $\mathcal{E}$ on a supermanifold $\mathcal{M}$, that is to *reverse its parity*. Indeed, let $\mathcal{E}$ be a rank $p|q$ sheaf, which is freely-generated in an open set U as follows

$$\mathcal{E}\lfloor_U \cong \mathcal{O}_{\mathcal{M}}\lfloor_U \cdot \{e_1^{(0)}, \ldots, e_p^{(0)} | e_1^{(1)}, \ldots, e_q^{(1)}\}, \tag{4.1}$$

where $\{e_1^{(0)}, \ldots, e_p^{(0)} | e_1^{(1)}, \ldots, e_q^{(1)}\}$ is a local frame of generators over U, the upper indices refer to the parity of the generators and, as a general convention in this paper, the even generator will be written in the first place. Then, acting with the parity-changing functor yields a rank $q|p$ locally-free sheaf, we call it $\Pi\mathcal{E}$. This is freely-generated over U by

$$\Pi\mathcal{E}\lfloor_U \cong \mathcal{O}_{\mathcal{M}}\lfloor_U \cdot \{\pi e_1^{(1)}, \ldots, \pi e_q^{(1)} | \pi e_1^{(0)}, \ldots, \pi e_p^{(0)}\}, \tag{4.2}$$

where we have denoted with $\pi e_i^{(0)}$ and $\pi e_j^{(1)}$ the images of the generators $e_i^{(0)}$ and $e_j^{(1)}$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$ under action of the parity-changing functor Π. That is, in other words,

$$\left(\begin{array}{c} e_1^{(0)} \\ \vdots \\ e_p^{(0)} \\ \hline e_1^{(1)} \\ \vdots \\ e_q^{(1)} \end{array} \right) \xmapsto{\quad \Pi \quad} \left(\begin{array}{c} \pi e_1^{(1)} \\ \vdots \\ \pi e_q^{(1)} \\ \hline \pi e_1^{(0)} \\ \vdots \\ \pi e_p^{(0)} \end{array} \right) \tag{4.3}$$

where as for the parity one has $|e_\ell^{(p)}| = p$ and $|\pi e_\ell^{(p)}| = (p+1) \mathrm{mod} 2$, for $p \in \mathbb{Z}_2$ and any ℓ.
Notice that, as observed above, given a covering $\{U_i\}_{i \in I}$ of a complex manifold $\mathcal{M}$, one can present a locally-free sheaf $\mathcal{E}$ by giving its transition functions $g_{ij}(\mathcal{E}) : \mathcal{E}(U_i)\lfloor_{U_i \cap U_j} \to \mathcal{E}(U_j)\lfloor_{U_i \cap U_j}$ in the intersections $U_i \cap U_j$, so that the sheaf $\mathcal{E}$ is identified by the pair $(\{U_i\}_{i\in I}, \{g_{ij}(\mathcal{E})\}_{i,j \in I})$. From this point of view it is easy to identify the sheaf $\Pi\mathcal{E}$. We denote by $M(g_{ij}(\mathcal{E}))$ the transition matrix related to $g_{ij}(\mathcal{E})$, so that one can write in general

$$M(g_{ij}(\mathcal{E})) = \left(\begin{array}{c|c} A_{p\times p} & B_{p \times q} \\ \hline C_{q \times p} & D_{q\times q} \end{array} \right) \in GL_{n|q}(\mathcal{O}_{\mathcal{M}}(U_i \cap U_j)) \tag{4.4}$$

for some invertible matrices $A_{p\times p} \in GL_p(\mathcal{O}_{\mathcal{M},0}(U_i \cap U_j)), D_{q\times q} \in GL_q(\mathcal{O}_{\mathcal{M},0}(U_i \cap U_j))$ and some matrices $B_{p\times q} \in Mat_{p\times q}(\mathcal{O}_{\mathcal{M},1}(U_i \cap U_j))$, $C_{q\times p} \in Mat_{q \times p}(\mathcal{O}_{\mathcal{M},1}(U_i \cap U_j))$. The transition functions $\{g_{ij}(\Pi \mathcal{E})\}_{i,j\in I}$ of the sheaf $\Pi\mathcal{E}$ are then immediately recovered from the $M(g_{ij}(\mathcal{E}))$'s via the *parity-transpose* operation:

$$M(g_{ij}(\Pi\mathcal{E})) = M(g_{ij}(\mathcal{E}))^{\Pi} := \left(\begin{array}{c|c} D_{q\times q} & C_{q \times p} \\ \hline B_{p \times q} & A_{p\times p} \end{array} \right) \in GL_{q|p}(\mathcal{O}_{\mathcal{M}}(U_i \cap U_j)), \tag{4.5}$$

in other words one finds that given $\mathcal{E} \leftrightarrow (\{U_i\}_{i\in I}, \{g_{ij}(\mathcal{E})\}_{i,j\in I})$, then the sheaf $\Pi\mathcal{E}$ is simply given by $\Pi\mathcal{E} \leftrightarrow (\{U_i\}_{i\in I}, \{g^{\Pi}_{ij}(\mathcal{E})\}_{i,j\in I})$, where we have indicated with $g^{\Pi}_{ij}(\mathcal{E})$ the parity-transpose operation on the transition functions, as explained above.

Let us see some instances of what explained above via some concrete examples. In particular, let us consider two locally-free sheaves that can be naturally defined on any supermanifold, the *tangent sheaf* $\mathcal{T}_{\mathcal{M}}$ and the *cotangent sheaf* $\mathcal{T}^*_{\mathcal{M}}$.
The tangent sheaf of $\mathcal{M}$ is defined, as usual, as the sheaf of superderivation on $\mathcal{M}$, where for a superalgebra A a *superderivation* is a homogeneous k-linear maps $D : A \to A$ of parity $|D| \in \mathbb{Z}_2$ that satisfies the $\mathbb{Z}_2$-graded Leibniz rule:

$$D(a \cdot b) = D(a) \cdot b + (-1)^{|D||a|} a \cdot D(b), \tag{4.6}$$

for any $a \in A$ homogeneous of parity $|a|$ and any $b \in A$. In particular, on the complex superspace $\mathbb{C}^{n|m}$ having coordinates $(z_1, \ldots, z_n | \theta_1, \ldots, \theta_m)$, the superderivations of the structure sheaf $\mathcal{O}_{\mathbb{C}^{n|m}}$ are written as $(\partial_{z_1}, \ldots, \partial_{z_n} | \partial_{\theta_1}, \ldots, \partial_{\theta_n})$, where the $\{\partial_{z_i}\}_{i=1,\ldots,n}$ are the *even* superderivations and the $\{\partial_{\theta_j}\}_{j=1,\ldots m}$ are the *odd* superderivations and it is an early result in the theory of supermanifolds due to Leites (see [12]) that the $\mathcal{O}_{\mathbb{C}^{p|q}}$-module of the $\mathbb{C}$-linear superderivations is *free* and has dimension $n|m$ with basis given indeed by $\{\partial_{z_1}, \ldots, \partial_{z_n} | \partial_{\theta_1}, \ldots, \partial_{\theta_m}\}$. It follows that, since a complex supermanifold $\mathcal{M}$ of dimension $n|m$ is by definition locally isomorphic to $\mathbb{C}^{n|m}$, the $\mathcal{O}_{\mathbb{C}^{n|m}}$-module of superderivations of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is actually a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $n|m$ and we denote if by $\mathcal{T}_{\mathcal{M}}$ and refer to as the tangent sheaf.

Once that the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is defined, one can use the construction above to introduce its parity-reversed version $\Pi\mathcal{T}_{\mathcal{M}}$, which is then a rank $m|n$ sheaf, locally-freely generated by $\{\pi\partial_{\theta_1}, \ldots, \pi\partial_{\theta_m}|\pi\partial_{z_1}, \ldots, \pi\partial_{z_n}\}$. This sheaf $\Pi\mathcal{T}_{\mathcal{M}}$ will play a fundamental role throughout this paper, as we shall see shortly.
As for the transition functions, the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ of a supermanifold $\mathcal{M}$ transforms with the supertranspose of the inverse of the Jacobian of the change of coordinate $\Phi_{ij} : \mathcal{O}_{\mathcal{M}}(U_i)\lfloor_{U_i\cap U_j} \rightarrow \mathcal{O}_{\mathcal{M}}(U_j)\lfloor_{U_i\cap U_j}$. Taking the parity-transpose yields the change of coordinates of $\Pi\mathcal{T}_{\mathcal{M}}$, that so that one has for short

$$M(\mathcal{T}_{\mathcal{M}}) = (\mathcal{J}ac(\Phi)^{-1})^{st} \qquad\qquad M(\Pi\mathcal{T}_{\mathcal{M}}) = ((\mathcal{J}ac(\Phi)^{-1})^{st})^{\Pi}. \tag{4.7}$$

Notice that this is exactly what one would find applying the chain-rule in the following form

$$\partial_{z_{ki}} = \sum_h \left(\frac{\partial z_{hj}}{\partial z_{ki}}\right)\partial_{z_{hj}} + \sum_\ell \left(\frac{\partial \theta_{\ell j}}{\partial z_{ki}}\right)\partial_{\theta_{\ell j}}, \tag{4.8}$$

that is moving the basis on the right.

Let us consider the example of projective superspaces $\mathbb{P}^{n|m}$ introduced above. In the conventions established in the previous sections, one has that the change of coordinates of $\mathcal{T}_{\mathbb{P}^{n|m}}$ in an intersection $U_i \cap U_j$ reads

$$\begin{aligned}
\partial_{z_{ji}} &= z_{ij}\left(-z_{ij}\partial_{z_{ij}} - \sum_{k\neq j,i} z_{kj}\partial_{z_{kj}} - \sum_\kappa \theta_{\kappa j}\partial_{\theta_{\kappa j}}\right)\\
\partial_{z_{\ell i}} &= z_{ij}\partial_{z_{\ell j}}\\
\partial_{\theta_{\kappa i}} &= z_{ij}\partial_{\theta_{\kappa j}}.
\end{aligned} \tag{4.9}$$

where here $|\partial_{z_{\ell i}}| = 0$ and $|\partial_{\theta_{\kappa i}}| = 1$.
In the same intersection, the transformations for the sheaf $\Pi\mathcal{T}_{\mathbb{P}^{n|m}}$ instead reads:

$$\begin{aligned}
\pi\partial_{\theta_{\kappa i}} &= z_{ij}\pi\partial_{\theta_{\kappa j}}\\
\pi\partial_{z_{ji}} &= z_{ij}\left(-z_{ij}\pi\partial_{z_{ij}} - \sum_{k\neq j,i} z_{kj}\pi\partial_{z_{kj}} - \sum_\kappa \theta_{\kappa j}\pi\partial_{\theta_{\kappa j}}\right)\\
\pi\partial_{z_{\ell i}} &= z_{ij}\pi\partial_{z_{\ell j}}.
\end{aligned} \tag{4.10}$$

where now $|\pi\partial_{\theta_{\kappa i}}| = 0$ and $|\pi\partial_{z_{\ell i}}| = 1$.

Let us now move to the cotangent sheaf of a supermanifold $\mathcal{M}$. This is defined starting from the tangent sheaf: one puts $\mathcal{T}^*_{\mathcal{M}} \coloneqq \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$. A local basis of the cotangent sheaf, dual to $\{\partial_{z_1}, \ldots, \partial_{z_n}|\partial_{\theta_1}, \ldots, \partial_{\theta_m}\}$ is written as usual as $\{dz_1, \ldots, dz_n|d\theta_1, \ldots, d\theta_m\}$, where the dz's are even and the $d\theta$'s are odd. The parity-reversed cotangent sheaf $\Pi\mathcal{T}^*_{\mathcal{M}}$ is then a rank $m|n$ sheaf, which is locally freely-generated by $\{\pi d\theta_1, \ldots, \pi d\theta_m|\pi dz_1, \ldots, \pi dz_n\}$, where now $\pi d\theta$'s are even and the πdz's are odd. We stress that this sheaf is usually called the *sheaf of* 1*-forms* and denoted with $\Omega^1_{\mathcal{M}}$. Actually, in a completely equivalent way, one can introduce $\Pi\mathcal{T}^*_{\mathcal{M}}$ as the sheaf defined by $\Pi\mathcal{T}_{\mathcal{M}} \coloneqq \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \Pi\mathcal{O}_{\mathcal{M}})$. Indeed

$$\mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \Pi\mathcal{O}_{\mathcal{M}}) = \mathcal{T}^*_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \Pi\mathcal{O}_{\mathcal{M}} = \Pi\mathcal{T}^*_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}} = \Pi\mathcal{T}^*_{\mathcal{M}}. \tag{4.11}$$

The sheaf $\Pi\mathcal{O}_{\mathcal{M}}$ is obviously locally-free of rank $0|1$ and the functor $- \otimes_{\mathcal{O}_{\mathcal{M}}} \Pi\mathcal{O}_{\mathcal{M}}$ acting on a generic sheaf of $\mathcal{O}_{\mathcal{M}}$-modules amount exactly to the parity-change of the sheaf itself. In other words, in general, one has that if $\mathcal{E}$ is a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $p|q$, then one finds that $\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{M}}} \Pi\mathcal{O}_{\mathcal{M}} = \Pi\mathcal{E}$ is of rank $q|p$ and the transition matrices of $\mathcal{E}$ and $\Pi\mathcal{E}$ are related

by a parity transposition.
As for the transition functions, the cotangent sheaf transforms with the Jacobian of the change of coordinates $\Phi_{ij} : \mathcal{O}_{\mathcal{M}}(U_i)\lfloor_{U_i\cap U_j} \to \mathcal{O}_{\mathcal{M}}(U_j)\lfloor_{U_i\cap U_j}$, so that one finds

$$M(\mathcal{T}^*_{\mathcal{M}}) = \mathcal{J}ac(\Phi) \qquad\qquad M(\Pi\mathcal{T}^*_{\mathcal{M}}) = \mathcal{J}ac(\Phi)^{\Pi}. \tag{4.12}$$

Again, this is what one would obtain by using

$$dz_{ki} = \sum_h \left(\frac{\partial z_{ki}}{\partial z_{hj}}\right) dz_{hj} + \sum_\ell \left(\frac{\partial z_{ki}}{\partial \theta_{\ell j}}\right) d\theta_{\ell j}. \tag{4.13}$$

Let us get back to the concrete example of the projective superspaces $\mathbb{P}^{n|m}$. The change of coordinates of $\mathcal{T}^*_{\mathbb{P}^{n|m}}$ reads

$$\begin{aligned} dz_{ji} &= -\frac{dz_{ij}}{z_{ij}^2} \\ dz_{\ell i} &= \frac{dz_{\ell j}}{z_{ij}} - \frac{z_{\ell j}}{z_{ij}^2} dz_{ji} \\ d\theta_{\kappa i} &= \frac{d\theta_{\kappa i}}{z_{ij}} - \frac{\theta_{\kappa i}}{z_{ij}^2} dz_{ij} \end{aligned} \tag{4.14}$$

where $|dz_{\ell i}| = 0$ and $|d\theta_{\kappa i}| = 1$. The transformations of $\Pi\mathcal{T}^*_{\mathbb{P}^{n|m}}$ instead are

$$\begin{aligned} \pi d\theta_{\kappa i} &= \frac{\pi d\theta_{\kappa j}}{z_{ij}} - \frac{\theta_{\kappa j}}{z_{ij}^2} \pi dz_{ij} \\ \pi dz_{ji} &= -\frac{\pi dz_{ij}}{z_{ij}^2} \\ \pi dz_{\ell i} &= \frac{\pi dz_{\ell j}}{z_{ij}} - \frac{z_{\ell j}}{z_{ij}^2} \pi dz_{ji} \end{aligned} \tag{4.15}$$

where now $|\pi dz_{\ell i}| = 1$ and $|\pi d\theta_{\kappa i}| = 1$.

Now, as should be suggested by the notation, the tangent $\mathcal{T}_{\mathcal{M}}$ and cotangent sheaf $\mathcal{T}^*_{\mathcal{M}}$, together with their parity-reversed version $\Pi\mathcal{T}_{\mathcal{M}}$ and $\Pi\mathcal{T}^*_{\mathcal{M}}$ are mutually *dual*.
Before seeing this, though, we have to recall the following important facts of super linear algebra, that actually makes difference in computations and might lead to mistakes. First of all, let us consider a supermatrix T as an *even* linear transformation between certain free supercommutative modules. Writing T in the block-form, one defines the supertransposition as

$$T^{st} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{st} := \left(\begin{array}{c|c} A^t & C^t \\ \hline -B^t & D^t \end{array}\right). \tag{4.16}$$

It is then immediate to see that the supertransposition has *not* period 2, but 4 instead, that is $T^{st^2} \neq T$, while $T^{st^4} = T$. Also, notice that the supertransposition does *not* commute with the parity transposition, $\Pi \circ st \neq st \circ \Pi$, but one finds instead the relations

$$\Pi \circ st \circ \Pi = st^2 \qquad\qquad st \circ \Pi \circ st = \Pi. \tag{4.17}$$

The previous discussion should warn about the issues one can encounter when dealing with the supertranspose and the parity transpose.
Indeed, let us now consider the case of $\mathcal{T}_{\mathcal{M}}$ and $\mathcal{T}^*_{\mathcal{M}}$. One finds

$$M(\mathcal{T}^*_{\mathcal{M}})^{st} \cdot M(\mathcal{T}_{\mathcal{M}}) = \mathcal{J}ac(\Phi)^{st} \cdot (\mathcal{J}ac(\Phi)^{-1})^{st} = (\mathcal{J}ac(\Phi)^{-1} \cdot \mathcal{J}ac(\Phi))^{st} = id \tag{4.18}$$

where we have used that $(AB)^{st} = B^{st}A^{st}$. One can thus define a pairing as follows,

$$
\begin{aligned}
\langle \cdot \, , \cdot \rangle : \mathcal{T}^*_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} &\longrightarrow \mathcal{O}_{\mathcal{M}} \\
\omega \otimes D &\longmapsto \langle \omega, D \rangle := \omega(D)
\end{aligned}
$$

for a general form ω and a vector field D.
Let us now pass to $\Pi\mathcal{T}_{\mathcal{M}}$ and $\Pi\mathcal{T}^*_{\mathcal{M}}$: one sees that it is no longer true that $M(\Pi\mathcal{T}^*_{\mathcal{M}})^{st} \cdot M(\Pi\mathcal{T}_{\mathcal{M}}) = id$. Instead, one finds that

$$
\begin{aligned}
M(\Pi\mathcal{T}_{\mathcal{M}})^{st} \cdot M(\Pi\mathcal{T}^*_{\mathcal{M}}) &= (\mathcal{J}ac(\Phi)^{-1})^{st\circ\Pi\circ st} \cdot \mathcal{J}ac(\Phi)^{\Pi} \\
&= (\mathcal{J}ac(\Phi)^{-1})^{\Pi} \cdot \mathcal{J}ac(\Phi)^{\Pi} \qquad (4.19)\\
&= (\mathcal{J}ac(\Phi)^{-1} \cdot \mathcal{J}ac(\Phi))^{\Pi} = id, \qquad (4.20)
\end{aligned}
$$

where we have used the second relation in (4.17) and that in general $(AB)^{\Pi} = A^{\Pi}B^{\Pi}$. The pairing is thus written as

$$
\begin{aligned}
\langle \cdot \, , \cdot \rangle_{\Pi} : \Pi\mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \Pi\mathcal{T}^*_{\mathcal{M}} &\longrightarrow \mathcal{O}_{\mathcal{M}} \\
D^{\pi} \otimes \omega^{\pi} &\longmapsto \langle D^{\pi}, \omega^{\pi} \rangle_{\Pi} := \omega^{\pi}(D^{\pi})
\end{aligned}
$$

where D^{π} and ω^{π} are related to D and ω by a parity change.
Usually, one sets $\omega^{\pi}(D^{\pi}) = \omega(D)$, and it is customary to take (see for example the Appendix of [21])

$$
\begin{cases}
dz_{\ell}(\pi\partial_{z_{\ell}}) := 1, \\
d\theta_{\kappa}(\pi\partial_{\theta_{\kappa}}) := -1 \\
dz_{\ell}(\pi\partial_{\theta_{\kappa}}) := 0 \\
d\theta_{\kappa}(\pi\partial_{z_{\ell}}) := 0.
\end{cases} \qquad (4.21)
$$

Also, notice that there is *no* natural pairing between $\mathcal{T}_{\mathcal{M}}$ and $\Pi\mathcal{T}^*_{\mathcal{M}}$ and, likewise, between $\Pi\mathcal{T}_{\mathcal{M}}$ and $\mathcal{T}^*_{\mathcal{M}}$: these sheaves have indeed also different rank. Nontheless, see again the Appendix in [21] for a definition of an "odd" pairing $\pi dz(\partial_z) = 1$ and $\pi d\theta(\partial_\theta) = -1$, which yields an isomorphism of vector spaces and *not* of vector superspaces.

Finally, there is one more very important natural sheaf that can be defined on a supermanifold, the so-called *Berezinian sheaf*, that can be looked at as a superanalog of the canonical sheaf of an ordinary manifold, whose sections are the elements that get integrated over. The key observation is that the sections of the canonical sheaf transform as *densities* under a change of local coordinates, we thus ask for a sheaf defined on the supermanifold $\mathcal{M}$ whose sections transform as densities as well. This calls for finding a supergeometric analog of the notion of determinant (of an automorphism) that enters the transformations of densities such as the sections of the canonical sheaf in ordinary geometry. The supergeometric analog of the determinant is known as *Berezianian.* Briefly, given a free $\mathbb{Z}_2$-graded module $A := A^{p|q}$, the Berezianian is a supergroup homomorphisms

$$
\mathrm{Ber} : GL(p|q; A) \longrightarrow GL(1|0; A_0) \qquad (4.22)
$$

that agrees with the determinant when $q = 0$ and it also proves to have similar properties (see [1] [12] [13]). Here $GL(p|q; A)$ are the invertible (even) automorphisms of A and A_0 stands for the even part of A.
Given a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules $\mathcal{E} \leftrightarrow (\{U_i\}_{i\in I}, \{g_{ij}(\mathcal{E})\}_{i,j\in I})$ of rank $p|q$, we thus

define the Berezinian sheaf of $\mathcal{E}$ - and we denote it by $\mathcal{B}er(\mathcal{E})$ - to be the locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules such that

$$\operatorname{rank} \mathcal{B}er(\mathcal{E}) = \begin{cases} 1|0 & n+m \text{ even,} \\ 0|1 & n+m \text{ odd.} \end{cases} \tag{4.23}$$

and whose section transforms with the Berezinian $\operatorname{Ber} g_{ij}(\mathcal{E})$ of the transition functions of $\mathcal{E}$. In particular, we employ the following definition (see also [13]): we call the Berezinian sheaf of a supermanifold $\mathcal{M}$ of dimension $n|m$ the sheaf

$$\mathcal{B}er\mathcal{M} := \mathcal{B}er(\Pi\mathcal{T}^*_{\mathcal{M}})^* = \mathcal{H}om(\mathcal{B}er(\Pi\mathcal{T}^*_{\mathcal{M}}), \mathcal{O}_{\mathcal{M}}). \tag{4.24}$$

Let us see why this apparently cumbersome definition makes sense, by discussing as usual the example of the projective superspaces $\mathbb{P}^{n|m}$.
It is well known, for example by adjunction theory, that the canonical sheaf $\mathcal{K}_{\mathbb{P}^n} := \bigwedge^n \mathcal{T}^*_{\mathbb{P}^n}$ of the n-dimensional projective space is given by $\mathcal{K}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$, and indeed projective spaces are Fano manifolds, having anti-ample canonical sheaf. If we wish to obtain this result back whenever reducing to an ordinary projective space $\mathbb{P}^n$ from a projective superspace $\mathbb{P}^{n|m}$, and we wish to use the sheaf $\Pi\mathcal{T}^*_{\mathbb{P}^{n|m}}$ as announced above, then we are then forced to employ the above definition. Indeed, if we are not taking the dual of the Berezinian sheaf above, we are led, because of parity reason to the *wrong* relation in the case of projective spaces, getting $\mathcal{K}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(n+1)$ instead. Let us see this in some details by computing the Berezinian sheaf explicitly (see [18] for a similar computation). We start from the dual of the Euler exact sequence for projective superspaces, which is a natural generalization of the well-known Euler exact sequence for projective spaces. Upon a parity-change one gets

$$0 \longrightarrow \Pi\mathcal{T}^*_{\mathbb{P}^{n|m}} \longrightarrow \mathbb{C}^{n+1|m} \otimes \Pi\mathcal{O}_{\mathbb{P}^{n|m}}(-1) \longrightarrow \Pi\mathcal{O}_{\mathcal{M}} \longrightarrow 0, \tag{4.25}$$

where the sheaves $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ are again direct generalizations of the usual invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$, actually they are the pullback sheaves $\pi^{-1}(\mathcal{O}_{\mathbb{P}^n}(\ell)) \otimes_{\pi^{-1}\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^{n|m}}$ of $\mathcal{O}_{\mathbb{P}^n}(\ell)$ via the projection $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$, see again [18] for details. Taking the Berezinian of the (4.25) yields

$$\mathcal{B}er(\Pi\mathcal{T}^*_{\mathcal{M}}) \cong \mathcal{B}er(\mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus m|n+1}) \cong \mathcal{O}_{\mathbb{P}^{n|m}}(-m+n+1), \tag{4.26}$$

so that, in turn

$$\mathcal{B}er(\Pi\mathcal{T}^*_{\mathcal{M}})^* \cong \mathcal{O}_{\mathbb{P}^{n|m}}(-m+n+1)^* \cong \mathcal{O}_{\mathbb{P}^{n|m}}(-n-1+m). \tag{4.27}$$

Notice that reversing parity and tensoring the (4.25) by $\mathcal{O}_{\mathbb{P}^n}$, one gets

$$0 \longrightarrow \mathcal{T}^*_{\mathbb{P}^n} \oplus (\mathcal{T}^*_{\mathbb{P}^{n|m}} \otimes \mathcal{O}_{\mathbb{P}^n})_1 \longrightarrow (\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})_0 \oplus (\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m})_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0, \tag{4.28}$$

where we have used that $(\mathcal{T}^*_{\mathbb{P}^{n|m}} \otimes \mathcal{O}_{\mathbb{P}^n})_0 \cong \mathcal{T}^*_{\mathbb{P}^n}$ and whose even-reduced parts reads

$$0 \longrightarrow \mathcal{T}^*_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0, \tag{4.29}$$

as it should, so that $\mathcal{K}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. The odd parts, actually yields the isomorphism $(\mathcal{T}^*_{\mathbb{P}^{n|m}} \otimes \mathcal{O}_{\mathbb{P}^n})_1 \cong \mathcal{O}(-1)^{\oplus m}_1$, which can be prove to be true from very general considerations [18].
Before we pass to the next section, it is important to stress, by the way, that there exist in literature more elegant and intrinsic characterizations of the Berezinian sheaf. This is the case, for example, of the very nice intrinsic construction given in [22], where the Berezinian sheaf is presented as a suitable quotient of locally-free sheaves of finite rank and a local basis is given, by means of forms and differential operators. More precisely, for a local coordinate system $z_1, \ldots, z_n | \theta_1, \ldots, \theta_m$ on a $n|m$ dimensional supermanifold $\mathcal{M}$ one has that the Berezinian is

generated (up to parity) by $dz_1 \wedge \dots \wedge dz_n \otimes \partial_{\theta_1} \circ \ldots \circ \partial_{\theta_m}$ over $\mathcal{O}_{\mathcal{M}}$, one can indeed check that this expression has the right transformation properties, thus matching on a rigorous basis the operative definition given above.

5. Superforms and Integral Forms Complex on a Supermanifold

Let $\mathcal{M}$ be a supermanifold of dimension $n|m$. It is possible to define the de Rham complex of differential superforms (henceforth superforms) associated to $\mathcal{M}$. This is constructed starting from the sheaf $\Omega^1_{\mathcal{M}} := \Pi \mathcal{T}^*_{\mathcal{M}}$, that it is locally freely-generated over an open set U by

$$\Omega^1_{\mathcal{M}} \lfloor_U := \Pi \mathcal{T}^*_{\mathcal{M}} \lfloor_U \cong \mathcal{O}_{\mathcal{M}} \lfloor_U \cdot \{d\theta_1, \ldots, d\theta_m | dz_1, \ldots, dz_n\}, \tag{5.1}$$

where we recall that $|d\theta_i| = 0$ and $|dz_j| = 1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, as seen above. There is a natural *odd* differential acting as follows

$$\begin{aligned} d : \mathcal{O}_{\mathcal{M}} &\longrightarrow \Pi \mathcal{T}^*_{\mathcal{M}} \\ f &\longmapsto df, \end{aligned} \tag{5.2}$$

where df is defined as

$$df := \sum_{i=1}^m d\theta_i \frac{\partial f}{\partial \theta_i} + \sum_{j=1}^n dz_j \frac{\partial f}{\partial z_j}, \tag{5.3}$$

in agreement with [1], page 17, and it satisfies the $\mathbb{Z}_2$-graded Leibniz rule, as one might check,

$$d(f \cdot g) = df \cdot g + (-1)^{|f|} f \cdot dg, \tag{5.4}$$

where we have used that $|d| = 1$. Importantly this differential can be lifted to an *exterior derivative*

$$\begin{aligned} d^i : Sym^i \Pi \mathcal{T}^*_{\mathcal{M}} &\longrightarrow Sym^{i+1} \Pi \mathcal{T}^*_{\mathcal{M}} \\ \omega &\longmapsto d\omega, \end{aligned} \tag{5.5}$$

having the properties that $d^i \circ d^{i+1} = 0$, therefore we have a complex $\Omega^\bullet_{\mathcal{M}} := (Sym^\bullet \Pi \mathcal{T}^*_{\mathcal{M}}, d^\bullet)$ of locally-free $\mathcal{O}_{\mathcal{M}}$-modules as follows

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}} \stackrel{d}{\longrightarrow} \Pi \mathcal{T}^*_{\mathcal{M}} \stackrel{d}{\longrightarrow} Sym^2 \Pi \mathcal{T}^*_{\mathcal{M}} \stackrel{d}{\longrightarrow} \cdots\cdots \stackrel{d}{\longrightarrow} Sym^n \Pi \mathcal{T}^*_{\mathcal{M}} \stackrel{d}{\longrightarrow} \cdots, \tag{5.6}$$

where we have dropped the index of the exterior derivative $d^i : Sym^i \Pi \mathcal{T}^*_{\mathcal{M}} \to Sym^{i+1} \Pi \mathcal{T}^*_{\mathcal{M}}$ for notational reasons.
A crucial fact should now be underlined: whilst the de Rham complex reduces - as it should - to the usual de Rham complex on a complex manifold if the the odd dimension m of the supermanifold $\mathcal{M}$ is zero, if $m \neq 0$ the de Rham complex on a supermanifold is *not bounded from above*. In other words, there is *no* notion of a top-form on a supermanifold, indeed one can actually take an arbitrary high power of the $d\theta$'s since they are commuting sections, *i.e* $d\theta \odot \ldots \odot d\theta = d\theta^{\odot i} \neq 0$ for any $i > 0$.
Let us consider for example the supermanifold $\mathbb{P}^{1|1}$. Then one will find that for any $i > 0$ the sheaf $Sym^i \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}$ is of rank $1|1$ and locally freely-generated over the open set U_0 by

$$Sym^i \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}(U_0) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \{d\theta_{10}^{\odot i} | dz_{10} \odot d\theta_{10}^{\odot i-1}\}, \tag{5.7}$$

where it is understood that $\Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}(U_0) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \{d\theta_{10} | dz_{10}\}$.

Also, notice that the Berezinian sheaf does *not* appear at any place in the de Rham complex $\Omega^\bullet_{\mathcal{M}}$ above, and therefore no sections of the sheaves appearing in the de Rham complex can be integrated over a supermanifold. In order to introduce a notion of top-form, suitable to define a geometric integration theory for supermanifolds, one has to resort to the notion of *integral forms*. We leave to the literature [1, 8, 4] and also [23, 24, 25, 26] a detailed discussion, here we will just sketch their main properties, in order to make the paper as self-consistent as possible.

In particular, an *integral top-form* is written locally as:

$$\omega^{(n|m)} = f(z,\theta)dz_1 \dots dz_n \delta(d\theta_1)\dots\delta(d\theta_m) \tag{5.8}$$

where $f(z,\theta)$ is a section of $\mathcal{O}_{\mathcal{M}}$ and a $\mathbb{Z}_2$-graded symmetric product is understood. A single symbol $\delta(d\theta)$ is formally defined as

$$\delta(d\theta) = \int_{\mathbb{R}} e^{id\theta t} dt, \tag{5.9}$$

where $t \in \mathbb{R}$ is an auxiliary variable, so that, referring to the expression (5.8), one has

$$\delta(d\theta_1)\dots\delta(d\theta_m) := \int_{\mathbb{R}^m} e^{i\sum_{i=0}^m d\theta_i t_i} dt_1 \wedge \dots \wedge dt_m, \tag{5.10}$$

together with their derivatives

$$(-i)^m \delta^{(1)}(d\theta_1)\dots\delta^{(1)}(d\theta_m) = \int_{\mathbb{R}^m} t_1 \dots t_m e^{i\sum_{i=0}^m d\theta_i t_i} dt_1 \wedge \dots \wedge dt_m. \tag{5.11}$$

Remarkably, the formal expression (5.8) transforms as a section of the Berezinian sheaf, and as such it can be integrated over.

More in general, an expression involving the dz's, $d\theta$'s, $\delta(d\theta)$'s and their derivatives is of the kind

$$\omega^{(p|q)} = f(z,\theta)dz_{a_1}\dots dz_{a_r} d\theta_{b_1}\dots d\theta_{b_s}\delta^{(r_1)}(d\theta_{c_1})\dots\delta^{(r_q)}(d\theta_{c_q}) \tag{5.12}$$

where the $\mathbb{Z}_2$-graded symmetric product between $dx, d\theta$ and δ's is understood, p and q correspond respectively to the *form* number and the *picture* number, with $0 \leq q \leq m$ and $p = r + s - \sum_{i=1}^{i=q} r_i$ and $0 \leq r \leq n$. In a given monomial, the $d\theta_i$ appearing in the product are different from those appearing in the delta's as

$$d\theta_i \delta(d\theta_i) = 0. \tag{5.13}$$

and $\omega(x,\theta)$ is a set of sections of the structure sheaf, having index structure[1] $\omega_{[a_1\dots a_r](b_1\dots b_s)[r_1\dots r_q]}(z,\theta)$. Also, we recall the following important rules, see for example [27]:

$$d(\delta^{(k)}(d\theta_i)) = 0 \text{ for } k \geq 0, \tag{5.14}$$

$$d\theta_i \delta^{(k)}(d\theta_i) = -k\delta^{(k-1)}(d\theta_i) \text{ for } k > 0. \tag{5.15}$$

Notice that the meaning of the first one is that $\delta^{(k)}(d\theta)$ is d-closed. With reference to the expression (5.12), the index r_i on the delta $\delta^{(r_i)}(d\theta_{b_j})$ denotes the degree of the derivative of the delta function with respect to its argument. The total picture q of $\omega^{(p|q)}$ corresponds to the total number of delta functions and its derivatives. The total form degree is given by

[1]The indices $a_1 \dots a_r$ and $b_1 \dots b_q$ are anti-symmetrized, the indices $r_1 \dots r_s$ are symmetrized because of the rules of the graded product:

$$\begin{aligned} dz_a dz_b &= -dz_b dz_a\,,\ dz_a d\theta_i = d\theta_i dz_a\,,\ d\theta_i d\theta_j = d\theta_j d\theta_i\,, \\ \delta(d\theta_i)\delta(d\theta_j) &= -\delta(d\theta_j)\delta(d\theta_i)\,, \\ dz_a\delta(d\theta_i) &= -\delta(d\theta_i)dz_a\,,\ d\theta_i\delta(d\theta_j) = \delta(d\theta_j)d\theta_i\ . \end{aligned}$$

$p = r + s - \sum_{i=1}^{i=q} r_i$ since the derivatives act effectively as negative degree forms and the delta functions carry zero form degree.

In this extended scenario, we call $\omega^{(p|q)}$ a *superform* if $q = 0$: in this case it belongs to the honest de Rham complex $\Omega^\bullet_{\mathcal{M}}$ we have introduced above. We call $\omega^{(p|q)}$ an *integral form* if $q = m$, and we shall discuss this case in a moment; otherwise $\omega^{(p|q)}$ for $q \neq 0, q \neq m$ is called *pseudoform*[2].

Let us now take on the case of *integral forms*, that is whence $q = m$. The theory of integral forms can be re-written in a manifest sheaf-theoretical formalism as to match and extend the above de Rham complex, that we will now call $\Omega^{\bullet;0}_{\mathcal{M}} := (Sym^\bullet \Pi \mathcal{T}^*_{\mathcal{M}}, d^\bullet)$, as to specify the picture of the forms involved. Indeed, we claim that the integral forms fits into a new complex, we call it $\Omega^{\bullet;m}_{\mathcal{M}}$, and we define it as follows

$$\Omega^{k;m}_{\mathcal{M}} := (\mathcal{B}er(\mathcal{M}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathcal{M}}, d^k), \qquad k \leq n, \tag{5.16}$$

where the operator $d^k : \mathcal{B}er(\mathcal{M}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathcal{M}} \to \mathcal{B}er(\mathcal{M}) \otimes Sym^{n-(k+1)}\Pi\mathcal{T}_{\mathcal{M}}$ is induced by that defined for the de Rham complex $\Omega^{\bullet;m}_{\mathcal{M}}$ above, as we shall see shortly.
Now, it is crucial to note that the complex terminates to the Berezinian sheaf, that is $\Omega^{n;m}_{\mathcal{M}} := \mathcal{B}er(\mathcal{M})$, so that one finds

$$\cdots \longrightarrow \mathcal{B}er(\mathcal{M}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathcal{M}} \longrightarrow \cdots \longrightarrow \mathcal{B}er(\mathcal{M}) \otimes \Pi\mathcal{T}_{\mathcal{M}} \longrightarrow \mathcal{B}er(\mathcal{M}) \longrightarrow 0. \tag{5.17}$$

In order to convince the reader about the correspondence between the mathematical sheaf-theoretic formalism of the complex $\Omega^{\bullet;m}_{\mathcal{M}}$ and the delta-function $\delta^{(k)}(d\theta)$'s formalism - which is preferred in the context of theoretical physics and string theory -, we now deal with the explicit and simple example of $\mathbb{P}^{1|1}$.
We aim to match the modules involving the delta's for a fixed total form degree k, with the sheaves

$$\Omega^{k;1}_{\mathbb{P}^{1|1}} := \mathcal{B}er(\mathbb{P}^{1|1}) \otimes_{\mathbb{P}^{1|1}} Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} \tag{5.18}$$

for any $k \leq 1$ appearing in the complex $\Omega^{\bullet;m}_{\mathcal{M}}$, by comparing their transition functions in the only intersection $U_0 \cap U_1$ of $\mathbb{P}^{1|1}$.
In order to do that we start dealing with the delta's formalism, as we first need the transformation properties the integral forms on projective superspaces. Generalizing the result of [27], and following the convention set above, on a general projective superspace $\mathbb{P}^{n|m}$ one finds

$$\begin{aligned}\delta^{(0)}(d\theta_{\ell i}) &= \delta^{(0)}\left(\frac{d\theta_{\ell j}}{z_{ij}} - \frac{\theta_{\ell j}}{z_{ij}^2}dz_{ij}\right) = z_{ij}\delta^{(0)}\left(d\theta_{\ell j} - \frac{\theta_{\ell j}}{z_{ij}}dz_{ij}\right) \\ &= z_{ij}\delta^{(0)}(d\theta_{\ell j}) - \theta_{\ell j}dz_{ij}\delta^{(1)}(d\theta_{\ell j}) \\ &= z_{ij}\delta^{(0)}(d\theta_{\ell j}) + \theta_{\ell j}\delta^{(1)}(d\theta_{\ell j})dz_{ij},\end{aligned} \tag{5.19}$$

where we recall that $|\delta^{(i)}(d\theta)| = 1$ and we have Taylor expanded (the increment has been put to the left) around the $d\theta$.
Generalizing this formula, for higher-derivatives one finds

$$\delta^{(k)}(d\theta_{\ell i}) = z_{ij}^{k+1}\delta^{(k)}(d\theta_{\ell j}) + z_{ij}^k\theta_{\ell j}\delta^{(k+1)}(d\theta_{\ell j})dz_{ij}. \tag{5.20}$$

Now, we have that in the delta's formalism the modules are locally generated by expressions of the kind

$$\Omega^{k;1}_{\mathbb{P}^{1|1}}(U_0) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \{dz_{10}\delta^{(1-k)}(d\theta_{10}) | \delta^{(-k)}(d\theta_{10})\} \qquad k < 1, \tag{5.21}$$

[2]Note that in Voronov's book the name pseudoform denotes a more general type of forms

and $\Omega^{1;1}_{\mathbb{P}^{1|1}}(U_0) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \{dz_{10}\delta^{(0)}(d\theta_{10})\}$. Using the transformation properties in the (5.19) and (5.20) adapted for $\mathbb{P}^{1|1}$, one finds the following transition matrix

$$M(\Omega^{1;1}_{\mathbb{P}^{1|1}}) = \left(-\frac{1}{z}\right) \qquad M(\Omega^{k;1}_{\mathbb{P}^{1|1}}) = \left(\begin{array}{c|c} -z^{-k} & 0 \\ \hline \theta z^{-k} & z^{1-k} \end{array}\right) \text{ for } k \leq 1, \tag{5.22}$$

where we have dropped for convenience the indices referring to the only intersection $U_0 \cap U_1$ on $\mathbb{P}^{1|1}$.

Let us now look at the sheaf-theoretic formalism. First of all, we have that, $\mathcal{B}er(\mathbb{P}^{1|1}) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(-1)$. Moreover, if $\mathcal{B}er(\mathbb{P}^{1|1})$ is locally-generated over U_0 by $\mathcal{D}[dz_{10}|d\theta_{10}]$ (see for example [1] for this notation), calculating explicitly the transition function of this rank $1|0$ locally-free sheaf, one has

$$\mathcal{D}[dz_{10}|d\theta_{10}] = \left(-\frac{1}{z_{01}}\right)\mathcal{D}[dz_{01}|d\theta_{01}], \tag{5.23}$$

thus matching $M(\Omega^{1;1}_{\mathbb{P}^{1|1}})$ above, as expected.
Also, locally, for $k < 1$, one finds that

$$\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}}(U_0) = \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \mathcal{D}[dz_{10}|d\theta_{10}] \otimes \left\{\pi\partial^{\odot 1-k}_{\theta_{10}} \middle| \pi\partial_{z_{10}} \odot \pi\partial^{\odot -k}_{\theta_{10}}\right\} \tag{5.24}$$

and, using the transformation rules introduced above for the sheaves of the kind $\Pi\mathcal{T}_{\mathbb{P}^{n|m}}$ specialized to the case of $\mathbb{P}^{1|1}$, one finds that the transformation matrix reads

$$M(\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}}) = -\frac{1}{z} \otimes \left(\begin{array}{c|c} z^{1-k} & 0 \\ \hline -\theta z^{1-k} & -z^{2-k} \end{array}\right) = \left(\begin{array}{c|c} -z^{-k} & 0 \\ \hline \theta z^{-k} & z^{1-k} \end{array}\right), \tag{5.25}$$

thus matching the remaining ones for $k < 1$.
We therefore have the following correspondence, realizing the actual isomorphism between the sheaf-theoretic and the delta's formalism:

$$\mathcal{D}[dz|d\theta] \otimes \pi\partial^{\odot 1-k}_{\theta} \longleftrightarrow dz\delta^{(1-k)}(d\theta), \tag{5.26}$$

$$\mathcal{D}[dz|d\theta] \otimes \pi\partial_z \odot \pi\partial^{\odot -k}_{\theta} \longleftrightarrow \delta^{(-k)}(d\theta), \tag{5.27}$$

for $k < 1$, together with $\mathcal{D}[dz|d\theta] \leftrightarrow dz\delta^{(0)}(d\theta)$, that are sections of the Berezinian sheaf $\Omega^{1;1}_{\mathbb{P}^{1|1}} \cong \mathcal{B}er(\mathbb{P}^{1|1})$.

Now that the correspondence is achieved at the level of the sheaves, we still have to deal with the coboundary operator d of the complex $\Omega^{\bullet;m}_{\mathbb{P}^{1|1}}$. First, we recall that looking at $\mathbb{P}^{1|1}$ one has a differential, acting locally as

$$\begin{aligned} d_U : \mathcal{O}_{\mathbb{P}^{1|1}}(U) &\longrightarrow \Pi\mathcal{T}^*_{\mathbb{P}^{1|1}}(U) \\ F &\longmapsto d_U F := dz\,\partial_z F + d\theta\,\partial_\theta F, \end{aligned} \tag{5.28}$$

that lifts to an exterior differential for the de Rham complex $\Omega^{\bullet;0}_{\mathbb{P}^{1|1}}$, as observed above. Now we recall that in (4.21) we have set a pairing on the local generator of $\Pi\mathcal{T}_{\mathbb{P}^{1|1}}$ and $\Pi\mathcal{T}^*_{\mathbb{P}^{1|1}}$ as follows:

$$\begin{cases} d\theta(\pi\partial_\theta) \equiv \langle \pi\partial_\theta, d\theta\rangle := -1 \\ dz(\pi\partial_z) \equiv \langle \pi\partial_z, dz\rangle := 1, \\ dz(\pi\partial_\theta) \equiv \langle \pi\partial_\theta, dz\rangle := 0 \\ d\theta(\pi\partial_z) \equiv \langle \pi\partial_z, d\theta\rangle := 0. \end{cases}$$

These relations will be used to extend the differential to the integral forms, in order to construct a honest complex.

Indeed, let $s^{(k)} \in \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} = \Omega^{k;1}_{\mathbb{P}^{1|1}}$ be a generic integral forms. Then, in a certain chart

$$s^{(k)} = \mathcal{D}[dz|d\theta] \otimes \big(F \cdot (\pi\partial_\theta^{\odot 1-k}) + G \cdot (\pi\partial_z \odot \pi\partial_\theta^{-k})\big) \tag{5.29}$$

for some $F, G \in \mathcal{O}_{\mathbb{P}^{1|1}}$.
We thus put

$$ds^{(k)} := \mathcal{D}[dz|d\theta] \otimes \big(dF \cdot (\pi\partial_\theta^{\odot 1-k}) + dG \cdot (\pi\partial_z \odot \pi\partial_\theta^{-k})\big) , \tag{5.30}$$

so that one finds

$$ds^{(k)} = \mathcal{D}[dz|d\theta] \otimes \big((dz\, \partial_z F + d\theta\, \partial_\theta F) \cdot (\pi\partial_\theta^{\odot 1-k}) + (dz\, \partial_z G + d\theta\, \partial_\theta G) \cdot (\pi\partial_z \odot \pi\partial_\theta^{\odot -k})\big) , \tag{5.31}$$

and upon using the pairing defined above, one has

$$ds^{(k)} = \mathcal{D}[dz|d\theta] \otimes \big(-|1-k| \partial_\theta F \cdot (\pi\partial_\theta^{\odot -k}) + (-1)^{|G|}\partial_z G \cdot (\pi\partial_\theta^{\odot -k}) - |k|\partial_\theta G \cdot (\pi\partial_z \odot \pi\partial_\theta^{\odot -k-1})\big) \tag{5.32}$$

Note that this defines a section in $\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} = \Omega^{k+1;1}_{\mathbb{P}^{1|1}}$, as it should.
Applying again d yields

$$d(ds^{(k)}) = \mathcal{D}[dz|d\theta] \otimes \big((-1)^{|G|} d\theta\, \partial_\theta \partial_z G \cdot (\pi\partial_\theta^{\odot -k}) - |k| dz\, \partial_z \partial_\theta G \cdot (\pi\partial_z \odot \pi\partial_\theta^{\odot -k-1})\big) . \tag{5.33}$$

Using again the pairings above, one gets:

$$\begin{aligned} d(ds^{(k)}) &= \mathcal{D}[dz|d\theta] \otimes \big(-|k|(-1)^{|G|} \partial_\theta \partial_z G \cdot (\pi\partial_\theta^{\odot -k-1}) - |k|(-1)^{|G|+1} \partial_z \partial_\theta G \cdot (\pi\partial_\theta^{\odot -k-1})\big) \\ &= |k|(-1)^{|G|} \mathcal{D}[dz|d\theta] \otimes (-\partial_z\partial_\theta G + \partial_z \partial_\theta G) \cdot (\pi\partial_\theta^{\odot -k-1}) = 0, \end{aligned} \tag{5.34}$$

as $[\partial_z, \partial_\theta] = 0$.
This shows that $d \circ d = 0$, so it can be promoted to a coboundary operator for the complex of integral forms $(\Omega^{k;1}_{\mathbb{P}^{1|1}} = \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}}, d^k)$, with

$$d^k : \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} \longrightarrow \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-(k+1)}\Pi\mathcal{T}_{\mathbb{P}^{1|1}}, \tag{5.35}$$

so that

$$\ldots \xrightarrow{d} \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi\mathcal{T}_{\mathbb{P}^{1|1}} \xrightarrow{d} \mathcal{B}er(\mathbb{P}^{1|1}) \longrightarrow 0 \tag{5.36}$$

This simple example can be generalized to any supermanifold $\mathcal{M}$, as to yield its integral forms complex.

Moreover, the fundamental relations characterizing the delta's, *i.e.* equation (5.13) and following, can be recovered in a more geometric fashion using the sheaf-theoretic formalism developed above. In particular, one finds that the basic relation $d\theta\delta^{(0)}(d\theta) = 0$ can be recover using the pairing defined above. Indeed, we recall that one has $\delta^{(0)}(d\theta) = \mathcal{D}(dz|d\theta) \otimes \pi\partial_z$, so that one finds

$$d\theta\delta^{(0)}(d\theta) = 0 \quad \longleftrightarrow \quad \mathcal{D}(dz|d\theta) \otimes \langle \pi\partial_z, d\theta \rangle = 0. \tag{5.37}$$

Even more, for higher-derivatives of the delta's, one has $dzd\theta\delta^{(1)}(d\theta) = -\delta^{(0)}(d\theta)dz$. Recalling that $dz\delta^{(1)}(d\theta) = \mathcal{D}(dz|d\theta) \otimes \pi\partial_\theta$, one finds again via the pairing

$$dzd\theta\delta^{(1)}(d\theta) = -dz\delta^{(0)}(d\theta) \quad \longleftrightarrow \quad \mathcal{D}(dz|d\theta) \otimes \langle \pi\partial_\theta, d\theta \rangle = -\mathcal{D}(dz|d\theta) \tag{5.38}$$

where we recall that $\mathcal{D}(dz|d\theta)$ is indeed a generating section of the Berezinian sheaf, corresponding to $\delta^{(0)}(d\theta)dz$ in the integral forms delta's notation.

Actually, as the attentive reader might have already noticed, it is fair to say that the whole construction of integral forms above can be obtained from first principles starting from the de Rham complex $\Omega^{\bullet;0}_{\mathcal{M}}$, upon using some homological algebra. This construction is completely natural and spare us from cumbersome choices. Leaving the details to future works, we just observe that, indeed, for a completely generic supermanifold $\mathcal{M}$ of dimension $n|m$, the locally-free sheaves making up the complex of integral forms $\Omega^{\bullet;m}_{\mathcal{M}}$ can be obtained from those appearing in the de Rham complex, $\Omega^{\bullet;0}_{\mathcal{M}} = Sym^{\bullet}\Pi\mathcal{T}^*_{\mathcal{M}}$, simply by applying the functor $h_{\mathcal{B}er(\mathcal{M})} := \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(-, \mathcal{B}er(\mathcal{M}))$ to them. In other words, one has

$$\Omega^{k;0}_{\mathcal{M}} \xmapsto{h_{\mathcal{B}er(\mathcal{M})}} \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\Omega^{k;0}_{\mathcal{M}}, \mathcal{B}er(\mathcal{M})) = \Omega^{n-k;m}_{\mathcal{M}} \tag{5.39}$$

for $k \geq 0$. Also, notice that this functor is *contravariant*: this means that if one has a morphism of sheaves $\phi : \Omega^{k,0}_{\mathcal{M}} \to \Omega^{k+1;0}_{\mathcal{M}}$, then applying the functor $h_{\mathcal{B}er(\mathcal{M})}$ to ϕ yields a morphism $h_{\mathcal{B}er(\mathcal{M})}(\phi) := \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\phi, \mathcal{B}er(\mathcal{M})) : \Omega^{n-k-1;m}_{\mathcal{M}} \to \Omega^{n-k}_{\mathcal{M}}$. In particular, recalling that by definition functors preserve composition, one has $h_{\mathcal{B}er(\mathcal{M})}(\phi \circ \psi) = h_{\mathcal{B}er(\mathcal{M})}(\phi) \circ h_{\mathcal{B}er(\mathcal{M})}(\psi)$.
The homological features of this construction and their implications will be elucidated in a forthcoming paper.

Before we go on, a remark is in order, though. While on the one hand we have already seen that superforms and integral forms are well-behaved and that they can be given a structure of complexes of locally-free *finitely generated* sheaves of $\mathcal{O}_{\mathcal{M}}$-modules, this is no longer true for *pseudoforms* - that is for middle-dimensional picture $0 < p < m$. Indeed, it can be seen that, for a fixed form number, pseudoforms are locally arranged in modules that are *not* finitely-generated, and therefore they cannot be described globally as locally-free sheaves of finite rank, such as for superforms and integral forms.
Let us see this by means of the easiest possible example, that of $\mathbb{P}^{1|2}$, which has already been discussed in [28].
Using the delta's formalism, one sees that those modules having picture number equal to 1 are generated over the open set U_0 by expressions of the kind

$$\Omega^{k;1}_{\mathbb{P}^{1|2}}(U_0) = \mathcal{O}_{\mathbb{P}^{1|2}}(U_0)\cdot\Big\{\delta^{(\ell+1)}(d\theta_{10})dz_{10}d\theta_{20}^{k+\ell+1}, \delta^{(\ell+1)}(d\theta_{20})dz_{10}d\theta_{10}^{k+\ell+1} \,| \\ |\, \delta^{(\ell)}(d\theta_{10})d\theta_{20}^{k+\ell}, \delta^{(\ell)}(d\theta_{20})d\theta_{10}^{k+\ell}\Big\} \qquad \ell \in \mathbb{N}\cup\{0\}. \tag{5.40}$$

This example suggests that these expressions can be are arranged in *quasi-coherent sheaves* of $\mathcal{O}_{\mathbb{P}^{1|2}}$-modules - and as such they might have an infinite Čech cohomology, see [28]. A more careful description of this particular quasi-coherent sheaves would be necessary in order to get a complete mathematically satisfying picture of the zoo of forms on a supermanifold.

6. Negative Degree Superforms and their Complex

We have seen that integral forms allow to define complexes of forms carrying a negative degree. Moreover, as explained in the section 2, in the framework of Large Hilbert Space, also ordinary superforms carrying a negative degree make their appearance: we will call these new special superforms: *inverse superforms*.
The fundamental observation is once again that, given a supermanifold $\mathcal{M}$, the local sections $d\theta$'s of the sheaf $\Pi\mathcal{T}^*_{\mathcal{M}}$ are *even*, and therefore they can be formally "inverted". Here, we

describe this phenomenology making use of our usual driving example of $\mathbb{P}^{1|1}$. Notice that since we are interested into the algebraic-geometric properties of these special superforms, we will describe them as sections of certain sheaves, and we will make no distinction between $1/d\theta$ and p.v.$(1/d\theta)$, as their transformation properties coincide.

Let us consider the sheaf $\Omega^{1;0}_{\mathbb{P}^{1|1}} = \Pi\mathcal{T}^*_{\mathbb{P}^{1|1}}$. As seen, over the open set U_0 of $\mathbb{P}^{1|1}$, $\Pi\mathcal{T}^*_{\mathbb{P}^{1|1}}$ is locally-freely generated by $\{d\theta_{10}|dz_{10}\}$. We would like to get a form of degree equal to -1, by "dividing" by $d\theta_{10}$, so we consider formal expressions like

$$\Omega^{-1;1}_{\mathbb{P}^{1|1}}(U_0) = \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \left\{ \frac{1}{d\theta_{10}} \Big| \frac{dz_{10}}{d\theta_{10}^2} \right\} . \tag{6.1}$$

Notice that, as for the sheaves of pseudoforms, only whenever the supermanifold $\mathcal{M}$ is of odd dimension equal to 1, the sheaves $\Omega^{k;p}_{\mathcal{M}}$ of any degree and picture number are coherent, actually locally-free sheaves of $\mathcal{O}_{\mathcal{M}}$-modules. Indeed, let us look again at the case of $\mathbb{P}^{1|2}$. One would find that the expressions with degree -1 can be generated by

$$\Omega^{-1;0}_{\mathbb{P}^{1|2}}(U_0) = \mathcal{O}_{\mathbb{P}^{1|2}}(U_0) \cdot \left\{ \frac{d\theta_{10}^\ell}{d\theta_{20}^{\ell+1}}, \frac{d\theta_{20}^\ell}{d\theta_{10}^{\ell+1}} \Bigg| dz_{10}\frac{d\theta_{10}^\ell}{d\theta_{20}^{\ell+2}}, dz_{10}\frac{d\theta_{20}^\ell}{d\theta_{10}^{\ell+2}} \right\} \qquad \ell \in \mathbb{N} \cup \{0\} . \tag{6.2}$$

One thus sees that these expressions are likely to make up a quasi-coherent sheaf, but not certainly a locally-free sheaf of finite rank, an issue that again makes the theory more difficult. Restricting ourself to the case having a single odd dimension, as in the previous section, we now aim to give this formal setting a sheaf-theoretic dignity. One can see that, in general, for $\mathbb{P}^{1|1}$

$$\Omega^{-k;0}_{\mathbb{P}^{1|1}}(U_0) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(U_0) \cdot \left\{ \frac{1}{d\theta_{10}^k} \Bigg| \frac{dz_{10}}{d\theta_{10}^{k+1}} \right\} \qquad k > 0. \tag{6.3}$$

In particular, for $k = 1$, the transformations read

$$\frac{1}{d\theta_{10}} = z_{01}\frac{1}{d\theta_{11}} + \theta_{11}\frac{dz_{01}}{d\theta_{11}^2}, \tag{6.4}$$

$$\frac{dz_{10}}{d\theta_{10}^2} = -\frac{dz_{01}}{d\theta_{11}^2}, \tag{6.5}$$

such that, generalizing to a generic $k > 0$, one gets a transformation matrix of the form:

$$M(\Omega^{-k;0}_{\mathbb{P}^{1|1}}) = \left(\begin{array}{c|c} z^k & kz^{k-1}\theta \\ \hline 0 & -z^{k-1} \end{array} \right). \tag{6.6}$$

Now, one can see that the case $k = 1$ corresponds to the transition functions of the sheaf $\Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi\mathcal{T}_{\mathbb{P}^{1|1}}$, indeed:

$$M(\Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi\mathcal{T}_{\mathbb{P}^{1|1}}) = \left(\begin{array}{c|c} z & \theta \\ \hline 0 & -1 \end{array} \right), \tag{6.7}$$

as one can readily verify, settling the first case. Notice that in this case $\Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi\mathcal{T}_{\mathbb{P}^{1|1}} \cong \mathcal{B}er(\mathbb{P}^{1|1}) \otimes \mathcal{T}_{\mathbb{P}^{1|1}}$.

More in general one finds that the correspondence we are looking for is

$$\Omega^{-k;0}_{\mathbb{P}^{1|1}} \cong \Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^k\Pi\mathcal{T}_{\mathbb{P}^{1|1}} \qquad k > 0, \tag{6.8}$$

where we observe that the functors Π and Sym^k do not commute (for $k > 1$), so that $\Pi \circ Sym^k \neq Sym^k \circ \Pi$. Actually, the transition matrix of the sheaf $\Pi \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^k \Pi \mathcal{T}_{\mathbb{P}^{1|1}}$ is given by

$$M(\Pi \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^k \Pi \mathcal{T}_{\mathbb{P}^{1|1}}) = \left(\begin{array}{c|c} z^k & z^{k-1}\theta \\ \hline 0 & -z^{k-1} \end{array} \right), \tag{6.9}$$

but clearly the numerical factor k in the upper-right entry of the matrix above can be recovered by a constant change of basis, actually a constant scaling of the generators - that does not modify the class in the cohomology set of the transition functions and, hence it does not change the sheaf we have identified.
In particular, choosing

$$A = \left(\begin{array}{c|c} \sqrt{k} & 0 \\ \hline 0 & 1/\sqrt{k} \end{array} \right) \tag{6.10}$$

does the job, as

$$A \left(\begin{array}{c|c} z^k & \theta z^{k-1} \\ \hline 0 & -z^{k-1} \end{array} \right) A^{-1} = \left(\begin{array}{c|c} z^k & k z^{k-1}\theta \\ \hline 0 & -z^{k-1} \end{array} \right). \tag{6.11}$$

Also, notice incidentally that $\det A = 1$. To conclude, the correspondence goes as follows:

$$\frac{1}{d\theta^k} \quad \longleftrightarrow \quad \sqrt{k}(\pi \mathcal{D}[dz|d\theta]) \otimes (\pi \partial_z \odot \pi \partial_\theta^{\odot k-1}) \tag{6.12}$$

$$\frac{dz}{d\theta^{k+1}} \quad \longleftrightarrow \quad \frac{1}{\sqrt{k}}(\pi \mathcal{D}[dz|d\theta]) \otimes (\pi \partial_\theta^{\odot k}), \tag{6.13}$$

thus giving the sheaf-theoretic identification:

$$\Omega^{-k;0}_{\mathbb{P}^{1|1}} \cong \Pi \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^k \Pi \mathcal{T}_{\mathbb{P}^{1|1}} \qquad k > 0. \tag{6.14}$$

The case $k = 0$ deserves a special attention, indeed the sheaf $\Omega^{0;0}_{\mathbb{P}^{1|1}}$ - that formerly corresponded to the structure sheaf $\mathcal{O}_{\mathbb{P}^{1|1}}$ - gets modified to

$$\Omega^{0;0}_{\mathbb{P}^{1|1}} \cong \mathcal{O}_{\mathbb{P}^{1|1}} \oplus \Pi \mathcal{B}er(\mathbb{P}^{1|1}), \tag{6.15}$$

as one has to take into account also an element which is locally of the form $\frac{dz}{d\theta}$, and therefore it is a superform of degree zero. It is straighforward to verity that such an element transforms as a section of parity-changed Berezinian sheaf $\Pi \mathcal{B}er(\mathbb{P}^{1|1})$, that is $\frac{dz}{d\theta} \equiv \pi \mathcal{D}[dz|d\theta]$. We will see in a moment why the structure sheaf gets extended in this way.

As we got this far, a crucial fact that has to be noted is that the sheaves making up the inverse superforms of a certain fixed degree, corresponds to certain sheaves of integral forms of certain fixed degree, but having *opposite parity*. In particular, it can be seen that for $k \leq 0$ one has

$$\underbrace{\Omega^{k;1}_{\mathbb{P}^{1|1}} = \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k} \Pi \mathcal{T}_{\mathbb{P}^{1|1}}}_{\text{Integral Forms}} \quad \longleftrightarrow \quad \underbrace{\Omega^{k-1;0}_{\mathbb{P}^{1|1}} = \Pi \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k} \Pi \mathcal{T}_{\mathbb{P}^{1|1}}}_{\text{Inverse Forms}}. \tag{6.16}$$

It follows that, altogether, one gets the following diagram:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \Pi \mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi \mathcal{T}_{\mathbb{P}^{1|1}} & \longrightarrow & \Pi \mathcal{B}er(\mathbb{P}^{1|1}) \oplus \mathcal{O}_{\mathbb{P}^{1|1}} & \longrightarrow & \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}} & \longrightarrow \ldots & & \\ \nwarrow^{\Theta} & & \nwarrow^{\Theta} & & \nwarrow^{\Theta} & & & & \\ \ldots \longrightarrow & \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^2 \Pi \mathcal{T}_{\mathbb{P}^{1|1}} & \longrightarrow & \mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi \mathcal{T}_{\mathbb{P}^{1|1}} & \longrightarrow & \mathcal{B}er(\mathbb{P}^{1|1}) & \longrightarrow 0 & & \end{array} \tag{6.17}$$

Here, the map $\Theta := -i\Theta(\iota_{\partial_\theta})$ introduced in sec 2, acts as the parity changing functor Π, and as such it is an *isomorphism up to the parity.* Note that the first map $\Theta : \mathcal{B}er(\mathbb{P}^{1|1}) \hookrightarrow \Pi\mathcal{B}er(\mathbb{P}^{1|1}) \oplus \mathcal{O}_{\mathbb{P}^{1|1}}$ is an immersion (again, up to parity) instead, acting as $s \mapsto (\pi s, 0)$. In this way, using the geometric language developed,

$$\begin{array}{c} \Theta \equiv \Pi : \Omega^{k;1} \equiv \mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} \longrightarrow \Omega^{k-1;0} \equiv \Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{1-k}\Pi\mathcal{T}_{\mathbb{P}^{1|1}} \\ s \longmapsto \pi s \end{array} \quad (6.18)$$

for $k < 1$. Explicitly, using the bases chosen above, one finds:

$$\begin{aligned} \mathcal{D}[dz|d\theta] \otimes \pi\partial_\theta^{\odot -k} &\xrightarrow{\Theta} \frac{1}{\sqrt{k}}(\pi\mathcal{D}[dz|d\theta]) \otimes (\pi\partial_\theta^{\odot -k}) \\ \mathcal{D}[dz|d\theta] \otimes (\pi\partial_\theta^{\odot 1-k} \odot \pi\partial_z) &\xrightarrow{\Theta} \sqrt{k}(\pi\mathcal{D}[dz|d\theta]) \otimes (\pi\partial_\theta^{\odot 1-k} \odot \pi\partial_z). \end{aligned} \quad (6.19)$$

The above discussion leads to the following picture: one sees that allowing for inverse superforms, that is expressions of the kind $\frac{1}{d\theta}$, corresponds to enlarge the usual complex of superforms on the supermanifold - in our case $\mathbb{P}^{1|1}$ - by a “copy” of the complex of integral forms having opposite parity, shifted to the left by a number of steps equal to the odd dimension of the supermanifold. Also, observe that the exterior differentials defined for the integral forms and the superforms as in the previous section can be used to make this sequence of sheaves into an actual complex, so that with abuse of notation, we write:

$$\cdots \longrightarrow \Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes \Pi\mathcal{T}_{\mathbb{P}^{1|1}} \xrightarrow{d} \Pi\mathcal{B}er(\mathbb{P}^{1|1}) \oplus \mathcal{O}_{\mathbb{P}^{1|1}} \xrightarrow{d} \Pi\mathcal{T}^*_{\mathbb{P}^{1|1}} \xrightarrow{d} Sym^2\Pi\mathcal{T}^*_{\mathbb{P}^{1|1}} \longrightarrow \cdots \quad (6.20)$$

Notice that, even if the transition functions will be more complicated, everything we have said above can be repeated almost identically also in the case of a generic supermanifold $\mathcal{M}$ of dimension $n|1$. Keeping on with the case of projective spaces, considering $\mathbb{P}^{n|1}$, one has that for the integral forms

$$\Omega^{k;1}_{\mathbb{P}^{n|1}} \longleftrightarrow \mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}} \quad (6.21)$$

for $k \leq n$. And likewise for the inverse superforms,

$$\Omega^{-k;0}_{\mathbb{P}^{n|1}} \longleftrightarrow \Pi\mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-1+k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}} \quad (6.22)$$

for $k > 0$.
Also, we have an odd morphism $\Theta : \Omega^{k;1}_{\mathbb{P}^{n|1}} \to \Omega^{k-1;0}_{\mathbb{P}^{n|1}}$ that for $k \leq 0$ is an *isomorphism up to the parity.* In the sheaf-theoretic formalism it reads

$$\begin{array}{c} \Theta_{k\geq 1} \equiv \Pi : \mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}} \longrightarrow \Pi\mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}} \\ s \longmapsto \pi s \end{array} \quad (6.23)$$

for $k \leq 0$ and it is indeed just a parity-inversion.
Differently, the morphism $\Theta : \Omega^{k;1}_{\mathbb{P}^{n|1}} \to \Omega^{k-1;0}_{\mathbb{P}^{n|1}}$ for $0 < k \leq n$ is just an *injective* morphism, as made clear by the sheaf-theoretic formalism, indeed:

$$\begin{array}{c} \Theta_{k>0} : \mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}} \longrightarrow \left(\Pi\mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}}\right) \oplus Sym^{k-1}\Pi\mathcal{T}^*_{\mathbb{P}^{n|1}} \\ s \longmapsto (\pi s, 0) \end{array} \quad (6.24)$$

This gives the following realization for picture number $p = 0$ of this extended de Rham complex, where the superforms have been supplemented by the inverse forms as well:

$$\Omega^{k;0}_{\mathbb{P}^{n|1}} \cong \left\{ \begin{array}{ll} Sym^{k-1}\Pi\mathcal{T}^*_{\mathbb{P}^{n|1}} & k > n \\ \left(\Pi\mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}}\right) \oplus Sym^{k-1}\Pi\mathcal{T}^*_{\mathbb{P}^{n|1}} & 0 < k \leq n \\ \Pi\mathcal{B}er(\mathbb{P}^{n|1}) \otimes Sym^{n-k}\Pi\mathcal{T}_{\mathbb{P}^{n|1}} & k \leq 0. \end{array} \right. \tag{6.25}$$

Getting back to the example of $\mathbb{P}^{1|1}$, we have that the sheaf-theoretic representation of the Large Hilbert Space related to the supermanifold $\mathbb{P}^{1|1}$, we denote it by $\mathcal{LHS}_{\mathbb{P}^{1|1}}$, is spanned by

$$\mathcal{LHS}_{\mathbb{P}^{1|1}} = \left\{ \begin{array}{ll} Sym^{k}\Pi\mathcal{T}^*_{\mathbb{P}^{1|1}} & k > 0 \\ \left(\mathcal{O}_{\mathbb{P}^{1|1}} \oplus \Pi\mathcal{B}er(\mathbb{P}^{1|1})\right) \oplus \mathcal{B}er(\mathbb{P}^{1|1}) & k = 0 \\ \left(\Pi\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{|k|}\Pi\mathcal{T}_{\mathbb{P}^{1|1}}\right) \oplus \left(\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^{|k|}\Pi\mathcal{T}_{\mathbb{P}^{1|1}}\right) & k < 0. \end{array} \right. \tag{6.26}$$

In agreement with what have been said in the section 2, we stress the "duplication" arising in this enlarged context: clearly, besides the case $k > 0$ where only usual positive-degree forms appears, the other cases displays all of the elements belonging to the superforms complex - enlarged by the inverse superforms - together with all of the elements belonging to the integral forms complex. The Large Hilbert Space $\mathcal{LHS}_{\mathbb{P}^{1|1}}$ therefore is spanned by copies of dentical sheaves, $\Omega^{k;0}_{\mathbb{P}^{1|1}} \oplus \Omega^{k+1;1}_{\mathbb{P}^{1|1}} \cong \Omega^{k;0}_{\mathbb{P}^{1|1}} \oplus \Pi\Omega^{k;0}_{\mathbb{P}^{1|1}}$, having opposite parity in the case $k < 0$. The case $k = 0$ is somehow "critical", as in the extended superforms complex the structure sheaf is supplemented by the Berezinian sheaf coming from the integral form complex lifted via $\Theta \equiv \Pi$, as shown in (6.15), and in the case $k > 0$ there are the usual superforms only, because there are no integral forms to be lifted.

Actually, before we conclude this section, it is fair to say that in String Field Theory the Large Hilbert Space related to a certain supermanifold is structured - better than just as sheaf of modules as above -, as a sheaf of algebras, with a formal notion of multiplication between superforms on the one hand and integral forms on the other hand. This, in turn, is to be viewed as a structure inherited by the extended superforms complex $\Omega^{k;0}_{\mathcal{M}}$, for $k \in \mathbb{Z}$, that is given as well a formal notion of multiplication between superforms, mimicking the exterior (or better, supersymmetric) product $\Omega^{k_1;0}_{\mathcal{M}} \wedge \Omega^{k_2;0}_{\mathcal{M}} \to \Omega^{k_1+k_2;0}_{\mathcal{M}}$, so that on the local generators, one formally puts

$$d\theta^{k_1} \wedge \frac{1}{d\theta^{k_2}} = d\theta^{k_1-k_2}, \qquad d\theta^{k_1} \wedge \frac{dz}{d\theta^{k_2}} = dzd\theta^{k_1-k_2},$$
$$dzd\theta^{k_1} \wedge \frac{1}{d\theta^{k_2}} = dzd\theta^{k_1-k_2}, \qquad dzd\theta^{k_1} \wedge \frac{dz}{d\theta^{k_2}} = 0,$$

where, clearly, $d\theta^{k_1-k_2} = 1/d\theta^{|k_1-k_2|}$ if $k_1 < k_2$. Recovering these relations and endowing the extended superforms complex with a honest algebra structure making the formal relations above rigorous, is not straightforward as one might expect given the modules description provided above, and we leave this to a future paper.

6.1. **Large Hilbert Space and Čech Cohomology.** It is not hard to provide the Čech cohomology of the Large Hilbert Space in the example of $\mathbb{P}^{1|1}$ we have dealt with so far. Indeed, by super Serre duality (see the last section of the present paper for an explanation), it is fully determined by the Čech cohomology of the case $k \geq 0$ in the (6.26) only - which in turn amount to compute the usual Čech cohomology of forms on a supermanifold.
We will do this in two ways. First, we treat the explicit example of $\mathbb{P}^{1|1}$, exploiting two facts: $\mathbb{P}^{1|1}$ is a projected supermanifold, and its reduced manifold is the Riemann sphere. Indeed, since $\mathbb{P}^{1|1}$ is a projected supermanifold, every locally-free sheaf of $\mathcal{O}_{\mathbb{P}^{1|1}}$-modules $\mathcal{E}_{\mathcal{O}_{\mathbb{P}^{1|1}}}$, such as

$Sym^k \Pi \mathcal{T}_{\mathbb{P}^{1|1}}$, is also a locally-free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules $\mathcal{E}_{\mathcal{O}_{\mathbb{P}^1}}$, and therefore, by virtue of the Grothendieck Theorem for locally-free sheaves on $\mathbb{P}^1$, it splits into a sum of invertible sheaves of the kind $\mathcal{O}_{\mathbb{P}^1}(k)$ for $k \in \mathbb{Z}$, see [32], that is,

$$\mathcal{E}_{\mathcal{O}_{\mathbb{P}^1}} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(k_i), \qquad k_i \in \mathbb{Z} \tag{6.27}$$

where we have forget about the parity and where $n = p + q$ is the rank of $\mathcal{E}_{\mathcal{O}_{\mathbb{P}^1}}$ if $\mathcal{E}_{\mathcal{O}_{\mathbb{P}^{1|1}}}$ is of rank $p|q$. In particular, we have that $Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}$ is locally freely-generated over $\mathcal{O}_{\mathbb{P}^1}$ by $\{d\theta^{\odot k}, \theta d\theta^{\odot k-1} \odot dz | dz \odot d\theta^{\odot k-1}, \theta d\theta^{\odot k}\}$, so that the matrix of the transition functions can be written using the usual rules as

$$[M(Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}})] = \left(\begin{array}{cc|cc} 1/z^k & -1/z^{k+1} & 0 & 0 \\ 0 & -1/z^{k+2} & 0 & 0 \\ \hline 0 & 0 & -1/k^{k+1} & 0 \\ 0 & 0 & 0 & -1/z^{k+1} \end{array} \right). \tag{6.28}$$

By simple (allowed) rows and columns operations this matrix can be brought to diagonal form

$$[M(Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}})] = \left(\begin{array}{cc|cc} 1/z^{k+1} & 0 & 0 & 0 \\ 0 & -1/z^{k+1} & 0 & 0 \\ \hline 0 & 0 & -1/k^{k+1} & 0 \\ 0 & 0 & 0 & -1/z^{k+1} \end{array} \right), \tag{6.29}$$

so that one can read from this expression the factorization into invertible sheaves over $\mathbb{P}^1$ of the sheaf $Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}$ for $k \geq 1$:

$$Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}} \cong \mathcal{O}_{\mathbb{P}^1}(-k-1)^{\oplus 2} \oplus \Pi \mathcal{O}_{\mathbb{P}^1}(-k-1)^{\oplus 2}. \tag{6.30}$$

The cohomology is easily computed:

$$H^0(Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}) \cong 0, \qquad H^1(Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}) \cong \mathbb{C}^{2k|2k} \tag{6.31}$$

where again $k \geq 1$. By super Serre duality, one gets that for $k \geq 1$

$$H^0(\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^k \Pi \mathcal{T}_{\mathbb{P}^{1|1}})^* \cong \mathbb{C}^{2k|2k}, \qquad H^1(\mathcal{B}er(\mathbb{P}^{1|1}) \otimes Sym^k \Pi \mathcal{T}_{\mathbb{P}^{1|1}}) \cong 0. \tag{6.32}$$

Also, $H^0(\mathcal{O}_{\mathbb{P}^{1|1}}) \cong \mathbb{C}^{1|0} \cong H^1(\mathcal{B}er(\mathbb{P}^{1|1}))$ and $H^1(\mathcal{O}_{\mathbb{P}^{1|1}}) \cong 0 \cong H^0(\mathcal{B}er(\mathbb{P}^{1|1}))$, so that one can see the cohomology of the extended de Rham complex and, in turns, of the Large Hilbert Space:

$$H^0(\mathcal{LHS}_{\mathbb{P}^{1|1}}) \cong \begin{cases} 0 & k > 0 \\ \mathbb{C}^{1|0} \oplus 0 & k = 0 \\ \mathbb{C}^{2|k|\,|\,2|k|} \oplus \mathbb{C}^{2|k|\,|\,2|k|} & k < 0, \end{cases} \tag{6.33}$$

$$H^1(\mathcal{LHS}_{\mathbb{P}^{1|1}}) \cong \begin{cases} \mathbb{C}^{2k\,|\,2k} & k > 0 \\ 0 \oplus \mathbb{C}^{0|1} & k = 0 \\ 0 & k < 0. \end{cases} \tag{6.34}$$

The second methods we introduce is more general and holds true for any projective superspace of the kind $\mathbb{P}^{n|m}$. Indeed, let us start from the super analog of the (dual of the) Euler exact sequence for the cotangent sheaf [18], [20]:

$$0 \longrightarrow \mathcal{T}^*_{\mathbb{P}^{n|m}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus n+1|m} \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}} \longrightarrow 0. \tag{6.35}$$

Its parity inverted version - which is the one we are interested into - reads

$$0 \longrightarrow \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus m|n+1} \longrightarrow \Pi \mathcal{O}_{\mathbb{P}^{n|m}} \longrightarrow 0. \tag{6.36}$$

Now, since we are interested into the cohomology of $Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}}$, we have to consider its k-symmetric power. We observe that since $\Pi \mathcal{O}_{\mathbb{P}^{n|m}}$ is of rank $0|1$, we will have that $Sym^k \Pi \mathcal{O}_{\mathbb{P}^{n|m}} \cong 0$ if $k \geq 2$ and that, by definition of exact sequence of sheaves, locally

$$\mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus m|n+1} \overset{loc}{\cong} \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \oplus \Pi \mathcal{O}_{\mathbb{P}^{n|m}}. \tag{6.37}$$

It follows that, taking the k-symmetric power, one gets

$$\begin{aligned}(Sym^k \mathcal{O}_{\mathbb{P}^{n|m}})^{\oplus m|n+1}(-k) &\overset{loc}{\cong} Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \oplus Sym^{k-1} \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \otimes \Pi \mathcal{O}_{\mathbb{P}^{n|m}} \\ &\cong Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \oplus \Pi Sym^{k-1} \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}},\end{aligned} \tag{6.38}$$

since $Sym^k \Pi \mathcal{O}_{\mathbb{P}^{n|m}} \cong 0$ for any $k \geq 2$, as observed above. This implies that the k-symmetric power of the exact sequence (6.35) reads

$$0 \longrightarrow Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \longrightarrow (Sym^k \mathcal{O}^{\oplus m|n+1}_{\mathbb{P}^{n|m}})(-k) \longrightarrow \Pi Sym^{k-1} \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}} \longrightarrow 0, \tag{6.39}$$

which tells that the cohomology of the sheaf $Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{n|m}}$ can be computed recursively. Notice incidentally that the short exact sequence (6.39) is actually completely general, it is deduced from completely general considerations, and it can be used to compute the sheaf cohomology of forms for any projective supermanifolds by restriction, once the embedding $\iota : \mathcal{M} \hookrightarrow \mathbb{P}^{n|m}$ is given.

Let us check that the result for $\mathbb{P}^{1|1}$ matches with what we have found above by means of Grothendieck Theorem. Noticing that, in general, for $\mathbb{P}^{1|1}$ we have that $(Sym^k \mathcal{O}^{\oplus 1|2}_{\mathbb{P}^{1|1}})(-k) \cong \mathcal{O}_{\mathbb{P}^{1|1}}(-k)^{\oplus 2|2}$, the short exact sequence (6.39) reads

$$0 \longrightarrow Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{1|1}}(-k)^{\oplus 2|2} \longrightarrow \Pi Sym^{k-1} \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}} \longrightarrow 0, \tag{6.40}$$

for $k \geq 1$. One finds that the only non trivial terms in the long exact cohomology sequence are those in the following short exact sequence of vector superspaces:

$$0 \longrightarrow H^1(Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^{1|1}}(-k)) \otimes \mathbb{C}^{\oplus 2|2} \longrightarrow H^1(\Pi Sym^{k-1} \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}) \longrightarrow 0. \tag{6.41}$$

It is very easy to see that

$$H^1(\mathcal{O}_{\mathbb{P}^{1|1}}(-k)) \otimes \mathbb{C}^{\oplus 2|2} \cong \mathbb{C}^{\oplus k-1|k} \otimes \mathbb{C}^{2|2} \cong \mathbb{C}^{\oplus 4k-2|4k-2}, \tag{6.42}$$

thus by recursion one sees that $H^1(\Pi Sym^{k-1} \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}) \cong \mathbb{C}^{\oplus 2k-2|2k-2}$, and in turns

$$H^1(Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^{1|1}}) \cong \mathbb{C}^{\oplus 2k|2k}, \tag{6.43}$$

just as above.

6.2. **Large Hilbert Space and Calabi-Yau Supermanifolds.** In this brief subsection we keep on looking at the extended forms complexes and its related Large Hilbert Space, by taking on a slightly different-flavored example, that of Calabi-Yau supermanifolds, *i.e.* supermanifolds having trivial Berezinian sheaf.

In doing so we will deal with possibly the easiest example of Calabi-Yau supermanifold in genus 0, which is given by the so-called Π-projective line $\mathbb{P}^1_\Pi$. As recently explained in [16] in much greater generality for a generic Π-projective space $\mathbb{P}^n_\Pi$, the Π-projective line can be looked at as the classifying space of the $1|1$-dimensional Π-symmetric sub superspaces of $\mathbb{C}^{2|2}$, that is all those sub superspaces S that are stable under the action of a morphism $p_\Pi : S \to \Pi S$, with $p_\Pi^2 = id$, which is a representation of the parity changing functor in the category of vector superspaces. Clearly, given a vector superspace $\mathbb{C}^{n|n} = \mathbb{C}^n \oplus \Pi \mathbb{C}^n$ we can choose a basis of even elements as follows $\mathbb{C}^n = \mathrm{Span}\{e_1, \ldots, e_n\}$. Starting from these elements, we can obtain a

basis for the whole $\mathbb{C}^{n|n}$ by putting $\mathbb{C}^{n|n} = \mathrm{Span}\{e_1, \ldots, e_n \,|\, p_\Pi e_1, \ldots, p_\Pi e_n\}$. Here, the action of $p_\Pi : \mathbb{C}^{n|n} \to \Pi\mathbb{C}^{n|n} \cong \mathbb{C}^{n|n}$ exchanges the generators of $\mathbb{C}^n$ with those of $\Pi\mathbb{C}^n$. This picture has a very simple consequence: if we are given a vector superspace $V^{n|n}$ together with a basis $\{e_1, \ldots, e_n \,|\, p_\Pi e_1, \ldots, p_\Pi e_n\}$, then a sub vector superspace of $V^{n|n}$ is Π-symmetric if and only if for every element $v = \sum_{i=1}^n z^i e_i + \theta^i p_\Pi e_i$ it also contains $v_\Pi = \sum_{i=1}^n (-\theta^i e_i + z^i p_\Pi e_i)$.
This last point of view is useful to realize $\mathbb{P}^1_\Pi$ as a closed supermanifold inside a super Grassmannian, namely $\mathbb{G}(1|1; \mathbb{C}^{2|2})$, see [13] or [14]: this is covered by two affine superspaces, each isomorphic to $\mathbb{C}^{1|1}$, having coordinates in the *super big-cells* notation given by

$$\mathcal{Z}_{\mathcal{U}_0} := \left(\begin{array}{cc||cc} 1 & z_0 & 0 & \theta_0 \\ \hline 0 & -\theta_0 & 1 & z_0 \end{array} \right) \qquad \mathcal{Z}_{\mathcal{U}_1} := \left(\begin{array}{cc||cc} z_1 & 1 & \theta_1 & 0 \\ \hline -\theta_1 & 0 & z_1 & 1 \end{array} \right). \tag{6.44}$$

The transition functions in the intersections of the charts can be found by (allowed) rows and column operation, yielding [16]:

$$\left(\begin{array}{cc||cc} 1 & z_0 & 0 & \theta_0 \\ \hline 0 & -\theta_0 & 1 & z_0 \end{array} \right) \xrightarrow{R_0/z_0, R_1/z_0} \left(\begin{array}{cc||cc} 1/z_0 & 1 & 0 & \theta_0/z_0 \\ \hline 0 & -\theta_0/z_0 & 1/z_0 & 1 \end{array} \right)$$

$$\left(\begin{array}{cc||cc} 1/z_0 & 1 & 0 & \theta_0/z_0 \\ \hline 0 & -\theta_0/z_0 & 1/z_0 & 1 \end{array} \right) \xrightarrow{R_0 - \theta_0/z_0 R_1} \left(\begin{array}{cc||cc} 1/z_0 & 1 & -\theta_0/z_0^2 & 0 \\ \hline 0 & -\theta_0/z_0 & 1/z_0 & 1 \end{array} \right)$$

$$\left(\begin{array}{cc||cc} 1/z_0 & 1 & -\theta_0/z_0^2 & 0 \\ \hline 0 & -\theta_0/z_0 & 1/z_0 & 1 \end{array} \right) \xrightarrow{R_1 + \theta_0/z_0 R_0} \left(\begin{array}{cc||cc} 1/z_0 & 1 & -\theta_0/z_0^2 & 0 \\ \hline \theta_0/z_0^2 & 0 & 1/z_0 & 1 \end{array} \right).$$

The transition functions characterizing the structure sheaf $\mathcal{O}_{\mathbb{P}^1_\Pi}$ of the Π-projective line can be read from the above expression and one gets

$$z_1 = \frac{1}{z_0}, \qquad \xi_1 = -\frac{\theta_1}{z_0^2}. \tag{6.45}$$

It follows that the Π-projective line $\mathbb{P}^{1|1}_\Pi$ is the $1|1$-dimensional supermanifold characterised by the pair $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \cong \Omega^1_{\mathbb{P}^1})$, and it is easy to get that the $\mathcal{B}er(\mathbb{P}^1_\Pi) \cong \mathcal{O}_{\mathbb{P}^1_\Pi}$, that is $\mathbb{P}^1_\Pi$ is a Calabi-Yau supermanifold. This has a particular nice consequence, that is the sheaves of integral forms simplifies to:

$$\Omega^{k;1}_{\mathbb{P}^1_\Pi} = Sym^{k-1}\Pi\mathcal{T}_{\mathbb{P}^1_\Pi}. \tag{6.46}$$

Notice that this is a peculiarity which is true - up to parity - for all of the Calabi-Yau supermanifold: for example, the well-known *super twistor-space* $\mathbb{P}^{3|4} (= \mathbb{CP}^{3|4})$, which is a Calabi-Yau supermanifold because $\mathcal{B}er(\mathbb{P}^{3|4}) \cong \Pi\mathcal{O}_{\mathbb{P}^{3|4}}$, has sheaves of integral forms given by $\Omega^{k;4}_{\mathbb{P}^{3|4}} = \Pi Sym^{k-3}\Pi\mathcal{T}_{\mathbb{P}^{3|4}}$.
Looking at the generators of the sheaves of integral forms for the Π-projective line, one gets the following correspondence between generators for $\Omega^{k;1}_{\mathbb{P}^1_\Pi} = Sym^k \Pi\mathcal{T}_{\mathbb{P}^1_\Pi}$:

$$\begin{aligned} &\pi\partial_\theta^{\odot 1-k} \longleftrightarrow dz\delta^{(1-k)}(d\theta), \\ &\pi\partial_z \odot \pi\partial_\theta^{\odot -k} \longleftrightarrow \delta^{(-k)}(d\theta), \end{aligned} \tag{6.47}$$

for $k \leq 0$, together with $\mathcal{B}er(\mathbb{P}^1_\Pi) \ni \mathcal{D}[dz|d\theta] = s \in \mathcal{O}_{\mathbb{P}^1_\Pi}$, as $\mathbb{P}^1_\Pi$ is a Calabi-Yau supermanifold. Likewise, the same arguments as above apply to recover the inverse forms characterizing the extended form complex:

$$\Omega^{k;0}_{\mathbb{P}^1_\Pi} \cong \begin{cases} Sym^k \Pi\mathcal{T}^*_{\mathbb{P}^1_\Pi} & k > 0 \\ \mathcal{O}_{\mathbb{P}^1_\Pi} \oplus \Pi\mathcal{O}_{\mathbb{P}^1_\Pi} & k = 0 \\ \Pi Sym^{|k|}\Pi\mathcal{T}_{\mathbb{P}^1_\Pi} & k < 0. \end{cases} \tag{6.48}$$

Switching to a more comfortable notation, notice in particular that for the sheaves $\Omega^{-k;0}_{\mathbb{P}^1_\Pi} = \Pi Sym^k \Pi \mathcal{T}_{\mathbb{P}^1_\Pi}$ when $k>0$ the correspondence between the generators is

$$\pi(\pi\partial_z \odot \pi\partial_\theta^{\odot k-1}) \longleftrightarrow \frac{1}{d\theta^k}, \tag{6.49}$$

$$\pi(\pi\partial_\theta^{\odot k}) \longleftrightarrow \frac{dz}{d\theta^{k+1}}. \tag{6.50}$$

Likewise, the Large Hilbert Space of the Calabi-Yau supermanifold $\mathbb{P}^1_\Pi$ is given by

$$\mathcal{LHS}_{\mathbb{P}^1_\Pi} = \begin{cases} Sym^k \Pi \mathcal{T}^*_{\mathbb{P}^1_\Pi} & k>0 \\ \left(\mathcal{O}_{\mathbb{P}^1_\Pi} \oplus \Pi \mathcal{O}_{\mathbb{P}^1_\Pi}\right) \oplus \mathcal{O}_{\mathbb{P}^1_\Pi} & k=0 \\ \left(\Pi Sym^{|k|}\Pi\mathcal{T}_{\mathbb{P}^1_\Pi}\right) \oplus \left(Sym^{|k|}\Pi\mathcal{T}_{\mathbb{P}^1_\Pi}\right) & k<0. \end{cases} \tag{6.51}$$

It is again easy to compute the Čech cohomology of the Large Hilbert Space using Grothendieck Theorem. Looking at the splitting as locally-free $\mathcal{O}_{\mathbb{P}^1}$-modules of the sheaf $Sym^k\Pi\mathcal{T}^*_{\mathbb{P}^1_\Pi}$ over $\mathbb{P}^1_\Pi$ for $k\geq 1$, one finds the following matrix of transition functions

$$[M(Sym^k\Pi\mathcal{T}^*_{\mathbb{P}^{1|1}})] = \left(\begin{array}{cc|cc} (-1)^k/z^{2k} & (-1)^{k+1}/z^{2k+1} & 0 & 0 \\ 0 & (-1)^{k+1}/z^{2k+2} & 0 & 0 \\ \hline 0 & 0 & (-1)^k/k^{2k+1} & 0 \\ 0 & 0 & 0 & (-1)^{k+1}/z^{2k+2} \end{array}\right), \tag{6.52}$$

that can be diagonalized by rows and columns operations to

$$[M(Sym^k\Pi\mathcal{T}^*_{\mathbb{P}^{1|1}})] = \left(\begin{array}{cc|cc} (-1)^k/z^{2k+1} & 0 & 0 & 0 \\ 0 & (-1)^{k+1}/z^{2k+1} & 0 & 0 \\ \hline 0 & 0 & (-1)^k/k^{2k} & 0 \\ 0 & 0 & 0 & (-1)^{k+1}/z^{2k+2} \end{array}\right), \tag{6.53}$$

so that one has the following decomposition

$$Sym^k\Pi\mathcal{T}^*_{\mathbb{P}^1_\Pi} \cong \mathcal{O}_{\mathbb{P}^1}(-2k-1)^{\oplus 2} \oplus \Pi\left(\mathcal{O}_{\mathbb{P}^1}(-2k)\oplus\mathcal{O}_{\mathbb{P}^1}(-2k-2)\right). \tag{6.54}$$

The cohomology is therefore given by

$$H^0(Sym^k\Pi\mathcal{T}^*_{\mathbb{P}^1_\Pi}) \cong 0, \qquad H^1(Sym^k\Pi\mathcal{T}^*_{\mathbb{P}^1_\Pi}) \cong \mathbb{C}^{4k|4k}, \tag{6.55}$$

that in turn yields the following cohomology for the Large Hilbert Space:

$$H^0(\mathcal{LHS}_{\mathbb{P}^1_\Pi}) = \begin{cases} 0 & k>0 \\ \left(\mathbb{C}^{1|0}\oplus\mathbb{C}^{0|1}\right)\oplus\mathbb{C}^{1|0} & k=0 \\ \mathbb{C}^{4k|4k}\oplus\mathbb{C}^{4k|4k} & k<0, \end{cases} \tag{6.56}$$

$$H^1(\mathcal{LHS}_{\mathbb{P}^1_\Pi}) = \begin{cases} \mathbb{C}^{4k|4k} & k>0 \\ \left(\mathbb{C}^{0|1}\oplus\mathbb{C}^{1|0}\right)\oplus\mathbb{C}^{0|1} & k=0 \\ 0 & k<0. \end{cases} \tag{6.57}$$

7. Superforms and Pseudo-forms for Higher Odd Dimensions

Let us now look at what happens in the case one deals with a supermanifold $\mathcal{M}$ having odd dimension greater than 1. To keep the discussion as concrete as possible we keep on using the example of $\mathbb{P}^{1|2}$. We have already seen in the expression (6.2) at the beginning of the previous section, that as soon as one allows superforms of negative degree, the sheaves $\Omega^{k;0}_{\mathbb{P}^{1|2}}$ for $k\in\mathbb{Z}$, making up the "extended" de Rham complex are no longer locally-free sheaves of a certain

(finite) rank, but infinitely generated quasi-coherent sheaves instead.
Including also *pseudo-forms* of middle dimensional picture number - which are similarly arranged into infinitely generated quasi-coherent sheaves $\Omega^{k;1}_{\mathbb{P}^{1|2}}$, for $k \in \mathbb{Z}$, as shown in (5.40) -, the whole picture goes as follows:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \Omega^{-2;0}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{-1;0}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{0;0}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{1;0}_{\mathbb{P}^{1|2}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \Omega^{-2;1}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{-1;1}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{0;1}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{1;1}_{\mathbb{P}^{1|2}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \Omega^{-2;2}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{-1;2}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{0;2}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{1;2}_{\mathbb{P}^{1|2}} & \longrightarrow & 0.
\end{array}
\tag{7.1}
$$

We recall that the bottom line, corresponding to integral forms, is a complex of locally-free sheaves of $\mathcal{O}_{\mathbb{P}^{1|2}}$-modules, and in particular, one has, for $k \in \mathbb{Z}$, $k \leq 1$,

$$
\Omega^{k;2}_{\mathbb{P}^{1|2}} := \mathcal{B}er(\mathbb{P}^{1|2}) \otimes Sym^{|k-1|}\Pi\mathcal{T}_{\mathbb{P}^{1|2}} \cong \Pi Sym^{|k-1|}\Pi\mathcal{T}_{\mathbb{P}^{1|2}}, \tag{7.2}
$$

since $\mathbb{P}^{1|2}$ is a Calabi-Yau supermanifold, in that $\mathcal{B}er(\mathbb{P}^{1|2}) \cong \Pi\mathcal{O}_{\mathbb{P}^{1|2}}$.
As explained in the previous section, one has that Θ_i for $i = 1, 2$ is a sheaf morphism as follows

$$
\Theta_i := \Theta(\iota_{\theta_i}) : \Omega^{k;p}_{\mathbb{P}^{1|2}} \longrightarrow \Omega^{k-1;p-1}_{\mathbb{P}^{1|2}}. \tag{7.3}
$$

Using the formal expressions involving the delta's and the inverse of superforms, they act in a certain chart as $\delta^{(i)}(d\theta_i) \overset{\Theta_i}{\longmapsto} 1/d\theta_i^{i+1}$, so that for example, starting from the bottom line of integral forms and acting with Θ_1 one has

$$
\begin{aligned}
\Theta_1 : \Omega^{1;2}_{\mathbb{P}^{1|2}} &\longrightarrow \Omega^{0;1}_{\mathbb{P}^{1|2}} \\
dz\delta^{(0)}(d\theta_1)\delta^{(0)}(d\theta_2) &\longmapsto \frac{dz}{d\theta_1}\delta^{(0)}(d\theta_2)
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
\Theta_1 : \Omega^{0;2}_{\mathbb{P}^{1|2}} &\longrightarrow \Omega^{-1;1}_{\mathbb{P}^{1|2}} \\
\left(\begin{array}{c} \delta^{(0)}(d\theta_1)\delta^{(0)}(d\theta_2) \\ dz\delta^{(1)}(d\theta_1)\delta^{(0)}(d\theta_2) \\ dz\delta^{(1)}(d\theta_1)\delta^{(0)}(d\theta_2) \end{array}\right) &\longmapsto \left(\begin{array}{c} \frac{\delta^{(0)}(d\theta_2)}{d\theta_1} \\ \frac{dz\delta^{(0)}(d\theta_2)}{d\theta_1^2} \\ \frac{dz\delta^{(1)}(d\theta_2)}{d\theta_1} \end{array}\right)
\end{aligned}
\tag{7.5}
$$

Studying carefully the transition functions of the above expressions one can see that, once again, as in the case of $\mathbb{P}^{1|1}$, the morphisms Θ_i are nothing but a change of parity, that is $\Theta_i \equiv \Pi$.
Less formally, recalling that $\Gamma\mathcal{B}er(\mathbb{P}^{1|2}) \ni \mathcal{D}[dz|d\theta_1, d\theta_2] \equiv dz\delta^{(0)}(d\theta_1)\delta^{(0)}(d\theta_2)$ and that $\Omega^{0;2}_{\mathbb{P}^{1|2}} \cong \mathcal{T}_{\mathbb{P}^{1|2}}$, so that $\partial_z \equiv \delta^{(0)}(d\theta_1)\delta^{(0)}(d\theta_2)$ and $\partial_{\theta_1} \equiv dz\delta^{(1)}(d\theta_1)\delta^{(0)}(d\theta_2)$ and $\partial_{\theta_2} \equiv dz\delta^{(0)}(d\theta_1)\delta^{(1)}(d\theta_2)$,

one finds that

$$\Theta_1 \equiv \Pi : \mathcal{B}er(\mathbb{P}^{1|2}) \longrightarrow \Omega^{0;1}_{\mathbb{P}^{1|2}} \tag{7.6}$$
$$\mathcal{D}[dz|d\theta_1, d\theta_2] \longmapsto \pi\mathcal{D}[dz|d\theta_1, d\theta_2]$$

$$\Theta_1 \equiv \Pi : \mathcal{T}_{\mathbb{P}^{1|2}} \longrightarrow \Omega^{-1;1}_{\mathbb{P}^{1|2}} \tag{7.7}$$
$$\begin{pmatrix} \partial_z \\ \partial_{\theta_1} \\ \partial_{\theta_2} \end{pmatrix} \longmapsto \begin{pmatrix} \pi\partial_z \\ \pi\partial_{\theta_1} \\ \pi\partial_{\theta_2} \end{pmatrix},$$

and the same applies to $\Theta_2 \equiv \Pi$. Notice that codomains of the map Θ_1 are still denoted as $\Omega^{0;1}_{\mathbb{P}^{1|2}}$ and $\Omega^{-1;1}_{\mathbb{P}^{1|2}}$ respectively since they are not identified yet as known sheaf, such as the domains instead. In particular, turning back to the formal language, they are locally freely-generated by the following formal expressions for $k \in \mathbb{Z}$ fixed and $i \in \mathbb{N} \cup \{0\}$

$$\Omega^{k;1}_{\mathbb{P}^{1|2}}(U_0) = \mathcal{O}_{\mathbb{P}^{1|2}}(U_0) \cdot \Big\{ dz d\theta_1^{i-1+k} \delta^{(i)}(d\theta_2), 1 \leftrightarrow 2 \,\Big|\, d\theta_1^{i+k}\delta^{(i)}(d\theta_2), 1 \leftrightarrow 2 \Big\}, \tag{7.8}$$

so that for example the element $\pi\mathcal{D}[dz|d\theta_1, d\theta_2]$ lifted by $\Theta_1 \equiv \Pi$ from $\mathcal{B}er(\mathbb{P}^{1|2})$ to $\Omega^{0;1}_{\mathbb{P}^{1|2}}$ is given by the choice $k = 0, i = 0$ in the previous expression and corresponds to $dz\delta^{(0)}(d\theta_2)/d\theta_1$, as we have seen.

It is possible to jump from the last line of the diagram (7.1) - that of integral forms -, to the first line - that of superforms, comprising also inverse superforms -, by composing the maps $\Theta_2 \circ \Theta_1 := \Theta(\iota_{\theta_2}) \circ \Theta(\iota_{\theta_1})$, and we call the morphism resulting from this composition Θ_{max}. Formally, this morphism converts both of the delta's appearing in an integral form in two inverse superforms, that is $\delta^{(\ell_1)}(d\theta_1)\delta^{(\ell_2)}(d\theta_2) \longmapsto \frac{1}{d\theta_1^{\ell_1+1} d\theta_2^{\ell_2+1}}$. Notice, that Θ_{max} is therefore an *even* morphism, and indeed by a sheaf-theoretic approach, looking again at the transition functions by a local computations, one can see that Θ_{max} acts nothing but the composition of two parity changing functor Π, so that one has

$$\Theta_{max} := \Theta(\iota_{\theta_2}) \circ \Theta(\iota_{\theta_1}) : \Omega^{k;2}_{\mathbb{P}^{1|2}} \xrightarrow{\ \Pi\ } \Omega^{k-1;1}_{\mathbb{P}^{1|2}} \xrightarrow{\ \Pi\ } \Omega^{k-2;0}_{\mathbb{P}^{1|2}} \tag{7.9}$$
$$s \longmapsto \pi s \longmapsto s,$$

where we have denoted by s a generic section. Notice also that Θ_{max} is injective but certainly not surjective, as indeed a generic $\Omega^{k;0}_{\mathbb{P}^{1|2}}$ is infinitely-generated by formal expression of the kind

$$\Omega^{k;0}_{\mathbb{P}^{1|2}}(U_0) = \mathcal{O}_{\mathbb{P}^{1|2}}(U_0) \cdot \Big\{ d\theta_1^{\kappa_1} d\theta_2^{\kappa_2} \,\Big|\, dz d\theta_1^{\ell_1} d\theta_2^{\ell_2} \Big\}, \tag{7.10}$$

for $k \in \mathbb{Z}$ and $\kappa_1 + \kappa_2 = k$, $\ell_1 + \ell_2 = k - 1$. In light of this, Θ_{max} *injects* all of the sheaves appearing in the complex of integral forms into the extended complex of superforms, that is for $k \leq 1$ we have:

$$\Theta_{max} : \Pi Sym^{|k-1|}\Pi\mathcal{T}_{\mathbb{P}^{1|2}} \hookrightarrow \Omega^{k-2;0}_{\mathbb{P}^{1|2}}, \tag{7.11}$$

or pictorially, getting back to (7.1), we have

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Omega^{-1;0}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{0;0}_{\mathbb{P}^{1|2}} & \longrightarrow & \Omega^{1;0}_{\mathbb{P}^{1|2}} & \longrightarrow \cdots \\ & & \nwarrow^{\Theta_{max}} & & \nwarrow^{\Theta_{max}} & & \\ \cdots \longrightarrow & \Pi Sym^2 \Pi \mathcal{T}_{\mathbb{P}^{1|2}} & \longrightarrow & \mathcal{T}_{\mathbb{P}^{1|2}} & \longrightarrow & \Pi\mathcal{O}_{\mathbb{P}^{1|2}} & \longrightarrow 0. \end{array} \tag{7.12}$$

In other words, taking for example a section of the Berezinian sheaf in the formal delta's representation, using the morphisms Θ_1, Θ_2 and Θ_{max}, one moves through the following commutative diagram

$$
\begin{array}{ccccc}
 & & \dfrac{dz}{d\theta_1 d\theta_2} \in \Omega^{-1;0}_{\mathbb{P}^{1|2}} & & \\
 & \overset{\Theta_2}{\nearrow} & \uparrow \Theta_{max} & \overset{\Theta_1}{\nwarrow} & \\
\Omega^{0;1}_{\mathbb{P}^{1|2}} \ni \dfrac{dz\delta^{(0)}(d\theta_2)}{d\theta_1} & \longleftarrow & \longrightarrow & & \dfrac{dz\delta^{(0)}(d\theta_1)}{d\theta_2} \in \Omega^{0;1}_{\mathbb{P}^{1|2}} \\
 & \underset{\Theta_1}{\nwarrow} & | & \underset{\Theta_2}{\nearrow} & \\
 & & dz\delta^{(0)}(d\theta_1)\delta^{(0)}(d\theta_2) \in \Omega^{1;2}_{\mathbb{P}^{1|2}}. & &
\end{array}
\tag{7.13}
$$

As already observed, $\Theta_{max} \equiv \Pi \circ \Pi = id$, and as such it maps a section of the Berezinian sheaf $\mathcal{D}[dz|d\theta_1 d\theta_2] \equiv dz\delta^{(0)}(d\theta_1)\delta^{(0)}(d\theta_2) \in \Pi\mathcal{O}_{\mathbb{P}^{1|2}}(U_0)$ to itself, as can be seen by looking at the transformation properties of the formal expression $dz/d\theta_1 d\theta_2$. The horizontal arrows means that one has a correspondence, $\Theta_1(\mathcal{D}[dz|d\theta_1 d\theta_2]) = \pi\mathcal{D}[dz|d\theta_1 d\theta_2] = \Theta_2(\mathcal{D}[dz|d\theta_1 d\theta_2])$. Both in the sheaves $\Omega^{k;1}_{\mathbb{P}^{1|2}}$ having middle-dimensional picture, and in the extended superform sheaf $\Omega^{k;0}_{\mathbb{P}^{1|2}}$ there are an infinite number of generators that do not come from liftings of generators of the locally-free sheaves $\Omega^{k;2}_{\mathbb{P}^{1|2}}$ via Θ_i, for $i = 1, 2$. This is the case, looking for example at $\Omega^{0;1}_{\mathbb{P}^{1|2}}$, of the formal generators $\{dzd\theta_1^{i-1}\delta^{(i)}(d\theta_2)\}, 1 \leftrightarrow 2\,|d\theta_1^j\delta^{(j)}(d\theta_2), 1 \leftrightarrow 2\,\}$ for $i > 0$ and $j \geq 0$ (recall that the case $i = 0$ corresponds to the lifting of a generating section of the Berezinian sheaf $\Omega^{1;2}_{\mathbb{P}^{1|2}} = \mathcal{B}er(\mathbb{P}^{1|2})$). These in turn lifts to $\{d\theta_1^j d\theta_2^{-j-1}, 1 \leftrightarrow 2\,|\,dzd\theta_1^{i-1}d\theta_2^{-i-1}, 1 \leftrightarrow 2\}$ for $i > 0$ and $j \geq 0$ in $\Omega^{-1;0}_{\mathbb{P}^{1|2}}$. One possibility that can be put forward in order to explain the insurgence of these infinity amount of somehow "spurious" elements is to look at these as *pure gauge*. Indeed, choosing $\Theta(\iota_1)$ and $\Theta(\iota_2)$ is actually a gauge choice, as taking a contraction $\iota_i := \iota_{\partial_{\theta_i}}$ along ∂_{θ_i} corresponds to pick up a privileged direction. Indeed, it can be seen that just by rotating the element $d\theta_1$ so that $d\theta_1 \to \alpha_1 d\theta_1 + \alpha_2 d\theta_2$, one gets

$$
\frac{1}{d\theta_1} \overset{R}{\longmapsto} \frac{1}{\alpha_1 d\theta_1 + \alpha_2 d\theta_2} = \frac{1}{\alpha_1 d\theta_1 (1 + \frac{\alpha_2}{\alpha_1}\frac{d\theta_2}{d\theta_1})} = \frac{1}{\alpha_1 d\theta_1}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\alpha_2}{\alpha_1}\right)^k\left(\frac{d\theta_2}{d\theta_1}\right)^k, \tag{7.14}
$$

and all of the powers of the element $d\theta_2/d\theta_1$ appears by expanding the rotated elements.
In support of this gauge interpretation, notice that this issue is bypassed as one considers $\Theta_{max} = \Theta_1 \circ \Theta_2$ instead, jumping directly to the top line of extended superforms, indeed no choice has been made in this case, as one performs a contraction $\iota_{\partial_{\theta_1}} \circ \iota_{\partial_{\theta_2}}$ along both ∂_{θ_1} and ∂_{θ_2}, saturating all of the available directions.

In this scenario, it is worth pointing out that the relationship between the maps given by Θ_i and Θ_{max} in particular, and the map η_0 introduced in section 2. Actually, working over $\mathbb{P}^{1|2}$, we consider two maps, $\eta_i : \Omega^{k;0} \to \Omega^{k+1;2}$ for $i = 1, 2$ and $k \in \mathbb{Z}$ whose action is given in the delta's formalism by

$$
\eta_i := \begin{cases} d\theta_i^k \longmapsto \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(d\theta) & k < 0 \\ d\theta_i^k \longmapsto 0 & k \geq 0, \end{cases} \tag{7.15}
$$

which implies that each of the η_i has an infinite dimensional kernel: looking at $\Omega^{-1;0}_{\mathbb{P}^{1|2}}$ for example, one sees that all the generators of the form $\{d\theta_1^j d\theta_2^{-j-1}\}_{i\geq 0}$ gets mapped to zero by η_1, and only choosing $i = 0$ one gets a non-zero element in $\Omega^{0;1}_{\mathbb{P}^{1|2}}$ by η_2, namely $\eta_2(d\theta_2^{-1}) = \delta^{(0)}(d\theta_2)$. Interestingly, just as before for the Θ_i's, one can consider the composition $\eta_{max} := \eta_1 \circ \eta_2 : \Omega^{k;0}_{\mathbb{P}^{1|2}} \to \Omega^{k+2;2}_{\mathbb{P}^{1|2}}$, which has been denoted η_0 in section 2 where just one odd dimension was taken into account. Considering a generic sheaf $\Omega^{k;0}_{\mathbb{P}^{1|2}}$, for $k_1 + k_2 = k$ one has that

$$\eta_{max} := \begin{cases} d\theta_1^{k_1} d\theta_2^{k_2} \longmapsto C\delta^{(|k_1|-1)}(d\theta_i)\delta^{(|k_2|-1)}(d\theta_2) & k_1, k_2 < 0 \\ d\theta_1^{k_1} d\theta_2^{k_2} \longmapsto 0 & \text{else.} \end{cases} \tag{7.16}$$

C is a numerical factor depending on k_1 and k_2.

Keep on looking at the delta's formalism, this implies that Θ_{max} and η_{max} are inverse *up to a kernel*: i.e. $\Theta_{max} : \Omega^{k+2;2}_{\mathbb{P}^{1|2}} \to \Omega^{k;0}_{\mathbb{P}^{1|2}}$ is an *injective* and *not surjective* sheaf morphism and, conversely, $\eta_{max} : \Omega^{k;0}_{\mathbb{P}^{1|2}} \to \Omega^{k+2;0}_{\mathbb{P}^{1|2}}$ is a *surjective* and *not injective* sheaf morphism. Here, once again, we recall that $\Omega^{k+2;0}_{\mathbb{P}^{1|2}}$ is a locally-free sheaf of a certain (finite) rank, so that the equation

$$\Theta_{max} \circ \eta_{max} \circ \Theta_{max} = \Theta_{max} \tag{7.17}$$

holds true as claimed in [33] from string field theory considerations. Notice also that the extended superforms sheaf $\Omega^{k;0}_{\mathbb{P}^{1|2}}$ fits into a short exact sequence as follows

$$0 \longrightarrow \ker \eta_{max} \longrightarrow \Omega^{k;0}_{\mathbb{P}^{1|2}} \longrightarrow \operatorname{Im} \eta_{max} \longrightarrow 0, \tag{7.18}$$

or

$$0 \longrightarrow \operatorname{Im} \Theta_{max} \longrightarrow \Omega^{k;0}_{\mathbb{P}^{1|2}} \longrightarrow \operatorname{coker} \Theta_{max} \longrightarrow 0, \tag{7.19}$$

where we recall that one has a correspondence $\operatorname{Im} \eta_{max} \cong \Omega^{k+2;2}_{\mathbb{P}^{1|2}}$ and $\operatorname{coker} \Theta_{max} \cong \ker \eta_{max}$. Notice also that the free equation of motions for string field theory [9, 33] can be recovered in this geometric setting by the cohomological equation

$$d(\operatorname{Im} \eta_{max}) = 0. \tag{7.20}$$

This is well-defined as the operator η_{max} maps the infinite dimensional Large Hilbert Space to the Small Hilbert Space, which is spanned by a locally-free sheaves of finite rank.

8. Serre Duality and Hodge Diamond of a Supermanifold

It is known, as remembered also early on in this paper, that in general the de Rham cohomology does not yield any information about the supergeometric structure and it coincides with the ordinary de Rham cohomology of the reduced manifold $\mathcal{M}_{red}$ [12, ?]. Things are clearly different for Čech cohomology instead. In this context, a very important early result due to Penkov [34] states that, if $\mathcal{M}$ is projective, *i.e.* if there exists an embedding morphism $\varphi : \mathcal{M} \to \mathbb{P}^{k|l}$, then the *dualizing sheaf* $\omega_{\mathcal{M}}$ is given by $\mathcal{B}er(\mathcal{M})$ and, just as in the classical commutative case, for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{M}}$-modules one has the isomorphism of vector superspaces $Ext^i(\mathcal{F}, \mathcal{B}er(\mathcal{M})) \cong H^{n-i}(\mathcal{M}, \mathcal{F})^*$ for $i \geq 0$, which is the generalization of *Serre duality* to the context of supergeometry. Notice by the way - in comparison with the ordinary commutative case - that in a supergeometric context it is no longer true in general that $H^n(\mathcal{M}, \mathcal{B}er(\mathcal{M}))$ is (even or odd) one-dimensional, isomorphic to $\mathbb{C}$ or $\Pi\mathbb{C}$: just consider for example the case of a split complex supermanifold $\mathcal{M}$ of dimension $1|1$ over $\mathbb{P}^1$, whose structure sheaf is given by $\mathcal{O}_{\mathbb{P}^1} \oplus \Pi\mathcal{O}_{\mathbb{P}^1}$. Computing, one finds that $H^1(\mathcal{M}, \mathcal{B}er(\mathcal{M})) \cong \mathbb{C}^{1|1}$.

To our limited aims, anyway, we will write Serre duality in the easier form

$$H^i(\mathcal{M}, \mathcal{F}) \cong \Pi^n H^{n-i}(\mathcal{M}, \mathcal{B}er(\mathcal{M}) \otimes \mathcal{F}^*)^*, \tag{8.1}$$

for n the even dimension of $\mathcal{M}$ and $i = 0, \dots, n$, and where $\Pi^n = \Pi$ if n is odd and $\Pi^n = id$ if n is even.

In the hypothesis we are working with a projective supermanifold $\mathcal{M}$, if one takes $\mathcal{F}$ to be the sheaf of superforms $Sym^k \Pi \mathcal{T}^*_{\mathcal{M}}$, Serre duality (8.1) yields the following interesting isomorphism, relating the Čech cohomology of superforms to the Čech cohomology of integral forms

$$\underbrace{H^i(\mathcal{M}, Sym^k \Pi \mathcal{T}^*_{\mathcal{M}})}_{\text{superforms}} \cong \underbrace{H^{n-i}(\mathcal{M}, \mathcal{B}er(\mathcal{M}) \otimes Sym^k \Pi \mathcal{T}_{\mathcal{M}})^*}_{\text{integral forms}} \tag{8.2}$$

for n the even dimension of the supermanifold $\mathcal{M}$, $i = 0, \dots, n$ and $k \geq 0$, and where we have forgotten about the parity for simplicity. In other words, the Čech cohomology of integral forms is fully determined by the Čech cohomology of superforms and viceversa.

It is interesting then to have a look at the supergeometric analog of the *Hodge diamond* of a supermanifold. Working in analogy with the ordinary complex geometric case, one set the *super Hodge numbers* of $\mathcal{M}$ to be

$$h^{p,q}_s(\mathcal{M}) = \dim_{\mathbb{C}} H^q(\mathcal{M}, Sym^p \Pi \mathcal{T}^*_{\mathcal{M}}), \tag{8.3}$$

where here $\mathcal{M}$ is again a generic complex supermanifold of dimension $n|m$. There is an obvious - yet striking - difference compared to the ordinary complex geometric case: one finds that in general $h^{p,q}_s(\mathcal{M}) \neq 0$ for $p > n = \dim \mathcal{M}_{red}$ since the de Rham complex is not bounded from above. On the other hand, it keeps being true that $h^{p,q}_s(\mathcal{M}) = 0$ for $q > n$, so that, on the most general ground, the super Hodge diamond, will not really be an actual diamond, but a heavily left-weighted shape instead. Let us consider for example the case of a supermanifold $\mathcal{M}$ having even dimension equal to 3.

$$\begin{array}{ccccccccccc}
\ddots & & \ddots & & \ddots & & \ddots & & & & \\
h^{7,0}_s & & h^{6,1}_s & & h^{5,2}_s & & h^{4,3}_s & & & & \\
& h^{6,0}_s & & h^{5,1}_s & & h^{4,2}_s & & h^{3,3}_s & & & \\
& & h^{5,0}_s & & h^{4,1}_s & & h^{3,2}_s & & h^{2,3}_s & & \\
& & & h^{4,0}_s & & h^{3,1}_s & & h^{2,2}_s & & h^{1,3}_s & \\
& & & & h^{3,0}_s & & h^{2,1}_s & & h^{1,2}_s & & h^{0,3}_s \\
& & & & & h^{2,0}_s & & h^{1,1}_s & & h^{0,2}_s & \\
& & & & & & h^{1,0}_s & & h^{0,1}_s & & \\
& & & & & & & h^{0,0}_s & & &
\end{array} \tag{8.4}$$

In the picture above, we have highlighted the region where the ordinary Hodge diamond of the complex manifold $\mathcal{M}_{red}$ is concentrated. Notice anyway that one should refrain to interpret the sum of the super Hodge numbers in each row as the *Betti numbers* $b_k(\mathcal{M})$ of the supermanifold $\mathcal{M}$. The Betti numbers b_k are indeed *topological invariants* and as such they only depend on the topology of the supermanifold - which actually corresponds with the topology of its reduced complex manifold $\mathcal{M}_{red}$ -, whilst super Hodge numbers are finer invariants that heavily depend of the supergeometric structure of the supermanifold $\mathcal{M}$. That is to say, homeomorphic supermanifolds - that yields identical Betti numbers b_k - might possibly give rise to very different super Hodge numbers. It is by the way opinion of the authors that it would be very interesting to generalize Hodge theory and the technology related to the so-called Hodge decomposition theorems to supergeometry: it is actually possible that the sum of super Hodge numbers might

acquire some significance in this extended framework.
Getting back to superforms, integral forms and Serre duality relating their Čech cohomologies, it is interesting to restore the *picture number* formalism. Serre duality then reads $H^q(\mathcal{M}, \Omega^{p;0}_{\mathcal{M}}) \cong \Pi^n H^{n-q}(\mathcal{M}, \Omega^{n-p;m}_{\mathcal{M}})^*$, for $p \geq 0$ and $q = 0, \ldots, n$, which (up to the parity) is actually what expected by similarity with the ordinary case in complex algebraic geometry in terms of Hodge numbers, but now the difference is that we have to take the picture number into account: $h_s^{p,q|0}(\mathcal{M}) = h_s^{n-p,n-q|m}(\mathcal{M})$, up to parity inversion depending on the dimension. Let us represent this symmetry pictorially, for a certain supermanifolds of dimension $2|m$:

$$
\begin{array}{ccccccccc}
\ddots & & \ddots & & \ddots & & & & \\
h_s^{4,0|0} & & h_s^{3,1|0} & & h_s^{2,2|0} & & & & \\
& h_s^{3,0|0} & & h_s^{2,1|0} & & h_s^{1,2|0} & & & \\
& & h_s^{2,0|0} & & h_s^{1,1|0} & & h_s^{0,2|0} & & \\
& & & h_s^{1,0|0} & & h_s^{0,1|0} & & & \\
& & & & h_s^{0,0|0} & & & & \\
& & & h_s^{2,1|m} & & h_s^{1,2|m} & & & \\
& & h_s^{2,0|m} & & h_s^{1,1|m} & & h_s^{0,2|m} & & \\
& & & h_s^{1,0|m} & & h_s^{0,1|m} & & h_s^{-1,2|m} & \\
& & & & h_s^{0,0|m} & & h_s^{-1,1|m} & & h_s^{-2,2|m} \\
& & & & \ddots & & \ddots & & \ddots
\end{array}
\tag{8.5}
$$

Where we have used that in general $h_s^{0,0|0}(\mathcal{M}) = h_s^{n,n|m}(\mathcal{M})$, as to achieve a more symmetric picture. Serre duality is represented as in the ordinary complex geometric context by a rotation by an angle π along $h_s^{0,0|0}(\mathcal{M})$ relating the upper with the lower figure.

Let us now consider the case of compact *super Riemann surfaces* $\mathcal{S}\Sigma_g$ of a fixed genus g (see [20], or also [35] for more details).
A compact super Riemann surface $\mathcal{S}\Sigma_g$ of genus g is the data of pair $(\mathcal{M}^{1|1}, \mathcal{D})$, where $\mathcal{M}^{1|1}$ is a complex supermanifold such that $\mathcal{M}^{1|1}_{red} = \Sigma_g$, where Σ_g is a compact Riemann surface, and $\mathcal{D}$ is a locally-direct (and hence locally-free) subsheaf of $\mathcal{T}_{\mathcal{M}^{1|1}}$ of rank $0|1$ such that $\mathcal{D}^{\otimes 2} \cong \mathcal{T}_{\mathcal{M}^{1|1}} / \mathcal{D}$, via $\mathcal{d}_1 \otimes \mathcal{d}_2 \longmapsto \{\mathcal{d}_1, \mathcal{d}_2\} \operatorname{mod} \mathcal{D}$, where $\{\cdot, \cdot\}$ is the (super) Lie bracket - notice that $\mathcal{d}_1$ and $\mathcal{d}_2$ are sections of $\mathcal{D}$ and as such are *odd* vector fields, so that the super Lie bracket here is actually the anticommutator $\{\mathcal{d}_1, \mathcal{d}_2\} = \mathcal{d}_1 \mathcal{d}_2 + \mathcal{d}_2 \mathcal{d}_1$.
In what follows we will employ an equivalent characterization for super Riemann surfaces, using *theta characteristics* Θ_g on Σ_g (see for example [36]). To this end we first recall that a theta characteristic is an element in $\mathrm{Th}(\Sigma_g) := \left\{ \Theta_g \in \mathrm{Pic}^{g-1}(\Sigma_g) : \Theta_g^{\otimes 2} \cong \mathcal{K}_{\Sigma_g} \right\}$, where $\mathcal{K}_{\Sigma_g}$ is the canonical sheaf of the compact Riemann surface Σ_g : in other words a theta characteristic Θ_g is the data of a pair (Θ_g, φ_g), where Θ_g is a line bundle on Σ_g and $\varphi_g : \Theta_g^{\otimes 2} \to \mathcal{K}_{\Sigma_g}$ is an isomorphism - this is why a theta characteristic is often denoted as a "square root" of the canonical sheaf $\Theta_g = \mathcal{K}_{\Sigma_g}^{\otimes 1/2}$.
In the following we will use that giving a compact super Riemann surface of genus g as above is the same as giving a pair (Σ_g, Θ_g), where Σ_g is an ordinary compact Riemann surface of genus

g and Θ_g a theta characteristic on it (see again [20, 35]), so that one can equivalently take this as a definition and indeed we will write $\mathcal{S}\Sigma_g := (\Sigma_g, \Theta_g)$. The structure sheaf of a super Riemann surface $\mathcal{S}\Sigma_g$ is given by $\mathcal{O}_{\mathcal{S}\Sigma_g} = \mathcal{O}_{\Sigma_g} \oplus \Theta_g$, which becomes a sheaf of superalgebras with multiplication law $(f_1, \theta_1) \cdot (f_2, \theta_2) = (f_1 f_2, f_1\theta_2 + f_2\theta_1)$. Clearly, the structure sheaf of a super Riemann surface is a sheaf of $\mathcal{O}_{\Sigma_g}$-algebras: it follows that the sheaf of 1-forms $\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}$ is a (locally-free) sheaf of $\mathcal{O}_{\Sigma_g}$-modules as well. More precisely, one finds that

$$\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g} = \Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g} \otimes \mathcal{O}_{\mathcal{S}\Sigma_g} \cong \Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g} \otimes (\mathcal{O}_{\Sigma_g} \oplus \Theta_g) \cong \mathcal{K}^{\otimes 1/2}_{\Sigma_g} \oplus \mathcal{K}^{\otimes 3/2}_{\Sigma_g} \oplus \Pi(\mathcal{K}^{\oplus 2}_{\Sigma_g}). \tag{8.6}$$

This gives the splitting of the sheaf of 1-forms of a super Riemann surface in terms of sheaves of $\mathcal{O}_{\Sigma_g}$-modules. In turns, the symmetric powers $Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}$, appearing in the de Rham complex are easily computed from the above expression, and one finds that

$$Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g} \cong \left(\mathcal{K}^{\otimes\frac{k}{2}}_{\Sigma_g} \oplus \mathcal{K}^{\otimes\frac{k+2}{2}}_{\Sigma_g}\right) \oplus \Pi\left(\mathcal{K}^{\otimes\frac{k+1}{2}}_{\Sigma_g}\right)^{\oplus 2}. \tag{8.7}$$

Given this decomposition, the cohomology can be computed using *Riemann-Roch theorem* for invertible sheaves over ordinary compact Riemann surfaces,

$$h^0(\mathcal{L}_{\Sigma_g}) - h^1(\mathcal{L}_{\Sigma_g}) = \deg(\mathcal{L}_{\Sigma_g}) - g + 1, \tag{8.8}$$

where we recall that for $\mathcal{L}_{\Sigma_g} = \mathcal{K}_{\Sigma_g}$, one has $\deg(\mathcal{K}_{\Sigma_g}) = 2g-2$, and whenever $\deg(\mathcal{L}_{\Sigma_g}) > 2g-2$, one has index of speciality $h^1(\mathcal{L}_{\Sigma_g}) = 0$, so that Riemann-Roch simplifies. In particular, with reference to the above equation (8.7), if one chooses $k > 2$, all of the summands in the decomposition have vanishing index of speciality, so that in particular

$$h^{k,1}_s(\mathcal{S}\Sigma_g) = h^1(Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) = 0|0 \qquad k > 2, \tag{8.9}$$

and one has

$$h^{k,0}_s(\mathcal{S}\Sigma_g) = h^0(Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) = h^0(\mathcal{K}^{\otimes\frac{k}{2}}_{\Sigma_g}) + h^0(\mathcal{K}^{\otimes\frac{k+2}{2}}_{\Sigma_g}) \,\big|\, 2h^0(\mathcal{K}^{\otimes\frac{k+1}{2}}_{\Sigma_g}) \qquad k > 2. \tag{8.10}$$

The global sections can then be computed by Riemann-Roch and in the case $k > 2$ one has:

$$h^0(\mathcal{K}^{\otimes\frac{k}{2}}_{\Sigma_g}) = (k-1)g - k + 1, \tag{8.11}$$

$$h^0(\mathcal{K}^{\otimes\frac{k+2}{2}}_{\Sigma_g}) = (k+1)g - k - 1, \tag{8.12}$$

$$h^0(\mathcal{K}^{\otimes\frac{k+1}{2}}_{\Sigma_g}) = k(g-1), \tag{8.13}$$

so that, altogether:

$$h^{k,0}_s(\mathcal{S}\Sigma_g) = h^0(Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) = 2k(g-1) \,|\, 2k(g-1) \qquad k > 2. \tag{8.14}$$

Also, recalling that in general $h^0(\mathcal{K}_{\Sigma_g}) = g$ and $h^1(\mathcal{K}_{\Sigma_g}) = 1$, and that for genus $g \geq 2$, $h^0(\mathcal{K}_{\Sigma_g}) = 3g-3$ and $h^0(\mathcal{K}_{\Sigma_g}) = 2g-2$ count the number of the even and odd moduli, one has that that

$$h^{2,0}_s(\mathcal{S}\Sigma_{g\geq 2}) = h^0(Sym^2\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_{g\geq 2}}) = 4g-3|4g-4 \tag{8.15}$$

$$h^{2,1}_s(\mathcal{S}\Sigma_{g\geq 2}) = h^1(Sym^2\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_{g\geq 2}}) = 1|0, \tag{8.16}$$

restricted to genus $g \geq 2$.
The remaining cohomologies depends from the particular theta characteristic chosen (recall

there are 2^{2g} inequivalent such choices over a compact Riemann surface). Indeed, by Riemann-Roch one sees that $h^0(\mathcal{K}_{\Sigma_g}^{\otimes\frac{1}{2}}) = h^1(\mathcal{K}_{\Sigma_g}^{\otimes\frac{1}{2}})$, moreover this dimension is bounded by Clifford theorem on special divisors, yielding that in general

$$h^0(\mathcal{K}_{\Sigma_g}^{\otimes\frac{1}{2}}) = h^1(\mathcal{K}_{\Sigma_g}^{\otimes\frac{1}{2}}) \leq \frac{g+1}{2}. \tag{8.17}$$

We define $h^0(\mathcal{K}_{\Sigma_g}^{\otimes\frac{1}{2}}) = h^1(\mathcal{K}_{\Sigma_g}^{\otimes\frac{1}{2}}) \coloneqq \nu_{\Theta_g}$, with $\nu_{\Theta_g} \leq \frac{g+1}{2}$. One can see that

$$h_s^{1,1}(\mathcal{S}\Sigma_g) = h^1(\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) = \nu_{\Theta_g} \,|\, 2, \tag{8.18}$$

$$h_s^{1,0}(\mathcal{S}\Sigma_{g\geq 2}) = h^0(\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_{g\geq 2}}) = \nu_{\Theta_g} + 2g - 2 \,|\, 2g, \tag{8.19}$$

$$h_s^{0,1}(\mathcal{S}\Sigma_g) = h^1(\mathcal{O}_{\mathcal{S}\Sigma_g}) = g \,|\, \nu_{\Theta_g}, \tag{8.20}$$

$$h_s^{0,0}(\mathcal{S}\Sigma_g) = h^0(\mathcal{O}_{\mathcal{S}\Sigma_g}) = 1 \,|\, \nu_{\Theta_g}. \tag{8.21}$$

Once this numerology is concluded, the respective dimensions can be inserted in a pictorial representation as above, as to get the Hodge diamond for a super Riemann surface of genus ≥ 2. By the way, it is possibly more instructive to represent this Hodge diamond by rotating it $\pi/4$ clockwise, as to get a *tower* better than a diamond:

$$\begin{array}{ccc}
\vdots & & \vdots \\
6g-6|6g-6 & & 0|0 \\
| & & | \\
4g-3|4g-4 & & 1|0 \\
| & & | \\
\nu_{\Theta_g} + 2g - 2 \,|\, 2g & & \nu_{\Theta_g}|2 \\
& \diagdown \quad \diagup & \\
1|\nu_{\Theta_g} & \times & g|\nu_{\Theta_g} \\
& \diagup \quad \diagdown & \\
\nu_{\Theta_g}|2 & & \nu_{\Theta_g} + 2g - 2 \,|\, 2g \\
| & & | \\
1|0 & & 4g-3|4g-4 \\
| & & | \\
0|0 & & 6g-6|6g-6 \\
\vdots & & \vdots
\end{array} \tag{8.22}$$

Let us now look at the upper part of the above tower. In this representation, the difference between the left and right "wall" of tower is nothing but the difference between the graded dimensions of the zeroth and the first cohomology group of the sheaf involved, in this case $Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}$ for $k \geq 1$. It can be observed that this difference is a *topological invariant* - just like in the ordinary case one has the Euler characteristic of a certain vector bundle - and it does *not* depend on the particular spin structure chosen, and therefore on ν_{Θ_g}: as we have seen above, this is a consequence of the Riemann-Roch theorem. In particular, looking at the superforms, one finds

$$\chi_s(Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) \coloneqq h^0(Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) - h^1(Sym^k\Pi\mathcal{T}^*_{\mathcal{S}\Sigma_g}) = k\,(2g-2)\,|k\,(2g-2)\,, \tag{8.23}$$

for $k \geq 1, g \geq 2$. Using similar argument as above (splitting and Riemann-Roch theorem), one can get the same conclusion for an arbitrary locally-free sheaf over a super Riemann surface. Notice by the way that in the first case considered above, given by $g = 2$, all of the reduced

manifolds are *hyperelliptic* Riemann surface and the possibilities for the theta characteristics on them are easily settled. Indeed, one finds in general that $\deg(\mathcal{K}^{\otimes 1/2}_{\Sigma_{g=2}}) = 1$, so that by Clifford theorem $h^0(\mathcal{K}^{\otimes 1/2}_{\Sigma_{g=2}}) \leq \lfloor 3/2 \rfloor$. This implies that for a choice of an *even* theta characteristic on $\Sigma_{g=2}$ (there are 10 such) one can only have $h^0(\mathcal{K}^{\otimes 1/2}_{\Sigma_{g=2}}) = 0$, while for an *odd* theta characteristic (there are 6 such) one can only find $h^0(\mathcal{K}^{\otimes 1/2}_{\Sigma_{g=2}}) = 1$.
The higher genus case is actually more complicated, as stressed for example in [37]: in genus $g = 3$, for example, $\mathcal{K}^{\otimes 1/2}_{\Sigma_{g=3}}$ has degree 2. A compact Riemann surfaces in genus $g = 3$ is either a quartic curve in $\mathbb{P}^2$ or hyperelliptic. If and only if it is hyperelliptic, a degree 2 line bundle such as $\mathcal{K}^{\otimes 1/2}_{\Sigma_{g=3}}$ admits a 2-dimensional space of global sections. It follows that an even theta characteristic over an hyperelliptic curve of genus 3 is such that $\nu_{\Theta_{g=3}} = 2$. If instead $\Sigma_{g=3}$ is a plane quartic curve, then an even theta characteristic on it will be such that $\nu_{\Theta_{g=3}} = 0$. In the case one chooses an odd theta characteristic, then the only possibility is $\nu_{\Theta_{g=3}} = 1$.

9. Conclusions and Outlooks

The analysis of the present work opens up new possible applications to physical systems. It has been shown in several papers (see for example [24]) how the PCO and the integral forms can be used to formulate a universal variational principle for supersymmetric theories. Given the Lagrangian superform in a supermanifold, the construction of the corresponding action and its variational principle goes through the multiplication by a suitable PCO. The choice of the latter determines the superspace representation of the final action and consequently its variational Euler-Lagrangian equations. Putting the theory of integral, pseudo forms and superforms into a very clear and geometrical perspective opens up new possible studies in quantum field theories. For example the relation between Harmonic Superspace Techniques and the geometry of supermanifolds is still lacking, even though some of the initial and fundamental ideas are coming from Pure Spinor String theory developments [40] and share some similarities with the Harmonic Superspace Approach.

A clear understanding of the geometry behind the complex of forms for a supermanifold is a pivotal tool for extending the comprehension of super string field theory. As we have shown, several ingredients used in the present work are imported from superstring field theory and translated in terms of supergeometry. We expect that the benefit would be specular and the present discuss would be useful to understand the quantization of super string field theory (BV formalism and the corresponding gauge fixing) and, as a very ambitious project, to study the moduli of super Riemann surfaces directly from superstring field theory.

At this stage it would be hard - and maybe a little dicey - to foresee immediate physical applications out of sections 8. More modestly, considerations can be made at the level of future perspectives and possible directions. The natural arena for the considerations regarding the super Hodge diamond (and its symmetries) would be indeed *geometrical mirror symmetry*, or better its generalization to a supergeometric context. By the way, it is fair to say that little is known about the intriguing possibility of promoting mirror symmetry at the level of supermanifolds. Back in the Nineties, Sethi proposed supermanifolds as mirror candidates for the elusive *rigid* manifolds, *i.e.* manifolds that do not posses complex moduli, whose mirror manifolds are therefore expected not to be *Kähler* manifolds [38]; on the same line, later on, Aganagic and Vafa proposed a technique to find mirrors of supermanifolds as super Landau-Ginzburg (LG) theories [39]. However, these interesting pieces of works are based on path-integral computations, better than on a deep understanding of the subtle geometry of supermanifolds. In this regard, it is opinion of the authors that there is still a long way to go before the sophisticated ideas of mirror symmetry can be given a mathematical consistence in a supergeometric context: as hinted in section 8, this would require to bring *Hodge-de Rham theory* to a supergeometric context, in order to provide a theoretical framework capable to make sense to the finer structural invariants related to supermanifolds. Also, it would be crucial to have a reliable *intersection theory* for supermanifolds developed, which in turn leads to further important questions. Indeed it is known that there exists large classes of non-projected or non-split complex supermanifolds that cannot be embedded into projective superpsaces $\mathbb{P}^{n|m}$ [15]: it follows that a generalized intersection theory for supermanifolds cannot be just a generalization of the ordinary one for complex manifolds, since, to a great extent, it relies on working in projective spaces $\mathbb{P}^n$. Settling these issues would be a remarkable advance in the theory of complex supermanifolds and a good starting point for further consideration on mirror symmetry for supermanifolds. After finishing the manuscript of this paper a first progress, inspired again by the analogy between forms on supermanifolds and string fields in superstring theory, has been made on the problem (briefly discussed in the introduction) of constructing multilinear non-associative products of forms that yield in the case of $1|1$ supermanifolds an A_∞-algebra of forms. [41].

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