

Integral Representations on Supermanifolds: super Hodge duals, PCOs and Liouville forms.

Leonardo Castellani ^{a,b,*}, Roberto Catenacci ^{a,c,†}, and Pietro Antonio Grassi ^{a,b,‡}

^(a) *Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale
Viale T. Michel, 11, 15121 Alessandria, Italy*

^(b) *INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino*

^(c) *Gruppo Nazionale di Fisica Matematica, INdAM, P.le Aldo Moro 5, 00185 Roma*

Abstract

We present a few types of integral transforms and integral representations that are very useful for extending to supergeometry many familiar concepts of differential geometry. Among them we discuss the construction of the super Hodge dual, the integral representation of picture changing operators of string theories and the construction of the super-Liouville form of a symplectic supermanifold.

Keywords: Geometric Integration Theory, 49Q15; Supermanifolds and graded manifolds, 58A50; Analysis on supermanifolds or graded manifolds, 58C50.

*leonardo.castellani@uniupo.it

†roberto.catenacci@uniupo.it

‡pietro.grassi@uniupo.it

1 Introduction

Pursuing the construction of supersymmetric Lagrangians based in the framework of supermanifold geometry, we proposed in [1] a new *Hodge operator* \star acting on (super)differential forms. For that aim, we have discussed a complete formalism (integral-, pseudo- e superforms, their complexes and the integration theory) in a series of papers [1, 2, 3] together with a suitable Hodge operator.

As a byproduct, this mathematical tool sheds also a new light on the Hodge operator in conventional differential geometry.

The theory of forms in supergeometry was extensively developed from a mathematical point of view (important general references are [4], [5], [6], [7]); the main aim of this paper is to present, in a formalism that make contact with the recent physical literature, a few subjects of interest mainly in physical applications.

The Hodge operator plays an essential role in differential geometry, yielding a fundamental relation between the exterior bundle of differential forms and the scalar product (\bullet, \bullet) on the manifold. The construction requires the existence of a metric g on the the manifold \mathcal{M} and is an involutive operation \star which satisfies the linearity condition $\star(f\omega) = f\star\omega$ with ω a given p -form.

In the case of supermanifolds (we refer for ex. to [1] for the basic ingredients of supergeometry with the notations and the definitions used in this paper; see also [8] for a recent extensive review), the definition of the Hodge dual turns out to be harder than expected since one has to deal with the infinite-dimensional complexes of superforms. The integral forms and pseudo-forms are crucial to establish the correct matching of elements between the different spaces of forms. This new type of differential forms requires the enlargement of the conventional space spanned by the fundamental 1-forms, admitting distribution-like expressions (essentially, Dirac delta functions and Heaviside step functions). This has triggered us to consider the *Fourier analysis* for differential forms (this was earlier considered in [9] and [10]), and leads to an integral representation of the Hodge operator as explained in [1, 3]. Such a representation can first be established in the case of a conventional manifold

M without any reference to supermanifolds, except for the notion of *Berezin integral*. A new set of anticommuting variables playing the role of dual variables to fundamental 1-forms dx^i is introduced and the Hodge operator is defined by a suitable Berezin integration on the new variables. The result is proven to coincide with the usual Hodge operator. When extended to supermanifolds, our construction yields a “good” definition of Hodge operator, satisfying all desired properties. Note that our fiberwise integral transform maps superforms into integral forms showing that the latter are fundamental ingredients for the Hodge theory in supermanifolds.

In the case of supermanifolds Voronov and Zorich studied in [9] a different type of fiberwise integral transform, that in the purely bosonic setting (usual differential forms) also gives the usual Hodge dual. Their transform maps usual differential forms defined on the parity reversed (i.e. even) normal bundle of the reduced bosonic manifold (embedded into the supermanifold setting to zero the anticommuting coordinates) to pseudoforms on the odd normal bundle. Depending on the nature of the forms transformed, in this setting the pseudoforms obtained are very general: polynomial and analytic forms in the commuting differentials and even distribution-like forms.

This type of transform is not suitable for extending the Hodge dual to supergeometry but it is very interesting because, in modern string theory language, it is strictly related to (the integral representation of) the picture changing operators.

As is well known, in conventional Fourier analysis, the Fourier transform of a product of two functions is the *convolution integral* of the Fourier transform of the two functions. In a reciprocal way, the Fourier transform of the convolution integral of two functions can be expressed as the product of the Fourier transforms of the latter. This simple formula can be imported in our framework where the Fourier transform represents the Hodge operator and the convolution integral is a suitable Berezin integral of two differential forms. With this observation we are able to express the Hodge dual of the wedge product of two differential forms as the (Berezin)-convolution of the Hodge duals of the differential forms.

In the case of integral forms (that are “distribution-like”) the convolution product is clearly the right one, because their graded wedge product vanishes.

Another problem encountered in extending to supermanifolds the concepts of the usual differential geometry is that of the Liouville form in a symplectic supermanifold. The problem here is that the super-symplectic form is naturally a superform of zero picture and hence the Liouville form cannot be defined simply as a graded exterior power of the symplectic form, because there is no a top exterior power and the correct Liouville form must be defined using instead integral forms. Also in this case an integral representation opens the way.

In this present paper we will elaborate on these subjects, on their definitions and properties.

Finally, the Fourier (Berezin) integral representation of the Hodge dual operator can be extended to noncommutative spaces. A very recent work on this appeared in [11], and the idea of a Fourier-Berezin transform can be found also in [12].

2 Forms and Integration

The usual integration theory of differential forms for bosonic manifolds can be conveniently rephrased to uncover its relation with Berezin integration [14],[16].

We start with a simple example: consider in \mathbb{R} the integrable 1-form $\omega = g(x)dx$ (with $g(x)$ an integrable function in \mathbb{R}). We have:

$$\int_{\mathbb{R}} \omega = \int_{-\infty}^{+\infty} g(x)dx .$$

Observing that dx is an anticommuting quantity, and denoting it by ψ , we could think of ω as a function on the superspace $\mathbb{R}^{1|1}$:

$$\omega = g(x)dx = f(x, \psi) = g(x)\psi \tag{2.1}$$

This function can be integrated *à la* Berezin reproducing the usual definition:

$$\int_{\mathbb{R}^{1|1}} f(x, \psi)[dxd\psi] = \int_{-\infty}^{+\infty} g(x)dx = \int_{\mathbb{R}} \omega$$

Note that the symbol of the formal measure $[dxd\psi]$ is written just to emphasize that we are integrating on the **two** variables x and ψ , hence the dx inside $[dxd\psi]$ is *not* identified with ψ .

Denoting by M a bosonic orientable differentiable manifold of dimension n , its exterior bundle $\Lambda(M) = \sum_{p=0}^n \Lambda^p(M)$ is the direct sum of $\Lambda^p(M)$ (their spaces of sections will be denoted as $\Omega(M)$ and $\Omega^p(M)$ respectively). A section ω of $\Lambda^p(M)$ can be written locally as

$$\omega = \sum \omega_{i_1 \dots i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p} \quad (2.2)$$

where the coefficients $\omega_{i_1 \dots i_p}(x)$ are functions on M and $i_1 < \dots < i_p$. The integral of $\omega \in \Omega^n(M)$ is defined as:

$$I[\omega] = \int_M \omega = \int_M \omega_{12 \dots n}(x) d^n x, \quad (2.3)$$

This opens the way to relating the integration theory of forms and the Berezin integral, by substituting every 1-form dx^i with a corresponding abstract Grassmann variable denoted again with dx^i . A section ω of $\Omega(M)$ is viewed locally as a function on a supermanifold $\mathcal{M} = T^*(M)$ with coordinates (x^i, dx^i) :

$$\omega(x, dx) = \sum \omega_{i_1 \dots i_p}(x) dx^{i_1} \dots dx^{i_p}; \quad (2.4)$$

such functions are polynomials in dx 's. Supposing now that the form ω is integrable, its Berezin integral gives:

$$\int_{\mathcal{M}=T^*(M)} \omega(x, dx) [d^n x d^n(dx)] = \int_M \omega \quad (2.5)$$

3 The Integral Representation of the Hodge Star and Convolutions

In the following, for a given set $\{\xi^i\}_{i=1}^n$ of Grassmann variables, our definition of the Berezin integral is $\int_{\mathbb{R}^0|n} \xi^1 \dots \xi^n [d^n \xi] = 1$ and not $\int_{\mathbb{R}^0|n} \xi^1 \dots \xi^n [d^n \xi] = (-1)^{\frac{n(n-1)}{2}}$. Moreover, if α is a monomial expression of some anticommuting variables α^k not depending on the ξ^i , we define: $\int_{\mathbb{R}^0|n} \alpha \xi^1 \dots \xi^n [d^n \xi] = \alpha$, where the product between α and the ξ^i is the usual \mathbb{Z}_2 graded wedge product in the superalgebra generated by the tensor product of the Grassmann algebra generated by the ξ^i and that generated by the α^k : if \mathcal{A} and \mathcal{B} are two \mathbb{Z}_2 -graded algebras with products $\cdot_{\mathcal{A}}$ and $\cdot_{\mathcal{B}}$, the \mathbb{Z}_2 -graded tensor product $\mathcal{A} \otimes \mathcal{B}$ is a \mathbb{Z}_2 -graded algebra with the

product (for homogeneous elements) given by :

$$(a \otimes b) \cdot_{\mathcal{A} \otimes \mathcal{B}} (a' \otimes b') = (-1)^{|a'| |b|} a \cdot_{\mathcal{A}} a' \otimes b \cdot_{\mathcal{B}} b'$$

In our case the algebras are Grassmann algebras and the products \cdot are wedge products. The symbols \otimes and \wedge will be, in general, omitted.

One can observe, see *e.g.* [18], [19], and also [1] that the usual Hodge dual in \mathbb{R}^n (for a metric¹ given by a matrix A with entries g_{ij}) can be obtained by means of the Fourier (Berezin)-integral transform \mathcal{T} . For $\omega(x, dx) \in \Omega^k(\mathbb{R}^n)$ we have:

$$\star \omega = i^{(k^2-n^2)} \frac{\sqrt{|g|}}{g} \mathcal{T}(\omega) = i^{(k^2-n^2)} \frac{\sqrt{|g|}}{g} \int_{\mathbb{R}^{0|n}} \omega(x, \eta') e^{i dx A \eta'} [d^n \eta'] \quad (3.1)$$

where $g = \det A$. The exponential series defining $e^{i dx A \eta'}$ is written using the \mathbb{Z}_2 graded wedge product quoted above. The Grassmann variables η' are defined as $\eta' = A^{-1} \eta$ where the η are the (*parity changed*)² variables dual to the dx . In this way the covariance properties of $\omega(x, \eta')$ are exactly those of a differential form and this is crucial in physical applications.

The factor $i^{(k^2-n^2)}$ can be obtained by computing the transformation of the monomial form $dx^1 dx^2 \dots dx^k$ in the simple case $A = I$.

The explicit computation gives:

$$i^{(k^2-n^2)} \mathcal{T}(dx^1 \dots dx^k) = \star(dx^1 \dots dx^k) \quad (3.2)$$

and

$$\mathcal{T}^2(\omega) = i^{(n^2-k^2)} i^{(k^2)} (\omega) = i^{n^2} (\omega) \quad (3.3)$$

yielding the usual duality relation:

$$\star \star \omega = i^{((n-k)^2-n^2)} i^{(k^2-n^2)} i^{n^2} (\omega) = (-1)^{k(k-n)} (\omega) \quad (3.4)$$

¹We consider metrics of arbitrary signature; The sign of g fixes the sign of the overall coefficient $\frac{\sqrt{|g|}}{g} = \pm \frac{1}{\sqrt{|g|}}$.

²Here and in the following we adopt the convention that d is an odd operator (so dx is a odd form but $\frac{\partial}{\partial x}$ is an even vector). A change of parity is necessary because we want η to be an odd variable.

As for functions, one can define (see [9]) a convolution product between differential forms on an ordinary manifold. The starting point is again the interpretation of differential forms as functions of the commuting variables x and the anticommuting variables dx . For $\alpha \in \Omega^p(\mathbb{R}^n)$ and $\beta \in \Omega^q(\mathbb{R}^n)$, the convolution product \bullet is defined using Berezin integration on the anticommuting variables:

$$\alpha \bullet \beta(x, dx) = \int_{\mathbb{R}^{0|n}} \alpha(x, \xi) \beta(x, dx - \xi) [d^n \xi] \quad (3.5)$$

where the ξ are auxiliary anticommuting variables. Note that this pointwise convolution product depends on a choice of a volume element (i.e. the ordering of the auxiliary variables). The convolution product ³ maps $\Omega^p \times \Omega^q \rightarrow \Omega^{p+q-n}$. To obtain (generically) non trivial results we must have $0 \leq p + q - n \leq n$. The algebra of this convolution is

$$\alpha \bullet \beta = (-1)^{(n^2+pq)} \beta \bullet \alpha$$

The convolution ‘interacts’ well with the integral transformation \mathcal{T} defined above and the wedge product. We will consider explicitly only the standard bosonic case in which the matrix A of the previous paragraph is the identity matrix I .

For example, in the case $n = 4$, we can compute $\mathcal{T}(dx^1 dx^2) = dx^3 dx^4$ and $\mathcal{T}(dx^1) = (-i) dx^2 dx^3 dx^4$, $\mathcal{T}(dx^2) = i dx^1 dx^3 dx^4$. The convolution is:

$$\begin{aligned} \mathcal{T}(dx^1) \bullet \mathcal{T}(dx^2) &= \int_{\mathbb{R}^{0|4}} (-i) \xi^2 \xi^3 \xi^4 (i) (dx^1 - \xi^1) (dx^3 - \xi^3) (dx^4 - \xi^4) [d^4 \xi] \\ &= dx^3 dx^4 = \mathcal{T}(dx^1 dx^2) \end{aligned}$$

Another simple example is the case $q = n - p$ where we find:

$$i^{n^2} (-1)^p (-1)^{p(n-p)} \mathcal{T}(\alpha \beta) = \mathcal{T}(\alpha) \bullet \mathcal{T}(\beta) \quad (3.6)$$

Indeed, recalling that:

$$\mathcal{T}(dx^1 \dots dx^p) = i^{(n^2-p^2)} (dx^{p+1} dx^{p+2} \dots dx^n) \quad (3.7)$$

³We must integrate generically monomials of the type $(\xi)^{p+q-k} (dx)^k$ and the Berezin integration selects $k = p + q - n$.

$$\mathcal{T}(dx^{p+1} \dots dx^n) = i^{(p^2)}(dx^1 dx^2 \dots dx^p) \quad (3.8)$$

$$\mathcal{T}(dx^1 \dots dx^n) = 1 \quad (3.9)$$

we find:

$$\mathcal{T}(dx^1 \dots dx^p) \bullet \mathcal{T}(dx^{p+1} \dots dx^n) = i^{n^2} \int_{\mathbb{R}^{0|n}} (\xi^{p+1} \dots \xi^n) (dx^1 - \xi^1) \dots (dx^p - \xi^p) [d^n \xi] = \quad (3.10)$$

$$i^{n^2} \int_{\mathbb{R}^{0|n}} (\xi^{p+1} \dots \xi^n) (-1)^p \xi^1 \dots \xi^p [d^n \xi] = i^{n^2} (-1)^p (-1)^{p(n-p)} \mathcal{T}(dx^1 dx^2 \dots dx^n) \quad (3.11)$$

The properties of the convolution reflect on corresponding properties of the Hodge star operator. Using $\star \omega = i^{(k^2 - n^2)} \mathcal{T}(\omega)$ for $\omega(x, dx) \in \Omega^k(\mathbb{R}^n)$, we obtain a simple formula for the Hodge dual of the wedge product of forms in the case $p + q = n$:

$$\star(\alpha\beta) = (-1)^p (\star\alpha) \bullet (\star\beta) \quad (3.12)$$

Considering now the general case of a p -form α and a q -form β in a n -dimensional space, one can prove the following relation:

$$\star(\alpha\beta) = (-1)^{n+q(n-p)} (\star\alpha) \bullet (\star\beta) \quad (3.13)$$

easily checked to be satisfied by the monomials

$$\alpha = dx^1 dx^2 \dots dx^p, \quad \beta = dx^{n-q+1} dx^{n-q+2} \dots dx^n \quad (3.14)$$

Indeed recall that

$$\star\alpha = dx^{p+1} \dots dx^n, \quad \star\beta = (-1)^{q(n-q)} dx^1 \dots dx^{n-q} \quad (3.15)$$

$$\star(\alpha\beta) = (-1)^{q(n-p-q)} dx^{p+1} \dots dx^{n-q} \quad (3.16)$$

Moreover, using the definition of the convolution, one finds

$$(\star\alpha) \bullet (\star\beta) = (-1)^{q(n-q)} (-1)^p (-1)^{n(n-q-p)} (-1)^{p(n-p)} dx^{p+1} \dots dx^{n-q} \quad (3.17)$$

Comparing the last two equations, relation (3.13) follows. By linearity the same relation (3.13) holds also for generic forms. Two particular cases provide nontrivial checks:

i) when $\alpha = 1 \in \Omega^0$:

$$\begin{aligned} \star(1\beta) &= (-1)^{n+qn} (\star 1) \bullet (\star\beta) = (-1)^{n+qn} \int_{\mathbb{R}^{0|n}} (\xi^1 \dots \xi^n) (\star\beta(dx - \xi)) [d^n \xi] \\ &= (-1)^{n+qn} (-1)^{n(n-q)} \star\beta = \star\beta \end{aligned} \quad (3.18)$$

ii) when $\beta = 1 \in \Omega^0$:

$$\begin{aligned} \star(\alpha 1) &= (-1)^n (\star\alpha) \bullet (\star 1) = (-1)^n \int_{\mathbb{R}^{0|n}} (\star\alpha)(\xi) (dx^1 - \xi^1) \dots (dx^n - \xi^n) [d^n \xi] \\ &= (-1)^n \int_{\mathbb{R}^{0|n}} (\star\alpha)(\xi + dx) (-1)^n \xi^1 \dots \xi^n [d^n \xi] = \star\alpha \end{aligned} \quad (3.19)$$

where we used the traslational invariance (under $\xi \rightarrow \xi + dx$) of the Berezin integral.

Similar relations hold (modulo some suitable multiplicative coefficient depending also on the metric) for the more general integral transform that gives the Hodge dual for a generic metric A .

The convolution defined in the formula (3.5) can be normalized as:

$$\alpha \bullet' \beta(x, dx) = (-1)^{(n+pm+pq)} \alpha \bullet \beta \quad (3.20)$$

where again p is the degree of α , q the degree of β , and n the dimension of the space.

With this normalization the formula (3.13) looks better:

$$\star(\alpha\beta) = (\star\alpha) \bullet' (\star\beta) \quad (3.21)$$

Indeed, noting that $(\star\alpha) \bullet (\star\beta) = (-1)^{n+(n-p)n+(n-p)(n-q)} (\star\alpha) \bullet' (\star\beta)$, we have:

$$\star(\alpha\beta) = (-1)^{n+q(n-p)} (-1)^{n+(n-p)n+(n-p)(n-q)} (\star\alpha) \bullet' (\star\beta) = (\star\alpha) \bullet' (\star\beta)$$

The algebra of this new convolution is:

$$\alpha \bullet' \beta = (-1)^{(n-p)(n-q)} \beta \bullet' \alpha \quad (3.22)$$

Clearly this normalized convolution product has a unit, the standard volume form $\star 1$.

Equation (3.21) and the associativity of the wedge product show that the convolution product \bullet' is associative:

$$(\star\alpha \bullet' \star\beta) \bullet' (\star\gamma) = \star[(\alpha\beta)\gamma] = \star[\alpha(\beta\gamma)] = \star\alpha \bullet' (\star\beta \bullet' \star\gamma)$$

As last remark, we point out that, using our Fourier representation of the Hodge dual, it is easy to deduce the standard formula:

$$\alpha \wedge \star \alpha = (\alpha, \alpha) \star 1, \quad (3.23)$$

where (\cdot, \cdot) is the scalar product associated to the metric g . Moreover, the same scalar product can be rewritten with the new convolution as

$$(\alpha, \alpha) = \alpha \bullet' \star \alpha = (-1)^{p(n-p)} \star \alpha \bullet' \alpha \quad (3.24)$$

where instead of the wedge product we have used the convolution product.

4 Super Hodge dual and super Convolutions

We start, as usual, from the real superspace $\mathbb{R}^{n|m}$ with n bosonic $(x^i, i = 1, \dots, n)$ and m fermionic $(\theta^\alpha, \alpha = 1, \dots, m)$ coordinates. We denote by T the tangent bundle and by T^* the cotangent bundle (see footnote below). To simplify the notations we will denote by the same letter a bundle and the \mathbb{Z}_2 -graded modules of its sections.

With ours conventions⁴ these modules are generated over the ring of superfunctions as follows ($i = 1 \dots n$; $\alpha = 1 \dots m$):

$$\begin{aligned} T & \text{ by the even vectors } \frac{\partial}{\partial x^i} \text{ and the odd vectors } \frac{\partial}{\partial \theta^\alpha} \\ T^* & \text{ by the even forms } d\theta^\alpha \text{ and the odd forms } dx^i \end{aligned}$$

If Π is the parity reversal symbol ($\Pi \mathbb{R}^{p|q} = \mathbb{R}^{q|p}$), we can consider the bundle ΠT . The \mathbb{Z}_2 -graded module of its sections is generated by the even vectors b_α and the odd vectors η_i

We consider now the \mathbb{Z}_2 -graded tensor product $T^* \otimes \Pi T$ and the invariant even section σ given by:

$$\sigma = dx^i \otimes \eta_i + d\theta^\alpha \otimes b_\alpha \quad (4.1)$$

⁴As pointed out in the previous section, in order to make contact with the standard physical literature we adopt the conventions that d is an odd operator and dx (an odd form) is dual to the even vector $\frac{\partial}{\partial x}$. The same holds for the odd variables θ . As clearly explained for example in the appendix of the paper [26] if one introduces also the natural concept of even differential (in order to make more contact with the standard definition of cotangent bundle of a manifold) our cotangent bundle (that we consider as the bundle of one-forms) should, more appropriately, be denoted by ΠT^* .

Introducing a (pseudo)riemannian metric $A = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and a symplectic form $B = \gamma\left(\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\beta}\right)$, the even matrix $\mathbb{G} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is a supermetric in $\mathbb{R}^{n|m}$ (with obviously m even). A and B are, respectively, a $n \times n$ matrix and a $m \times m$ matrix with even entries, $\det A \neq 0$ and $\det B \neq 0$.

In matrix notations, omitting (here and in the following) the tensor product symbol, the section σ can be written as:

$$\sigma = dxAA^{-1}\eta + d\theta BB^{-1}b = dxA\eta' + d\theta Bb' = dZ\mathbb{G}W'$$

where $\eta' = A^{-1}\eta$ and $b' = B^{-1}b$ are the covariant forms corresponding to the vectors η and b , and $dZ = (dx \ d\theta)$ and $W' = \begin{pmatrix} \eta' \\ b' \end{pmatrix}$.

If $\omega(x, \theta, dx, d\theta)$ is a superform in $\Omega^{(p|0)}$ (i.e. a section locally given by a function $\omega(x, \theta, dx, d\theta)$ with polynomial dependence in the variables θ, dx and $d\theta$, of total degree p in the last two variables), the even section σ can be used to generate an integral transform that can be considered as a fiberwise integration on the fibers of T^* :

$$\mathcal{T}(\omega)(x, \theta, dx, d\theta) = \int_{\mathbb{R}^{m|n}} \omega(x, \theta, \eta', b') e^{i(dxA\eta' + d\theta Bb')} [d^m \eta' d^m b'] \quad (4.2)$$

where $\omega(x, \theta, \eta', b')$ has polynomial dependence in the variables θ, η' and b' and $e^{i\sigma}$ is defined as the usual power series.

This integral transform clearly depends on the choice of a supermetric and, from the point of view (relevant for physical applications) of covariance properties, maps forms to forms. We recall that other important types of integral transforms (depending on the choice of a volume element but not on a supermetric) were defined and studied in [18], [16], [15], and [17].

The integral over the odd η' variables is a Berezin integral and the integral over the even b' variables is defined by formal rules, for example:

$$\int_{\mathbb{R}^m} e^{id\theta Bb'} d^m b' = \frac{1}{\det B} \delta^m(d\theta) \quad (4.3a)$$

$$\int_{\mathbb{R}^m} b'_1 \dots b'_m e^{id\theta Bb'} d^m b' = (-i)^m \frac{1}{(\det B)^{m+1}} \left(\frac{d}{d\theta} \delta(d\theta) \right)^m \quad (4.3b)$$

The products $\delta^m(d\theta)$ and $(\frac{d}{d\theta}\delta(d\theta))^m$ (m here denotes the number of factors) are wedge products ordered as in $d^m b$. In other words this kind of integrals depends on the choice of an oriented basis. For example, we obtain the crucial anticommuting property of the delta forms (no sum on α, β):

$$\delta(d\theta^\alpha)\delta(d\theta^\beta) = \int_{\mathbb{R}^2} e^{i(d\theta^\alpha b'^\alpha + d\theta^\beta b'^\beta)} db'^\alpha db'^\beta = - \int_{\mathbb{R}^2} e^{i(d\theta^\alpha b'^\alpha + d\theta^\beta b'^\beta)} db'^\beta db'^\alpha = -\delta(d\theta^\beta)\delta(d\theta^\alpha) \quad (4.4)$$

We can generalize the Hodge dual to superforms of zero picture (note that the spaces of superforms or of integral forms are all finite dimensional) where we have the two types of differentials, $d\theta$ and dx .

A zero picture p -superform ω is a combination of a **finite number** of monomial elements of the form:

$$\rho_{(r,l)}(x, \theta, dx, d\theta) = f(x, \theta) dx^{i_1} dx^{i_2} \dots dx^{i_r} (d\theta^1)^{l_1} (d\theta^2)^{l_2} \dots (d\theta^s)^{l_s} \quad (4.5)$$

of total degree equal to $p = r + l_1 + l_2 + \dots + l_s$. We denote by l the sum of the l_i . We have also $r \leq n$.

The super Hodge dual on the monomials can be defined as:

$$\star\rho_{(r,l)} = (i)^{r^2-n^2} (i)^{\alpha(l)} \mathcal{T}(\rho_{(r,l)}) = (i)^{r^2-n^2} (i)^{\alpha(l)} \frac{\sqrt{|S\det\mathbb{G}|}}{S\det\mathbb{G}} \int_{\mathbb{R}^{m|n}} \rho_{(r,l)}(x, \theta, \eta', b') e^{i(dx A \eta' + d\theta B b')} [d^n \eta' d^m b'] \quad (4.6)$$

We recall that:

$$S\det\mathbb{G} = \frac{\det A}{\det B}$$

The normalization coefficient is given by: $\alpha(l) = 2pl - l^2 - nl - l$ (with $l = p - r$) if n is even and $\alpha(l) = l$ if n is odd. These coefficient was computed in [1]

The \star operator on monomials can be extended by linearity to generic forms in $\Omega^{(p|0)}$:

$$\star : \Omega^{(p|0)} \longrightarrow \Omega^{(n-p|m)}$$

Both spaces are **finite dimensional** and \star is an isomorphism⁵.

⁵The normalization coefficients chosen in the definitions of the duals of $\rho_{(r,l)}$ and $\rho_{(r|j)}$ lead to the usual

An important example in $\mathbb{R}^{n|m}$ is $1 \in \Omega^{(0|0)}$:

$$\star 1 = \sqrt{\left| \frac{\det A}{\det B} \right|} d^n x \delta^m(d\theta) \in \Omega^{(n|m)}$$

In the case of $\Omega^{(p|m)}$, a p -integral⁶ form ω is a combination of a **finite number** of monomial elements of the form:

$$\rho_{(r|j)}(x, \theta, dx, d\theta) = f(x, \theta) dx^{i_1} dx^{i_2} \dots dx^{i_r} \delta^{(j_1)}(d\theta^1) \delta^{(j_2)}(d\theta^2) \dots \delta^{(j_m)}(d\theta^m) \quad (4.7)$$

where $p = r - (j_1 + j_2 + \dots + j_m)$. We denote by j the sum of the j_i . We have also $r \leq n$.

The Hodge dual is:

$$\star \rho_{(r|j)} = (i)^{r^2-n^2} (i)^{\alpha(j)} \frac{\sqrt{|S\det \mathbb{G}|}}{S\det \mathbb{G}} \int_{\mathbb{R}^{m|n}} \rho_{(r|j)}(x, \theta, \eta, b) e^{i(dx A \eta' + d\theta B b')} [d^n \eta' d^m b'] \quad (4.8)$$

Note that we could consider also a more general **even** super metric:

$$\mathbb{G} = \begin{pmatrix} G_{(ab)}(x, \theta) & G_{a\beta}(x, \theta) \\ G_{\alpha b}(x, \theta) & G_{[\alpha\beta]}(x, \theta) \end{pmatrix} \equiv \begin{pmatrix} A & C \\ D & B \end{pmatrix} \quad (4.9)$$

where $G_{(ab)}(x, \theta), G_{[\alpha\beta]}(x, \theta)$ are even matrices and $G_{a\beta}(x, \theta), G_{\alpha b}(x, \theta)$ are odd matrices. In matrix notation the even section σ is in this case given by:

$$\sigma = dZ \mathbb{G} \mathbb{G}^{-1} W = dx A \eta' + d\theta B b' + dx C b' + d\theta D \eta'$$

In general, the super matrix \mathbb{G} can be expressed in terms of the supervielbein \mathbb{V} as follows

$$\mathbb{G} = \mathbb{V} \mathbb{G}_0 \mathbb{V}^T$$

where \mathbb{G}_0 is an invariant constant super matrix characterizing the tangent space of the supermanifold $\mathbb{R}^{(n|m)}$. The overall coefficient of the Hodge dual becomes

$$\frac{\sqrt{|S\det \mathbb{G}_0|}}{S\det \mathbb{V} S\det \mathbb{G}_0}$$

duality on $\Omega^{(p|0)}$:

$$\star \star \rho_{(r,p-r)} = (-1)^{p(p-n)} \rho_{(r,p-r)}$$

⁶In the literature, see [13] and also [18], one finds pseudodifferential forms of distributional type which belong to the spaces $\Omega^{(p|q)}$ where p denote the form degree and q the picture number with $0 \leq q \leq m$ (for *picture number* we intend the number of Dirac delta functions assuming that a given pseudodifferential form can be decomposed in term of them). Those with $q = m$ denote the Bernstein-Leites integral forms.

where $\text{Sdet}\mathbb{V}$ is the superdeterminant of the supervielbein.

In the case of integral forms (that are “distribution-like”) the convolution product is clearly the right one, because the product of integral forms vanishes using the graded wedge product. The convolution between two integral forms α and β can be defined as before:

$$\alpha \bullet \beta(x, dx, \theta, d\theta) = \int_{\mathbb{R}^{n|m}} \alpha(x, \xi, \theta, b) \beta(x, dx - \xi, \theta, d\theta - b) [d^n \xi d^m b] \quad (4.10)$$

Again the integral is a Berezin integral for the variables ξ and a usual integral for the variables b . In addition, we recall that the definition depends upon a choice of a volume. What is important here is that the convolution of two integral forms is again an integral form (i.e. the total number of delta forms and derivatives of delta forms is conserved). This convolution product maps $\Omega^{p|m} \times \Omega^{q|m} \rightarrow \Omega^{p+q-n|m}$. Note that, in this case, the form number can be negative.

5 Integral Representation of PCOs

Another interesting integral transform (see [9]) is a fiberwise integration not on T^* but on the parity changed normal bundle N of the reduced bosonic manifold $M \simeq \mathbb{R}^n$. The reduced manifold M with tangent bundle $T(M)$ is embedded into \mathcal{M} by setting all the anticommuting coordinates to zero; its normal bundle N is defined as:

$$0 \longrightarrow T(M) \longrightarrow T_M \longrightarrow N \longrightarrow 0 \quad (5.1)$$

where T_M is the tangent bundle of the supermanifold \mathcal{M} restricted to the reduced manifold M . The fibers of N are odd and its sections are generated by the odd vectors $\frac{\partial}{\partial \theta^\alpha}$. It is known that the total space of N is a supermanifold isomorphic to \mathcal{M} ; any real supermanifold can be described as a vector bundle with even base and odd fibers. We consider now the parity changed bundle ΠN with sections generated by the even vectors b_α . The total space of ΠN is a bosonic manifold with coordinates x and b . A differential form on ΠN is a function

$\omega(x, b, dx, db)$ and we can define an integral fiberwise transform:

$$\begin{aligned} \mathcal{T}(\omega)(x, \theta, dx, d\theta) &= \int_{\mathbb{R}^{m|m}} \omega(x, b, dx, db) e^{i(\theta db - d\theta b)} [d^m(db) d^m b] \\ &= \int_{\mathbb{R}^{m|m}} \omega(x, b, dx, db) e^{-id(\theta b)} [d^m(db) d^m b] \end{aligned} \quad (5.2)$$

where the even variables $d\theta$ are dual to the even variables b and the odd variables θ are dual to the odd variables db . The integral over the odd variables db is a Berezin integral. We see that this integral transform maps differential forms on the parity changed normal bundle into pseudoforms on N and does not depend on the choice of a metric.

Depending on the nature of the forms transformed, the pseudoforms obtained can be very general: polynomial or forms in the commuting differentials and even distribution-like forms when $\omega(x, b, dx, db)$ is polynomial in the commuting variables b . If $\omega(x, b, dx, db)$ is generalized to a distribution valued form its transformation is a superform⁷.

Comparing this transformation with the transformation defined in the formula (4.2) we observe that it is not suitable for defining a super Hodge dual but it is connected to the picture changing operators.

The Picture Changing Operators (PCOs) were introduced in [20] in string theory. In supergeometry they were introduced and studied by Belopolsky ([21, 22]), For another geometrical interpretation of PCOs as Poincaré duals of bosonic submanifolds embedded in a supermanifold see [2] and [24]). These operators are non trivial elements of $H_d^{(0|r)}$, act connecting the complexes of (super, pseudo and integral)-forms on supermanifolds, gives isomorphisms in cohomology, and can be described in our context as follows.

Given a constant commuting vector v we define the following form:

$$Y_v = v_\alpha \theta^\alpha \delta(v_\alpha d\theta^\alpha), \quad (5.3)$$

which is d -closed but not d -exact. Y_v belongs to $\Omega^{(0|1)}$ and by choosing m independent vectors $v^{(\alpha)}$, we have

$$\prod_{\alpha=1}^m Y_{v^{(\alpha)}} = \det(v_\beta^{(\alpha)}) \theta^{\alpha_1} \dots \theta^{\alpha_m} \delta(d\theta^{\alpha_1}) \dots \delta(d\theta^{\alpha_m}), \quad (5.4)$$

⁷For example, if $\omega = bdb$ then $\mathcal{T}(\omega) = i\delta'(d\theta)$ and if $\omega = \delta'(b)db$ then $\mathcal{T}(\omega) = -id\theta$. The imaginary factors could be eliminated introducing a normalization factor in the definition of the integral transformation.

where $v_\beta^{(\alpha)}$ is the β -component of the α -vector. We can apply the PCO on a given form by taking the graded wedge product.

For example, given ω in $\Omega^{(p|r)}$ (as discussed in footnote 6, p denotes the form degree and r denote the picture number), we have

$$\omega \longrightarrow \omega \wedge Y_v \in \Omega^{(p|r+1)}, \quad (5.5)$$

Notice that if $r = m$, then $\omega \wedge Y_v = 0$; on the other hand, if v does not depend on the arguments of the delta forms in ω , we obtain a non-vanishing form. In addition, if $d\omega = 0$ then $d(\omega \wedge Y_v) = 0$ (by applying the Leibniz rule), and if $\omega \neq dK$ then it follows that also $\omega \wedge Y_v \neq dU$ where U is a form in $\Omega^{(p-1|r+1)}$. The Y_v are non trivial elements of the de Rham cohomology and they are globally defined. So, given an element of the cohomology $H_d^{(p|r)}$, the new form $\omega \wedge Y_v$ is an element of $H_d^{(p|r+1)}$.

An integral representation of these operators is obtained acting with the transformation (5.2) on suitable forms of type $\omega(x, dx, b, db)$.

For example

$$\mathbb{Y}^{(0|1)} = \theta \delta(d\theta) = \int_{\mathbb{R}^{1|1}} e^{i(\theta db - d\theta b)} [d(db)db] \quad (5.6)$$

with $\omega(x, dx, b, db) = 1$.

The PCO $\mathbb{Y}^{(0|1)}$ is an example of this kind of operators, acting on $\Omega^{(p|0)}$ and increasing the number of delta forms, and therefore increasing the picture. As was shown in [25], $\mathbb{Y}^{(0|1)}$ generates the cohomology $H_d^{(0|1)}$; therefore, any other cohomology representative can be expressed in terms of $\mathbb{Y}^{(0|1)}$ up to d -exact terms. For example, we can consider the following form

$$\omega(x, dx, b, db) = 1 + ibdxdb \quad (5.7)$$

whose transformation gives

$$\widehat{\mathbb{Y}}^{(0|1)} = -dx\delta'(d\theta) + \theta\delta(d\theta) = d(-x\delta'(d\theta)) + \mathbb{Y}^{(0|1)}, \quad (5.8)$$

which clearly differs from $\mathbb{Y}^{(0|1)}$ by a d -exact term. The new PCO $\widehat{\mathbb{Y}}^{(0|1)}$ can also be written as

$$\widehat{\mathbb{Y}}^{(0|1)} = -(dx + \theta d\theta)\delta'(d\theta). \quad (5.9)$$

It is invariant under the supersymmetry transformations $\delta_\epsilon x = \epsilon\theta$ and $\delta_\epsilon\theta = \epsilon$ with ϵ a constant anticommuting parameter. Therefore, even though $\widehat{\mathbb{Y}}^{(0|1)}$ belongs to the same cohomology class of $\mathbb{Y}^{(0|1)}$, it has interesting properties, lacking for $\mathbb{Y}^{(0|1)}$. The exact term in (5.8) is not supersymmetric and the corresponding variation of $\mathbb{Y}^{(0|1)}$ is d -exact: $\delta_\epsilon\mathbb{Y}^{(0|1)} = d(-\epsilon\theta\delta'(d\theta))$.

The PCO's of the type \mathbb{Y} are needed to increase the number of delta's in the differential forms, passing from zero-pictures to the highest possible picture.⁸ However given an integral form, we need to be able to construct a superform by acting with another operator decreasing the picture number. That can be achieved by considering the following operator:

$$\delta(\iota_D) = \int_{-\infty}^{\infty} \exp(it\iota_D) dt \quad (5.10)$$

where D is an odd vector field on $T(\mathcal{M})$ with $[D, D] \neq 0$ and ι_D is the contraction along the vector D .

For example, if we decompose D on a basis $D = D^\alpha\partial_{\theta^\alpha}$, where the D^α are even coefficients and $\{\partial_{\theta^\alpha}\}$ is a basis of the odd vector fields, and take $\omega = \omega_\beta d\theta^\beta \in \Omega^{(1|0)}$, we have

$$\iota_D\omega = D^\alpha\omega_\alpha = D^\alpha\frac{\partial\omega}{\partial d\theta^\alpha} \in \Omega^{(0|0)}. \quad (5.11)$$

In addition, due to $[D, D] \neq 0$, we have also that $\iota_D^2 \neq 0$. The differential operator $\delta(\iota_\alpha) \equiv \delta(\iota_D) -$ with $D = \partial_{\theta^\alpha}$ - acts on the space of integral forms as follows (we neglect possible introduction of derivatives of delta forms, but that generalization can be easily done):

$$\begin{aligned} \delta(\iota_\alpha) \prod_{\beta=1}^m \delta(d\theta^\beta) &= \int_{-\infty}^{\infty} \exp(it\iota_\alpha) \delta(d\theta^\alpha) \prod_{\beta=1 \neq \alpha}^m \delta(d\theta^\beta) dt \\ &= \int_{-\infty}^{\infty} \delta(d\theta^\alpha + it) \prod_{\beta=1 \neq \alpha}^m \delta(d\theta^\beta) dt = -i \prod_{\beta=1 \neq \alpha}^m \delta(d\theta^\beta) \end{aligned} \quad (5.12)$$

The result contains $m - 1$ delta forms, and therefore it has picture $m - 1$. The picture number is decreased. Acting several times with $\delta(\iota_D)$, we can remove all the delta's. Note that $\delta(\iota_\alpha)$ is an odd operator because maps an even (odd) product of delta forms in an odd (even) one.

⁸There is also the possibility to increase the picture to a number between zero and the maximum value. In that case we have pseudo-differential forms (i.e. forms with picture $< m$), however, since we do not use them in the present work, we leave aside such a possibility.

Let us consider now, in the simplest case $\mathcal{M} = \mathbb{R}^{(1|1)}$, the following “double” differential operator

$$\mathbb{Z}^{(0|-1)} = -i\partial_\theta\delta(\iota_{\partial_\theta}) = i\delta(\iota_{\partial_\theta})\partial_\theta \quad (5.13)$$

where ∂_θ is the partial derivative along θ and ι_{∂_θ} is the contraction along that vector.

The operator $\mathbb{Z}^{(0|-1)}$ is the product of two operators acting on different quantities: ∂_θ acts only on functions, and $\delta(\iota_{\partial_\theta})$ acts only on the delta forms. Then, we can easily check that:

$$\mathbb{Z}^{(0|-1)} \circ \mathbb{Y}^{(0|1)} = 1. \quad (5.14)$$

A more general form for $\mathbb{Z}^{(0|-1)}$ could be constructed, but we are not interested here in such a generalization. Moreover, for several variables θ^α , we can consider the product of single operators .

Finally, we note that the Voronov integral transform can be used to produce a representation of the operator $\mathbb{Z}^{(0|-1)}$ as a multiplication operator in the space of ”dual forms” i.e. of type $\omega(x, b, dx, db)$.

The usual Fourier transform \mathcal{F} in \mathbb{R} (with coordinate x) gives a representation of the derivative operator $\frac{d}{dx}$ as a multiplication⁹ in the momentum space (with coordinate p).

In our simple case the operator $\mathbb{Z}^{(0|-1)}$ acts on the spaces $\Omega^{(0|1)}$ and $\Omega^{(1|1)}$ producing elements of $\Omega^{(0|0)}$ and $\Omega^{(1|0)}$ respectively.

A generic form $\omega \in \Omega^{(0|1)} \oplus \Omega^{(1|1)}$ but $\notin \ker \mathbb{Z}^{(0|-1)}$ can be written as:

$$\omega(x, dx, \theta, d\theta) = f(x)\theta\delta(d\theta) + g(x)\theta dx\delta(d\theta). \quad (5.15)$$

because $\mathbb{Z}^{(0|-1)}(\delta(d\theta)) = \mathbb{Z}^{(0|-1)}(dx\delta'(d\theta)) = \mathbb{Z}^{(0|-1)}(\theta dx\delta'(d\theta)) = 0$

The action of the operator $\mathbb{Z}^{(0|-1)}$ gives:

$$\mathbb{Z}^{(0|-1)}(\omega) = f(x) - g(x)dx \in \Omega^{(0|0)} \oplus \Omega^{(1|0)} \quad (5.16)$$

Denoting again by ΠN the even normal bundle of the embedding $\mathbb{R} \rightarrow \mathbb{R}^{(1|1)}$, acting with the Voronov *antitransform* (that we call here again \mathcal{T}) we obtain:

$$\mathcal{T}(\omega) = f(x) - g(x)dx \in \Omega^0(\Pi N) \oplus \Omega^1(\Pi N) \quad (5.17)$$

⁹For a function $f(x)$ we have $\left[\mathcal{F}\frac{df}{dx}\right](p) = ip\mathcal{F}(f)(p)$ and this is usually written as $\mathcal{F}\left(\frac{d}{dx}\right) = ip$

and

$$\mathcal{T}\left(\mathbb{Z}^{(0|-1)}(\omega)\right)(x, dx, b, db) = [f(x) - g(x)dx] \delta(b) db \in \Omega^0(\Pi N) \oplus \Omega^1(\Pi N) \quad (5.18)$$

where, as above, we allow also distributional valued differential forms. Finally, from this we get the desired *right* multiplicative representation:

$$\mathcal{T}\left(\mathbb{Z}^{(0|-1)}\right) = \delta(b) db \quad (5.19)$$

Similar considerations and computations yield the representation in more general cases.

6 Super Liouville Measure for super Kähler Manifolds.

Another interesting integral representation leads to riemannian and symplectic volumes of supermanifolds. Many examples of riemannian and symplectic volumes for supermanifolds have been also recently studied by Voronov [26]

Let us consider a bosonic compact Kähler manifold M , characterised by a Kähler potential \mathcal{K} , depending on the complex variables (Z^I) . From the Kähler potential, one extracts the Kähler 2-form

$$K^{(2)} = dZ^I \wedge d\bar{Z}^{\bar{J}} \partial_I \bar{\partial}_{\bar{J}} \mathcal{K} = g_{I\bar{J}}(Z, \bar{Z}) dZ^I \wedge d\bar{Z}^{\bar{J}}. \quad (6.1)$$

The matrix $g = (g_{I\bar{J}})$ is a $n \times n$ matrix depending on Z^I and the complex conjugate $\bar{Z}^{\bar{J}}$. The Liouville measure is given by

$$dV^{(n)} = \underbrace{K^{(2)} \wedge \cdots \wedge K^{(2)}}_n = \det(g) \prod_{I=1}^n dZ^I d\bar{Z}^{\bar{I}} \quad (6.2)$$

and $\int_M dV^{(n)} = \text{vol}(M)$ computes the symplectic volume of the Kähler manifold that coincides with the riemannian one.

This expression can be rewritten as a Berezin integral by introducing two sets of new anticommuting dual variables $\eta^I, \bar{\eta}^{\bar{I}}$ and $\xi_I, \bar{\xi}_{\bar{I}}$

$$\det(g) \prod_{I=1}^n dZ^I d\bar{Z}^{\bar{I}} = \int e^{g_{I\bar{J}}(Z, \bar{Z}) \eta^I \bar{\eta}^{\bar{J}} + \xi_I dZ^I + \bar{\xi}_{\bar{I}} d\bar{Z}^{\bar{I}}} \left[\prod_{I=1}^n d\eta^I d\bar{\eta}^{\bar{I}} d\xi_I d\bar{\xi}_{\bar{I}} \right] \quad (6.3)$$

The left hand side is a factorized expression and, under a reparametrization of the manifold, the combination of the two factors is invariant. The right hand side has the same property as it can be shown by observing that the variables η^I transform covariantly, while ξ_I transform contravariantly.

Let us move to a supermanifold \mathcal{M} with dimensions $(n|m)$. We consider a super-Kähler 2-superform $K^{(2|0)}$:

$$K^{(2|0)} = g_{I\bar{J}} dZ^I \wedge d\bar{Z}^{\bar{J}} + g_{I\bar{\beta}} dZ^I \wedge d\bar{\theta}^{\bar{\beta}} + g_{\alpha\bar{J}} d\theta^\alpha \wedge d\bar{Z}^{\bar{J}} + g_{\alpha\bar{\beta}} d\theta^\alpha \wedge d\bar{\theta}^{\bar{\beta}} \quad (6.4)$$

where the matrices $g_{I\bar{J}}, g_{I\bar{\beta}}, g_{\alpha\bar{J}}, g_{\alpha\bar{\beta}}$ form a supermatix

$$G_{A\bar{B}} = \begin{pmatrix} g_{I\bar{J}} & g_{I\bar{\beta}} \\ g_{\alpha\bar{J}} & g_{\alpha\bar{\beta}} \end{pmatrix}, \quad A = (I, \alpha), \quad \bar{B} = (\bar{J}, \bar{\beta}), \quad (6.5)$$

whose entries are superfields, functions of (Z^I, θ^α) and of their conjugated.

In this case one cannot simply define the super-Liouville form as $\underbrace{K^{(2|0)} \wedge \dots \wedge K^{(2|0)}}_n$ mainly because this expression would be a (non integrable) superform of zero picture and not a top integral form. Using a generalization of the integral representation given before, one can instead define the correct super-Liouville form as:

$$\int e^{-(g_{I\bar{J}}\eta^I\bar{\eta}^{\bar{J}}+g_{I\bar{\beta}}\eta^I\bar{\theta}^{\bar{\beta}}+g_{\alpha\bar{J}}b^\alpha\bar{\eta}^{\bar{J}}+g_{\alpha\bar{\beta}}b^\alpha\bar{\theta}^{\bar{\beta}})+\xi_I dZ^I+\bar{\xi}_{\bar{I}} d\bar{Z}^{\bar{I}}+c_\alpha d\theta^\alpha+\bar{c}_{\bar{\alpha}} d\bar{\theta}^{\bar{\alpha}}} \left[\prod_{I=1}^n \prod_{\alpha=1}^m d\eta^I d\bar{\eta}^{\bar{I}} db^\alpha d\bar{b}^{\bar{\alpha}} d\xi_I d\bar{\xi}_{\bar{I}} dc_\alpha d\bar{c}_{\bar{\alpha}} \right] \quad (6.6)$$

$$= \text{Sdet}(G_{A\bar{B}}) \prod_{I=1}^n dZ^I d\bar{Z}^{\bar{I}} \prod_{\alpha=1}^m \delta(d\theta^\alpha) \delta(d\bar{\theta}^{\bar{\alpha}})$$

That is a top integral form. See also Witten [27].

The integral is a Berezin integral in the variables η^I and ξ_I and it is a usual integral over the commuting variables b^α and c_α . Integrating over η^I and $\bar{\eta}^{\bar{I}}$ and over b^α and $\bar{b}^{\bar{\alpha}}$ leads to the Berezinian. The integral over $\xi_I, \bar{\xi}_{\bar{I}}$ and over $c_\alpha, \bar{c}_{\bar{\alpha}}$ produces the second and the third factor in the left hand side. Both sides are invariant under reparametrization. Notice that the Berezin integral produces the numerator of the superdeterminant while the integral over

the bosonic variables gives the denominator.

$$\text{Sdet}(G_{A\bar{B}}) = \frac{\det \left(g_{I\bar{J}} - g_{I\bar{B}}(g^{-1})^{\alpha\bar{\beta}} g_{\alpha\bar{J}} \right)}{\det (g_{\alpha\bar{\beta}})} \quad (6.7)$$

So, the Liouville measure for a Kähler supermanifold is

$$dV^{(n|m)} = \text{Sdet}(G_{A\bar{B}}) \prod_{I=1}^n dZ^I d\bar{Z}^I \prod_{\alpha=1}^m \delta(d\theta^\alpha) \delta(d\bar{\theta}^{\bar{\alpha}}) \quad (6.8)$$

The integral of this measure gives the super volume. The volume element $dV^{(n|m)}$ is an integral top form and (as in the usual bosonic case) it is equal to the superHodge dual $\star 1$ as explained in the previous paragraph.

The same procedure can clearly be implemented in the more general case of a symplectic supermanifold to produce the *symplectic* volume element, showing again the power of integral representations for generalizing to supergeometry many familiar concepts of differential geometry.

Acknowledgements

We thank Paolo Aschieri for valuable discussions.

References

- [1] L. Castellani, R. Catenacci and P. A. Grassi, *The Geometry of Supermanifolds and new Supersymmetric Actions*, Nucl. Phys. B **899**, 112 (2015) [arXiv:1503.07886 [hep-th]].
- [2] L. Castellani, R. Catenacci and P. A. Grassi, *Supergravity Actions with Integral Forms*, Nucl. Phys. B **889**, 419 (2014) [arXiv:1409.0192 [hep-th]].
- [3] L. Castellani, R. Catenacci and P. A. Grassi, *Hodge Dualities on Supermanifolds*, Nucl. Phys. B **899**, 570 (2015) [arXiv:1507.01421 [hep-th]].
- [4] Yu. I. Manin, *Gauge fields and complex geometry*, Springer, 1977 (second edition).

- [5] P. Deligne, J. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*. In Quantum fields and strings: a course for mathematicians, vol 1,2. Princeton, NJ 1996-1997.
- [6] A. Rogers, *Supermanifolds. Theory and applications*. World Scientific Publ. 2007.
- [7] C. Bartocci, U. Bruzzo, D.H. Hernandez Ruiperez, *The geometry of supermanifolds*, Kluwer Academic Publishers, Dordrecht, 1991.
- [8] E. Witten, *Notes on Supermanifolds and Integration*, [arXiv:1209.2199 [hep-th]].
- [9] T. Voronov and A. Zorich *Integration on Vector Bundles*, Functional analysis and its applications , 22, (1988) 94-103.
- [10] J. Kalkman, *BRST Model for Equivariant Cohomology and Representatives for the Equivariant Thom Class*, Commun. Math. Phys. **153** (1993) 447.
- [11] S. Majid, *Hodge Star as Braided Fourier Transform* [arXiv:1511.00190 [math.QA]].
- [12] S. Majid, *The self-representing Universe* in: Mathematical Structures of the Universe, Copernicus Center (2014) CRC Press, eds. M. Eckstein, M. Heller, S. Szybka. We thank the author for making us aware of this reference.
- [13] J. Bernstein, D. Leites, *Integral forms and the Stokes formula on supermanifolds*, Funct. Anal. Appl. 11 (1977), no.1, 55-56.
- [14] J. Bernstein, D. Leites, *How to integrate differential forms on supermanifolds*, Functional analysis and its applications , 11, (1977) 70-71.
- [15] F. F. Voronov, *Quantization on supermanifolds and an analytic proof of the Atiyah-Singer index theorem*, Translated in J. Soviet Math. 64 (1993), no. 4, 9931069. Original: Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Nov. Dostizh., 38, VINITI, Moscow, 1990, 3-118 .

- [16] T. Voronov, *Geometric integration theory on supermanifolds*. Soviet Scientific Review, Section C: Mathematical Physics, 9, Part 1. Harwood Academic Publisher, Chur. 1991 (Second edition 2014).
- [17] T. Voronov, *Quantization of Forms on the Cotangent Bundle*, Communications in Mathematical Physics, Volume 205, Issue 2, pp. 315-336 (1999), [arXiv:math/9809130 [math.DG]].
- [18] F.A.Berezin, *Differential forms on supermanifolds*, Soviet J. Nuclear Phys. 30 (1979) 605-609.
- [19] T. Voronov, *Quantization of forms on the cotangent bundle*, Comm. Math. Phys. 205 (1999), no.2, 315-336.
- [20] D. Friedan, E. J. Martinec and S. H. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, Nucl. Phys. B **271** (1986) 93.
- [21] A. Belopolsky, *New geometrical approach to superstrings*, [arXiv:hep-th/9703183].
- [22] A. Belopolsky, *Picture changing operators in supergeometry and superstring theory*, [arXiv:hep-th/9706033].
- [23] A. Belopolsky, *De Rham Cohomology of the Supermanifolds and Superstring BRST Cohomology*, Phys.Lett. B403 (1997), 47-50.
- [24] L. Castellani, R.Catenacci and P.A.Grassi, *The Integral Form of Supergravity*, arXiv:1607.05193 [hep-th].
- [25] R. Catenacci, M. Debernardi, P. A. Grassi and D. Matessi, *Čech and de Rham Cohomology of Integral Forms*, J. Geom. Phys. **62** (2012) 890 [arXiv:1003.2506 [math-ph]].
- [26] T. Voronov, *On Volumes of Classical Supermanifolds* [arXiv:1503.06542v1 [math.DG]]
- [27] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. **252** (2004) 189 [hep-th/0312171].