# The Hodge Operator Revisited

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**Abstract**

We present a new construction for the Hodge operator for differential manifolds based on a Fourier (Berezin)-integral representation. We find a simple formula for the Hodge dual of the wedge product of differential forms, using the (Berezin)-convolution. The present analysis is easily extended to supergeometry and to non-commutative geometry.

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# 1 Introduction

Pursuing the construction of supersymmetric Lagrangians based in the framework of supermanifold geometry, we proposed in [1] a new *Hodge operator* $\star$ acting on (super)differential forms. For that aim, we have developed a complete formalism (integral-, pseudo- e superforms, their complexes and the integration theory) in a series of papers [2, 1, 3] together with a suitable Hodge operator. As a byproduct, this new mathematical tool sheds also a new light on the Hodge operator in conventional differential geometry. The present work illustrates this new point of view.

The Hodge operator plays an essential role in differential geometry, yielding a fundamental relation between the exterior bundle $\Omega^\bullet(M)$ of differential forms and the scalar product $(\bullet, \bullet)$ on the manifold. The construction requires the existence of a metric $g$ on the the manifold $\mathcal{M}$ and is an involutive operation $\star$ which satisfies the linearity condition $\star(f\omega) = f \star \omega$ with $\omega$ a given $p$-form of $\Omega^\bullet(M)$.

In the case of supermanifolds (we refer for ex. to [1] for the basic ingredients of supergeometry; see also [4] and for a recent extensive review see [5]), the definition of the Hodge dual turns out to be harder than expected since one has to deal with the infinite-dimensional complexes of superforms. The integral forms and pseudo-forms are crucial to establish the correct matching of elements between the different spaces of forms. This new type of differential forms requires the enlargement of the conventional space spanned by the fundamental 1-forms, admitting distribution-like expressions (essentially, Dirac delta functions and Heaviside step functions). This has triggered us to consider the *Fourier analysis* for differential forms (this was also considered in [6]), and leads to an integral Fourier representation of the Hodge operator as explained in [1, 3]. Such a representation can first be established in the case of a conventional manifold $M$ without any reference to supermanifolds, except for the notion of *Berezin integral*. A new set of anticommuting variables playing the role of dual variables to fundamental 1-forms $dx^i$ are introduced and the Hodge operator is defined by a suitable Berezin integration on the new variables. The result is proven to coincide with the usual Hodge operator. When extended to supermanifolds, our construction yields a "good" definition of Hodge operator, satisfying all desired properties.

As is well known, in conventional Fourier analysis, the Fourier transform of a product of two functions is the *convolution integral* of the Fourier transform of the two functions. In a reciprocal way, the Fourier transform of the convolution integral of two functions can be expressed as the product of the Fourier transforms of the latter. This simple formula can be imported in our framework where the Fourier transform represents the Hodge operator and the convolution integral is a suitable Berezin integral of two differential forms. With this observation we are able to express the Hodge dual of the wedge product of two differential forms as the (Berezin)-convolution of the Hodge duals of the differential forms. In the present paper, we show the consistency of these definitions, derive their properties and make contact with the conventional results.

Finally, our Fourier (Berezin) integral representation of the Hodge dual operator can be extended to noncommutative spaces. A very recent work on this appeared in [7].

The letter is organized as follows: in Sec. 2, we review the integration theory of differential forms as Berezin integration. In Sec. 3 we give the integral Fourier representation of the Hodge dual. In Sec. 4, we present the Berezin-convolution and the Hodge dual of the product of differential forms.

## 2 Forms and Integration

The usual integration theory of differential forms for bosonic manifolds can be conveniently rephrased to uncover its relation with Berezin integration.

We start with a simple example: consider in $\mathbb{R}$ the integrable 1-form $\omega = g(x)dx$ (with $g(x)$ an integrable function in $\mathbb{R}$ ). We have:

$$\int_{\mathbb{R}} \omega = \int_{-\infty}^{+\infty} g(x)dx \,.$$

Observing that $dx$ is an anticommuting quantity, and denoting it by $\psi$, we could think of $\omega$ as a function on the superspace $\mathbb{R}^{1|1}$:

$$\omega = g(x)dx = f(x, \psi) = g(x)\psi \tag{2.1}$$

This function can be integrated *à la* Berezin reproducing the usual definition:

$$\int_{\mathbb{R}^{1|1}} f(x,\psi)[dxd\psi] = \int_{-\infty}^{+\infty} g(x)dx = \int_{\mathbb{R}} \omega$$

Note that the symbol of the formal measure $[dxd\psi]$ is written just to emphasize that we are integrating on the **two** variables $x$ and $\psi$, hence the $dx$ inside $[dxd\psi]$ is *not* identified with $\psi$.

Denoting by $M$ a bosonic orientable differentiable manifold of dimension $n$, its exterior bundle $\Omega^{\bullet}(M) = \sum_{p=0}^{n} \bigwedge^{p}(M)$ is the direct sum of $\bigwedge^{p}(M)$ (sometimes denoted also by $\Omega^{p}(M)$). A section $\omega$ of $\Omega^{p}(M)$ can be written locally as

$$\omega = \sum \omega_{i_1 \dots i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p} \tag{2.2}$$

where the coefficients $\omega_{i_1 \dots i_p}(x)$ are functions on $M$ and $i_1 < \dots < i_p$. The integral of $\omega \in \Omega^{n}(M)$ is defined as:

$$I[\omega] = \int_M \omega = \int_M \omega_{i_1 \dots i_n}(x)\, d^n x\,, \tag{2.3}$$

This opens the way to relating the integration theory of forms and the Berezin integral, by substituting every 1-form $dx^i$ with a corresponding abstract Grassmann variable denoted again with $dx^i$. A section $\omega$ of $\Omega^{\bullet}(M)$ is viewed locally as a function on a supermanifold $\mathcal{M} = T^*(M)$ with coordinates $(x^i, dx^i)$ :

$$\omega(x, dx) = \sum \omega_{i_1 \dots i_p}(x) dx^{i_1} \dots dx^{i_p}\,; \tag{2.4}$$

such functions are polynomials in $dx$'s. Supposing now that the form $\omega$ is integrable, its Berezin integral gives:

$$\int_{\mathcal{M}=T^*(M))} \omega(x, dx)[d^n x d^n(dx)] = \int_M \omega \tag{2.5}$$

# 3 The Integral Representation of the Hodge Star

In the following, for a given set $\{\xi^i\}_{i=1}^{n}$ of Grassmann variables, our definition of the Berezin integral is $\int_{\mathbb{R}^{0|n}} \xi^1 \dots \xi^n \, [d^n\xi] = 1$ and not $\int_{\mathbb{R}^{0|n}} \xi^1 \dots \xi^n \, [d^n\xi] = (-1)^{\frac{n(n-1)}{2}}$. Moreover, if $\alpha$ is a monomial expression of some anticommuting variables $\alpha^k$ not depending on the $\xi^i$, we define: $\int_{\mathbb{R}^{0|n}} \alpha\xi^1 \dots \xi^n \, [d^n\xi] = \alpha$, where the product between $\alpha$ and the $\xi^i$ is the usual $\mathbb{Z}_2$ graded wedge product in the superalgebra generated by the tensor product of the Grassmann algebra generated by the $\xi^i$ and that generated by the $\alpha^k$ : if $\mathcal{A}$ and $\mathcal{B}$ are two $\mathbb{Z}_2$-graded algebras with products $\cdot_{\mathcal{A}}$and $\cdot_{\mathcal{B}}$, the $\mathbb{Z}_2$-graded tensor product $\mathcal{A}\otimes\mathcal{B}$ is a $\mathbb{Z}_2$-graded algebra with the product (for homogeneous elements) given by :

$$(a\otimes b)\cdot_{\mathcal{A}\otimes\mathcal{B}}(a'\otimes b')=(-1)^{|a'||b|}a\cdot_{\mathcal{A}}a'\otimes b\cdot_{\mathcal{B}}b'$$

In our case the algebras are Grassmann algebras and the products $\cdot$ are wedge products. The symbols $\otimes$ and $\wedge$ will be, in general, omitted.

As observed in [1] one can obtain the usual Hodge dual in $\mathbb{R}^n$ (for a metric given by a matrix $A$ with entries $g_{ij}$ ) by means of the Fourier (Berezin)-integral transform $\mathcal{T}$ . For $\omega(x,dx)\in\Omega^k(\mathbb{R}^n)$ we have:

$$\star\omega=i^{\left(k^2-n^2\right)}\frac{\sqrt{|g|}}{g}\mathcal{T}(\omega)=i^{\left(k^2-n^2\right)}\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|n}}\omega(x,\eta')e^{idxA\eta'}[d^n\eta'] \tag{3.1}$$

where $g=\det A$ and the exponential series defining $e^{idxA\eta'}$ is written using the $\mathbb{Z}_2$ graded wedge product quoted above. The Grassmann variables $\eta'$ are defined as $\eta'=A^{-1}\eta$ where the $\eta$ are the *(parity changed)* variables dual to the $dx$. In this way the covariance properties of $\omega(x,\eta')$ are exactly those of a differential form.

For example, in $\mathbb{R}^2$ we can compute:

$$e^{idxA\eta'}=1+ig_{11}dx^1\eta'^1+ig_{21}dx^2\eta'^1+ig_{12}dx^1\eta'^2+ig_{22}dx^2\eta'^2+gdx^1dx^2\eta'^1\eta'^2 \tag{3.2}$$

and the definition (3.1) gives the usual results:

$$\begin{aligned}
\star 1 &= i^{\left(0^2-2^2\right)}\mathcal{T}(1)=\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}e^{idxA\eta'}[d^2\eta']=\sqrt{|g|}dx^1dx^2\\
\star dx^1dx^2 &= i^{\left(2^2-2^2\right)}\mathcal{T}(\eta'^1\eta'^2)=\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}\eta'^1\eta'^2e^{idxA\eta}[d^2\eta']=\frac{\sqrt{|g|}}{g}\\
\star dx^1 &= i^{\left(1^2-2^2\right)}\mathcal{T}(\eta'^1)=i^{\left(1^2-2^2\right)}\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}\eta'^1e^{idxA\eta'}[d^2\eta']=-g^{12}\sqrt{|g|}dx^1+g^{11}\sqrt{|g|}dx^2\\
\star dx^2 &= i^{\left(1^2-2^2\right)}\mathcal{T}(\eta'^2)=i^{\left(1^2-2^2\right)}\frac{\sqrt{|g|}}{g}\int_{\mathbb{R}^{0|2}}\eta'^2e^{idxA\eta'}[d^2\eta']=-g^{22}\sqrt{|g|}dx^1+g^{21}\sqrt{|g|}dx^2
\end{aligned}$$

The factor $i^{\left(k^2-n^2\right)}$ can be obtained by computing the transformation of the monomial form $dx^1dx^2...dx^k$ in the simple case $A=I$.

Noting that in the Berezin integral only the higher degree term in the $\eta$ variables is involved, and that the monomials $dx^i\eta^i$ are even objects, we find:

$$\begin{aligned}\mathcal{T}\left(dx^1...dx^k\right) &= \int_{\mathbb{R}^{0|n}} \eta^1...\eta^k e^{idx\eta}[d^n\eta] = \\ &= \int_{\mathbb{R}^{0|n}} \eta^1...\eta^k e^{i\left(\sum_{i=1}^k dx^i\eta^i + \sum_{i=k+1}^n dx^i\eta^i\right)}[d^n\eta] = \\ &= \int_{\mathbb{R}^{0|n}} \eta^1...\eta^k e^{i\sum_{i=1}^k dx^i\eta^i} e^{i\sum_{i=k+1}^n dx^i\eta^i}[d^n\eta] = \\ &= \int_{\mathbb{R}^{0|n}} \eta^1...\eta^k e^{i\sum_{i=k+1}^n dx^i\eta^i}[d^n\eta] = \\ &= \int_{\mathbb{R}^{0|n}} \frac{i^{n-k}}{(n-k)!}\eta^1...\eta^k \left(\sum_{i=k+1}^n dx^i\eta^i\right)^{n-k} [d^n\eta]\end{aligned}$$

Rearranging the monomials $dx^i\eta^i$ one obtains:

$$\left(\sum_{i=k+1}^n dx^i\eta^i\right)^{n-k} = (n-k)!\left(dx^{k+1}\eta^{k+1})(dx^{k+2}\eta^{k+2})...(dx^n\eta^n\right) =$$

$$= (n-k)!(-1)^{\frac{1}{2}(n-k)(n-k-1)}\left(dx^{k+1}dx^{k+2}...dx^n)(\eta^{k+1}\eta^{k+2}...\eta^n\right)$$

and finally:

$$\mathcal{T}\left(dx^1...dx^k\right) =$$

$$= \int_{\mathbb{R}^{0|n}} \frac{i^{n-k}}{(n-k)!}\eta^1...\eta^k\,(n-k)!(-1)^{\frac{1}{2}(n-k)(n-k-1)}\left(dx^{k+1}dx^{k+2}...dx^n)(\eta^{k+1}\eta^{k+2}...\eta^n\right)[d^n\eta] =$$

$$= \int_{\mathbb{R}^{0|n}} i^{n-k}(-1)^{\frac{1}{2}(n-k)(n-k-1)}(-1)^{k(n-k)}\left(dx^{k+1}dx^{k+2}...dx^n)(\eta^1...\eta^k)(\eta^{k+1}\eta^{k+2}...\eta^n\right)[d^n\eta] =$$

$$= i^{\left(n^2-k^2\right)}(dx^{k+1}dx^{k+2}...dx^n)$$

The computation above gives immediately:

$$i^{\left(k^2-n^2\right)}\mathcal{T}\left(dx^1...dx^k\right) = \star\left(dx^1...dx^k\right) \tag{3.3}$$

and

$$\mathcal{T}^2(\omega) = i^{\left(n^2-k^2\right)} i^{\left(k^2\right)}(\omega) = i^{n^2}(\omega) \tag{3.4}$$

yielding the usual relation:

$$\star\star\omega = i^{\left(\left(n-k\right)^2-n^2\right)} i^{\left(k^2-n^2\right)} i^{n^2}(\omega) = (-1)^{k(k-n)}(\omega) \tag{3.5}$$

# 4 Convolution Product of Forms

As for functions, one can define a convolution product between differential forms on an ordinary manifold. The starting point is the interpretation of differential forms as functions of the commuting variables $x$ and the anticommuting variables $dx$. For $\alpha \in \Omega^p(\mathbb{R}^n)$ and $\beta \in \Omega^q(\mathbb{R}^n)$, the convolution product $\bullet$ is defined using Berezin integration on the anticommuting variables:

$$\alpha \bullet \beta(x, dx) = \int_{\mathbb{R}^{0|n}} \alpha(x,\xi)\beta(x, dx-\xi)[d^n\xi] \tag{4.1}$$

where $\xi$ is an auxiliary anticommuting variable. The convolution product maps $\Omega^p \times \Omega^q \to \Omega^{p+q-n}$ [1]. To obtain (generically) non trivial results we must have $0 \leq p+q-n \leq n$. The algebra of this convolution is

$$\alpha \bullet \beta = (-1)^{\left(n^2+pq\right)} \beta \bullet \alpha$$

The convolution 'interacts" well with the integral transformation $\mathcal{T}$ defined above and the wedge product. We will consider explicitly only the standard bosonic case in which the matrix $A$ of the.previous paragraph is the identity matrix $I$.

For example, in the case $n = 4$, we can compute $\mathcal{T}(dx^1dx^2) = dx^3dx^4$ and $\mathcal{T}(dx^1) = (-i)\,dx^2dx^3dx^4$, $\mathcal{T}(dx^2) = i dx^1dx^3dx^4$. The convolution is:

$$\begin{aligned}\mathcal{T}\left(dx^1\right) \bullet \mathcal{T}\left(dx^2\right) &= \int_{\mathbb{R}^{0|4}} (-i)\,\xi^2\xi^3\xi^4\,(i)\left(dx^1-\xi^1\right)\left(dx^3-\xi^3\right)\left(dx^4-\xi^4\right)[d^4\xi] \\ &= dx^3dx^4 = \mathcal{T}\left(dx^1dx^2\right)\end{aligned}$$

[1] We must integrate generically monomials of the type $(\xi)^{p+q-k}\,(dx)^k$ and the Berezin integration selects $k = p+q-n$.

Another simple example is the case $q = n - p$ where we find:

$$i^{n^2} (-1)^p (-1)^{p(n-p)} \mathcal{T} (\alpha\beta) = \mathcal{T} (\alpha) \bullet \mathcal{T} (\beta) \tag{4.2}$$

Indeed, recalling that:

$$\mathcal{T} \left(dx^1 ... dx^p\right) = i^{\left(n^2 - p^2\right)} (dx^{p+1} dx^{p+2} ... dx^n) \tag{4.3}$$

$$\mathcal{T} \left(dx^{p+1} ... dx^n\right) = i^{\left(p^2\right)} (dx^1 dx^2 ... dx^p) \tag{4.4}$$

$$\mathcal{T} \left(dx^1 ... dx^n\right) = 1 \tag{4.5}$$

we find:

$$\mathcal{T} \left(dx^1 ... dx^p\right) \bullet \mathcal{T} \left(dx^{p+1} ... dx^n\right) = i^{n^2} \int_{\mathbb{R}^{0|n}} (\xi^{p+1} ... \xi^n)(dx^1 - \xi^1) ... (dx^p - \xi^p) \left[d^n \xi\right] = \tag{4.6}$$

$$i^{n^2} \int_{\mathbb{R}^{0|n}} (\xi^{p+1} ... \xi^n) (-1)^p \xi^1 ... \xi^p \left[d^n \xi\right] = i^{n^2} (-1)^p (-1)^{p(n-p)} \mathcal{T} \left(dx^1 dx^2 ... dx^n\right) \tag{4.7}$$

The properties of the convolution reflect on corresponding properties of the Hodge star operator. Using $\star\omega = i^{\left(k^2 - n^2\right)} \mathcal{T}(\omega)$ for $\omega(x, dx) \in \Omega^k(\mathbb{R}^n)$, we obtain a simple formula for the Hodge dual of the wedge product of forms in the case $p + q = n$:

$$\star (\alpha\beta) = (-1)^p (\star\alpha) \bullet (\star\beta) \tag{4.8}$$

Considering now the general case of a $p$-form $\alpha$ and a $q$-form $\beta$ in a $n$-dimensional space, one can prove the following relation:

$$\star (\alpha\beta) = (-1)^{n + q(n-p)} (\star\alpha) \bullet (\star\beta) \tag{4.9}$$

easily checked to be satisfied by the monomials

$$\alpha = dx^1 dx^2 ... dx^p, \quad \beta = dx^{n-q+1} dx^{n-q+2} ... dx^n \tag{4.10}$$

Indeed recall that

$$\star\alpha = dx^{p+1} ... dx^n, \quad \star \beta = (-1)^{q(n-q)} dx^1 ... dx^{n-q} \tag{4.11}$$

$$\star(\alpha\beta) = (-1)^{q(n-p-q)} dx^{p+1}...dx^{n-q} \tag{4.12}$$

Moreover, using the definition of the convolution, one finds

$$(\star\alpha) \bullet (\star\beta) = (-1)^{q(n-q)}(-1)^{p}(-1)^{n(n-q-p)}(-1)^{p(n-p)} dx^{p+1}...dx^{n-q} \tag{4.13}$$

Comparing the last two equations, relation (4.9) follows. By linearity the same relation (4.9) holds also for generic forms. Two particular cases provide nontrivial checks:

i) when $\alpha = 1 \in \Omega^0$ :

$$\begin{aligned} \star(1\beta) = (-1)^{n+qn} (\star 1) \bullet (\star\beta) &= (-1)^{n+qn} \int_{\mathbb{R}^{0|n}} (\xi^1...\xi^n) (\star\beta(dx-\xi)) [d^n\xi] \\ &= (-1)^{n+qn} (-1)^{n(n-q)} \star \beta = \star\beta \end{aligned} \tag{4.14}$$

ii) when $\beta = 1 \in \Omega^0$:

$$\begin{aligned} \star(\alpha 1) = (-1)^{n} (\star\alpha) \bullet (\star 1) &= (-1)^{n} \int_{\mathbb{R}^{0|n}} (\star\alpha)(\xi) (dx^1 - \xi^1)...(dx^n - \xi^n) [d^n\xi] \\ &= (-1)^{n} \int_{\mathbb{R}^{0|n}} (\star\alpha)(\xi + dx) (-1)^{n} \xi^1...\xi^n [d^n\xi] = \star\alpha \end{aligned} \tag{4.15}$$

where we used the traslational invariance (under $\xi \to \xi + dx$) of the Berezin integral.

Similar relations hold (modulo some suitable multiplicative coefficient depending also on the metric) for the more general integral transform that gives the Hodge dual for a generic metric $A$.

The convolution defined in the formula (4.1) could be normalized as:

$$\alpha \bullet' \beta(x, dx) = (-1)^{(n+pn+pq)} \alpha \bullet \beta \tag{4.16}$$

Where again $p$ is the degree of $\alpha$, $q$ the degree of $\beta$, and $n$ the dimension of the space.

With this normalization the formula (4.9) looks better:

$$\star(\alpha\beta) = (\star\alpha) \bullet' (\star\beta) \tag{4.17}$$

Indeed, noting that $(\star\alpha) \bullet (\star\beta) = (-1)^{n+(n-p)n+(n-p)(n-q)} (\star\alpha) \bullet' (\star\beta)$, we have:

$$\star(\alpha\beta) = (-1)^{n+q(n-p)} (-1)^{n+(n-p)n+(n-p)(n-q)} (\star\alpha) \bullet' (\star\beta) = (\star\alpha) \bullet' (\star\beta)$$

The algebra of this new convolution is:

$$\alpha \bullet' \beta = (-1)^{(n-p)(n-q)} \beta \bullet' \alpha \tag{4.18}$$

Clearly this normalized convolution product has a unit, the standard volume form $\star 1$.

As last remark, we point out that, using our Fourier representation of the Hodge dual, it is easy to deduce the standard formula

$$\alpha \wedge \star\alpha = (\alpha, \alpha) \star 1 \,, \tag{4.19}$$

where $(\cdot,\cdot)$ is the scalar product associated to the metric $g$ introduced in the previous section. Moreover, the same scalar product can be rewritten with the new convolution as

$$(\alpha, \alpha) = \alpha \bullet' \star\alpha = (-1)^{p(n-p)} \star \alpha \bullet' \alpha \tag{4.20}$$

where instead of the wedge product we have used the convolution product.

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