Super Quantum Mechanics in the Integral Form Formalism

L. Castellani, R. Catenacci and P.A. Grassi

Abstract. We reformulate Super Quantum Mechanics in the context of integral forms. This framework allows to interpolate between different actions for the same theory, connected by different choices of Picture Changing Operators (PCO). In this way we retrieve component and superspace actions, and prove their equivalence. The PCO are closed integral forms, and can be interpreted as super Poincaré duals of bosonic submanifolds embedded into a supermanifold.. We use them to construct Lagrangians that are top integral forms, and therefore can be integrated on the whole supermanifold. The $D = 1$, $N = 1$ and the $D = 1$, $N = 2$ cases are studied, in a flat and in a curved supermanifold. In this formalism we also consider coupling with gauge fields, Hilbert space of quantum states and observables.

Keywords. Supersymmetry, Supermanifold, Super Quantum Mechanics.

1. Introduction

Since the invention of supersymmetry, several authors provided useful mathematical tools for its geometrical formulation, based essentially on the interpretation of supersymmetry as a coordinate transformation in fermionic directions, described by Grassmann coordinates θ. Still, there remained several problems, mostly related to integration theory on supermanifolds.

The first formulations of supersymmetric models were given in terms of a *component* action, containing bosonic and fermionic fields, and invariant under supersymmetry transformations mixing bosons and fermions.

The same dynamics can be derived in a more efficient way from an action which is *manifestly* invariant under supersymmetry. This framework is known as *superspace* approach and the various fields of the spectrum are contained in some *superfields* (or superforms). The action is obtained as an integral of products of superfields and their derivatives. In this approach the set of coordinates x of the worldvolume is augmented by a set of fermionic

coordinates θ, and a superfield is a function of x and θ. These coordinates parametrize an open set of a supermanifold (which locally is denoted by $\mathbb{R}^{(n|m)}$) which is a generalization of a differential manifold. In the section 2 we summarize the theory of supermanifolds, on which a vast literature exists (see for ex. [1, 2, 3, 4, 6]). Fermionic derivatives are needed in this context, and they form an algebra representing the supersymmetry algebra. A supersymmetry variation of a superfield is obtained by means of a differential operator, representing the supersymmetry generators on the ring of superfunctions on the supermanifold. In this framework, the action is manifestly supersymmetric since the supersymmetry variation of the Lagrangian is a total derivative on superspace, and its integral vanishes. Although the superspace framework has several advantages w.r.t. to the component formalism, the geometry behind it still needs some clarification.

Motivated by string theory (both in RNS formalism [23, 24] and in Pure Spinor formalism [25]) new geometrical elements, known as integral forms, were introduced. They are essential to provide a sensible theory of geometric integration for supermanifolds and they are the natural generalizations of differential forms of a conventional manifold. Their properties and their integration theory are briefly described in the text and we refer to the literature [22, 23, 26, 7] and the book by Voronov [5]) for more details.

Once integration on supermanifolds has been established on a sound geometrical basis, we can finally rewrite the action in the component formalism and the action in superspace as different representations of the same geometrical action. This is achieved by constructing an interpolating action, known in the literature as a *rheonomic* action (see the main reference [29]). The Picture Changing Operators can be interpreted as integral forms $\mathbb{Y}$ representing the super Poincaré dual of the embedding of a bosonic submanifold into a supermanifold, and are used to construct a Lagrangian (a n-superform multiplied by a PCO $\mathbb{Y}$ to give a top integral form) that can be integrated on the whole supermanifold. It turns out that by choosing different Poincaré duals $\mathbb{Y}$ one can interpolate between different equivalent actions.

To illustrate these features, we consider in this paper the simple example of Super Quantum Mechanics, viewed as a $D = 1$ quantum field theory. The application of the formalism of integral forms to theories in higher dimensions will be the subject of a forthcoming paper. The case of $D = 3$ $N = 1$ supergravity was analyzed in [28]). We consider both $N = 1$ and the $N = 2$ cases, since they have different characteristics worth to be described. First, we build the rheonomic action (which was not present in the literature), then we show how the different choices of $\mathbb{Y}$ interpolate between the different realisations (component action or superspace action). In the case of SQM everything is clear and easy to compute and provides a perfect introductory example for the use of these techniques. In the last section, we also argue that the observables of the theory share the same properties of the action and that also for them one can use different representations corresponding to suitable $\mathbb{Y}$.

The paper is structured as follows: in sec.2 we collect some introductory material about supermanifold theory and we give also a few mathematical details about the super particle model. This section can be skipped by experts on supermanifold theory. In sec.3, we review the most important points of the integration theory of integral forms. In sec.4 we discuss some of the properties of the Picture Changing Operators. In sec.5 we study the model of SQM $N = 1$ and in sec.6 the model $N = 2$. In sec.7 we discuss the Hilbert space and in sec.8 the observables.

2. Supermanifolds and the Supersymmetric point particle.

We give in this introductory section a very short review of the definitions and the concepts of the theory of supermanifolds. The definitions and the notations are mainly taken from [1] and [2] to which we refer for a more complete treatment.

This section also contains some comments and some examples that might help to gain intuition on the topic.

A **supercommutative** ring is a $\mathbb{Z}_2$-graded ring $A = A_0 \oplus A_1$ such that if $i, j \in \mathbb{Z}_2$, then $a_i a_j \in A_{i+j}$ and $a_i a_j = (-1)^{i+j} a_j a_i$, where $a_k \in A_k$. Elements in A_0 (resp. A_1) are called **even** (resp. **odd**).

A **super ringed space** is a topological space X together with a sheaf $\mathcal{O}_X$ of supercommutative rings. If the stalks are local[1] rings, the super ringed space is called a **superspace**.

A **superdomain** $U^{n|m}$ is the superspace $\left(U^n, \mathcal{C}^{\infty n|m}\right)$, where $U^n \subseteq \mathbb{R}^n$ is open and $\mathcal{C}^{\infty n|m}$ is the sheaf of supercommutative rings given by:

$$V \mapsto \mathcal{C}^\infty \left(V\right) \left[\theta^1, \theta^2, ..., \theta^m\right],$$

where $V \subseteq U^n$ is open and $\theta^1, \theta^2, ..., \theta^m$ are the generators of a Grassmann algebra. The grading is the natural $\mathbb{Z}_2$ grading in even and odd elements.

Every element of $\mathcal{C}^{\infty n|m}\left(V\right)$ is called a **superfunction** and may be written as $\sum_I f_I \theta^I$, where I is a multi-index and $f_I \in \mathcal{C}^\infty \left(V\right)$ is an ordinary function.

A smooth **supermanifold** $\mathcal{M}$ of dimension $n|m$ is a superspace locally isomorphic to the superspace $U^{n|m}$. In this section we will denote by $\mathcal{M} \equiv (M, \mathcal{O}_M)$ the supermanifold whose underlying topological space is M and whose sheaf of supercommutative rings is $\mathcal{O}_M$. In the following sections the supermanifold will be denoted by $\mathcal{M}^{(n|m)}$.

This definition means that given a point $x \in M$ (the underlying topological space) there is an open set $U \subset M$ such that U is homeomorphic to $U_0 \subset \mathbb{R}^n$ and $\mathcal{O}_M|_U$ is isomorphic to $\mathcal{C}^{\infty n|m}|_{U_0}$. The coordinates x^i of U^n are called even (or bosonic) coordinates, while the elements θ^α are called odd (or fermionic) coordinates.

This definition has a difficulty that arises because, in order to use supermanifolds in physical applications, we would like to think in terms of points

---

[1] A ring is called local if it has a unique maximal ideal.

and functions, but ordinary (topological) points here are only the points of the topological space M , and the (super)functions are really sections of the sheaf.

To a section s of $\mathcal{O}_M$ on an open set U of M containing x :

$$s : U \to \mathcal{O}_M|_U$$

one can associate its **value** in x as the unique real number $s^\sim(x)$ such that $s - s^\sim(x)$ is not invertible on every neighborhood of x contained in U.

The sheaf of algebras $\mathcal{O}^\sim$, whose sections are the functions $s^\sim$, defines the structural sheaf of an ordinary differentiable manifold on the space M. This manifold is called the **reduced manifold** $\mathcal{M}_{\mathrm{red}}$ of the supermanifold $\mathcal{M}$. The **points** of $\mathcal{M}$ are just the ordinary points of $\mathcal{M}_{\mathrm{red}}$.

For example, in the simple case of the supermanifold $\mathbb{R}^{1|1} = (\mathbb{R}, \mathcal{C}^\infty(\mathbb{R})[\theta])$, the points are the ordinary points of $\mathbb{R}$. The global section $s = x\theta$ is nilpotent and for any real number $a \neq 0$, $x\theta - a$ is invertible (its inverse being $-a^{-2}x\theta - a^{-1}$) and hence the value of s is **zero** at any point $x \in \mathbb{R}^{1|1}_{\mathrm{red}} = \mathbb{R}$. In other terms the **value** at the point x of a generic section $\sigma = f + g\theta$ is simply $f(x)$, the value in x of the ordinary real function f. This means that the topological points cannot see the nilpotent objects because we cannot reconstruct a section from its values at the topological points, and this is not what is needed to support the intuition in physical applications, because we would like to give a meaning to odd functions.

We can consider also the case of the purely odd supermanifold $\mathbb{R}^{0|1} = \left(\mathbb{R}^0, \mathcal{C}^\infty(\mathbb{R}^0)[\theta]\right)$. The ring of "smooth" functions is simply $\mathbb{R}[\theta]/\theta^2$. The reduced manifold is just a single point[2]. If we want to analyze "geometrically" $\mathbb{R}^{0|1}$ we must study the maps from $\mathbb{R}^{0|1}$ to an ordinary manifold M. These maps can be represented by ring omomorphisms going in the opposite direction:

$$\mathcal{C}^\infty(M) \to \mathbb{R}[\theta]/\theta^2$$

A generic homomorphism is given by $f \to A(f) + B(f)\theta$. This gives two equations, the first one is $A(fg) = A(f)A(g)$ from which we conclude that $A(f) = f(m)$ with m a point of M. The second one is $B(fg) = B(f)g(m) + B(g)f(m)$ which states that $B(f)$ is a derivation over functions and hence is given by a tangent vector at m: $B(f) = \xi_m(f)$. We can describe $\mathbb{R}^{0|1}$ in M as a point with a set of tangent vectors or, more abstractly, as a "nilpotent cloud" surrounding a single abstract point.

Maps in the opposite direction, from an ordinary manifold M to $\mathbb{R}^{0|1}$ are given by homomorphisms:

$$\mathbb{R}[\theta]/\theta^2 \to \mathcal{C}^\infty(M)$$

We have that θ must go to zero (because $\mathcal{C}^\infty(M)$ has no nilpotents) and hence any such map simply maps the manifold M to the single point in $\mathbb{R}^{0|1}$

---

[2]The ring $\mathbb{R}[\theta]/\theta^2$ has only one prime (and maximal) ideal (the ideal generated by θ) and hence we verify that the set of points of $\mathbb{R}^{0|1}$ is the spectrum of the ring of smooth functions.

and we see again that we cannot give a meaning to odd functions. The same argument holds true also in the general case of $\mathbb{R}^{0|m}$.

This problem can be solved using the idea of the **functor of points** which is the formalization of the concept of *auxiliary fermionic parameters* often used in physical applications.

This functor can be used for giving a definite meaning to the elusive concept of odd functions i.e. "classical fermions" (following the terminology of [3])

We now wish to explain how the intuitive geometrical interpretation of the x^i's as "even coordinates" and the θ^α's as "odd coordinates" can be obtained from the super ringed space definition of supermanifolds through the concept of **functor of points**.

Given two supermanifolds $\mathcal{M}$ and $\mathcal{S}$, the $\mathcal{S}$-points of $\mathcal{M}$ (or the points of $\mathcal{M}$ parametrized by $\mathcal{S}$) are given by the set

$$\mathcal{M}(\mathcal{S}) = \mathrm{Hom}(\mathcal{S}, \mathcal{M}) = \{\text{set of morphisms } \mathcal{S} \to \mathcal{M}\}.$$

$\mathcal{M}$ is the supermanifold we want to describe and $\mathcal{S}$ is the model on which we base the description of $\mathcal{M}$. Changing $\mathcal{S}$ modifies the description of $\mathcal{M}$. The functor which associates $\mathcal{S}$ to $\mathcal{M}(\mathcal{S})$ is a functor between the category of supermanifolds and the category of sets.

The set of morphisms $\mathcal{S} \to \mathcal{M}$ is, in this construction, the set of "points" of the supermanifolds. See also [4] for more details.

Let us recall now some fundamental properties of morphisms. A morphism f between two superdomains $U^{p|q}$ and $V^{r|s}$ is given by a smooth map $f^\sim : U^p \to V^r$ and a homomorphism f^* of superalgebras that respects the parity:

$$f^* : \mathcal{C}^{\infty\, r|s}(V^r) \to \mathcal{C}^{\infty\, p|q}(U^p).$$

It must satisfy the following properties:

- If $t = (x^1, \ldots, x^r)$ are coordinates on V^r, each component x^i can also be interpreted as a section of $\mathcal{C}^{\infty\, r|s}(V^r)$. If $f^i = f^*(x^i)$, then f^i is an **even** element of the algebra $\mathcal{C}^{\infty\, p|q}(U^p)$.
- If θ^α is a generator of $\mathcal{C}^{\infty\, r|s}(V^r)$, then $g^\alpha = f^*(\theta^\alpha)$ is an **odd** element of the algebra $\mathcal{C}^{\infty\, p|q}(U^p)$.
- The smooth map $f^\sim : U^p \to V^r$ is $f^\sim = (f^{1\sim}, \ldots, f^{r\sim})$, where the $f^{i\sim}$ are the values of the even elements f^i.

The following fundamental result (see for example [1]) gives the local characterizations of morphisms:

If $\phi : U^p \to V^r$ is a smooth map and f^i, g^α, with $i = 1, \ldots, r$, $\alpha = 1, \ldots, s$, are given elements of $\mathcal{C}^{\infty\, p|q}(U^p)$, with f^i even, g^α odd, and satisfying $\phi = (f^{1\sim}, \ldots, f^{r\sim})$, there exists a unique morphism $f : U^{p|q} \to V^{r|s}$ with $f^\sim = \phi$, $f^*(x^i) = f^i$ and $f^*(\theta^\alpha) = g^\alpha$.

A morphism $f \in \mathrm{Hom}(U^{p|q}, V^{r|s})$ is then uniquely determined by a choice of r even sections and s odd sections of $\mathcal{C}^{\infty\, p|q}(U^p)$, i.e. morphisms are in one to one correspondence with $(r + s)$-tuples $(f^1, \ldots, f^r, g^1, \ldots, g^s)$, where the f^i's are even and the g^α's are odd in the algebra $\mathcal{C}^{\infty\, p|q}(U^p)$. If we

denote by $\Gamma_q^0(U^p)$ and $\Gamma_q^1(U^p)$ respectively the set of even and odd sections of $C^{\infty\, p|q}(U^p)$, then the above fact is expressed as

$$\mathrm{Hom}(U^{p|q}, V^{r|s}) = (\Gamma_q^0(U^{p|q}))^r \times (\Gamma_q^1(U^{p|q}))^s.$$

Where q denotes the number of odd generators of the algebra we are considering.

In particular, if $\mathcal{S} = \mathbb{R}^{0|q}$, then

$$\mathrm{Hom}(\mathbb{R}^{0|q}, V^{r|s}) = (\Gamma_q^0)^r \times (\Gamma_q^1)^s$$

where Γ_q^0 and Γ_q^1 represent respectively the even and the odd components of a Grassmann algebra with q generators, and their cartesian powers the r "bosonic coordinates" and the s "fermionic coordinates" of $V^{r|s}$.

One could say that the super ringed space structure of $\mathcal{M}$ encodes the information of how the even and odd coordinates (x^i, θ^α) glue together, but independently on the number of generators of the underlying super algebra. The number of generators (q in the above case) can be fixed by taking a supermanifold $\mathcal{S}$ and constructing $\mathrm{Hom}(\mathcal{S}, \mathcal{M})$. This procedure is the formalization of the concept of "auxiliary fermionic parameters" often used in physical applications.

2.1. The Supersymmetric point particle.

We now describe the "supersymmetric point particle" as an example of the general theory. This one-dimensional model is simple with respect to computations, but it is not at all simple from the mathematical point of view because the naive interpretation of the supermanifold $\mathbb{R}^{(1|1)}$ as a space in which there are "points" with commuting and anticommuting coordinates (x, θ) is not adequate. The main reason is that in the naive interpretation of $\mathbb{R}^{(1|1)}$ there is only one real coordinate x and only one fermionic coordinate θ, so for supersymmetry we are forced to write down equations that apparently are not allowed or meaningful.

A supersymmetric particle is described by a map $\mathbb{R} \to V$. The space V is a real $\mathbb{Z}_2$-graded vector space $V = V^0 \oplus V^1$. We will consider the simple case in which $V^0 \oplus V^1 = \mathbb{R} \oplus \mathbb{R}^{0|1}$. The bosonic part is an ordinary function $\varphi(t) \in \mathbb{R}$, the fermionic part $\nu(t)$ must be defined using the functor of points.

Let us consider again the example of $\mathbb{R}^{0|1} = (\mathbb{R}^0, C^\infty(\mathbb{R}^0)[\theta])$; we have seen that there are no odd functions from an ordinary manifold to $\mathbb{R}^{0|1}$ (or $\mathbb{R}^{0|m}$), so we must consider instead maps from a supermanifold to $\mathbb{R}^{0|1}$. We take[3] the supermanifold $\mathbb{R}^{1|2} = (\mathbb{R}, C^\infty(\mathbb{R})[\eta^1, \eta^2])$. In terms of the functor of points description the "$\mathbb{R}^{1|2}-$ points" of $\mathbb{R}^{0|1}$ will be labelled by **zero** even sections and **one** odd section of $C^\infty(\mathbb{R}^{1|2})$. A map $\mathbb{R}^{1|2} \to \mathbb{R}^{0|1}$ is represented by a morphism

$$C^\infty(\mathbb{R}^0)[\theta] \to C^\infty(\mathbb{R})[\eta^1, \eta^2]$$

---

[3]Note that $\mathbb{R}^{1|2}$ is just the minimal choice, we could have taken $\mathbb{R}^{1|m}$ with $m \geq 2$.

given by: $\theta \rightarrow \sum f_i \eta^i$ for some functions $f_i \in \mathcal{C}^\infty (\mathbb{R})$. For any $t \in \mathbb{R}$ we can consider the family of sections ν_t of $\mathcal{C}^\infty (\mathbb{R}) [\eta^1, \eta^2]$ that we can now write:

$$\nu_t = \nu(t) = \sum f_i(t) \eta^i \tag{2.1}$$

$\nu(t)$ can be interpreted as an *odd function* that can be used like an ordinary function and might be called a *classical fermion*.

The Lagrangian of the model is the sum of a "bosonic part" and a "fermionic" one:

$$L = \left(\frac{d\varphi(t)}{dt} \right)^2 + \nu(t) \frac{d}{dt} \nu(t) \tag{2.2}$$

Note that $\nu^2 = 0$ and hence $\nu(t) \frac{d}{dt} \nu(t)$ is not a total derivative. This term is bosonic and $\neq 0$.

The supersymmetry transformations are defined as:

$$\delta_\epsilon \varphi(t) = \epsilon \nu(t), \;\; \delta_\epsilon \nu(t) = -\epsilon \frac{d\varphi(t)}{dt}. \tag{2.3}$$

Where ϵ is the constant anticommuting parameter of the supersymmetry interpreted now as a linear combination of η^1 and η^2. The Lagrangian transforms as:

$$\delta_\epsilon L = \epsilon \frac{d}{dt} \left(\frac{d\varphi(t)}{dt} \nu(t) \right) \tag{2.4}$$

We see that the action S is invariant (with suitable boundary conditions):

$$\delta_\epsilon S = \int_{\mathbb{R}} \delta_\epsilon L \, dt = 0 \tag{2.5}$$

3. Integral forms, integration theory and Poincaré duals.

The integral forms are the crucial ingredients to define a geometric integration theory for supermanifolds inheriting all good properties of differential forms integration theory in conventional (purely bosonic) geometry. In this section we briefly describe the notations and the most relevant definitions (see [23], [5] and also [7, 8, 9]).

We consider a supermanifold with n bosonic dimensions and m fermionic dimensions, denoted here and in the following by $\mathcal{M}^{(n|m)}$, locally isomorphic to the superspace $\mathbb{R}^{(n|m)}$. The local coordinates in an open set are denoted by (x^a, θ^α). A $(p|q)$ pseudoform $\omega^{(p|q)}$ has the following structure:

$$\omega^{(p|q)} = \omega(x, \theta) dx^{a_1} \dots dx^{a_r} d\theta^{\alpha_1} \dots d\theta^{\alpha_s} \delta^{(b_1)} (d\theta^{\beta_1}) \dots \delta^{(b_q)} (d\theta^{\beta_q}) \tag{3.1}$$

where, in a given monomial, the $d\theta^a$ appearing in the product are different from those appearing in the delta's $\delta(d\theta)$ and $\omega(x, \theta)$ is a set of superfields with index structure $\omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)[\beta_1 \dots \beta_q]}(x, \theta)$.

The two integer numbers p and q correspond respectively to the *form number* and the *picture number*, and they range from $-\infty$ to $+\infty$ for p and $0 \leq q \leq m$. The index b on the delta $\delta^{(b)}(d\theta^\alpha)$ denotes the degree of the derivative of the delta function with respect to its argument. The total picture

of $\omega^{(p|q)}$ corresponds to the total number of delta functions and its derivatives. We call $\omega^{(p|q)}$ a *superform* if $q = 0$ and an *integral form* if $q = m$; otherwise it is called *pseudoform*. The total form degree is given by $p = r + s - \sum_{i=1}^{i=q} b_i$ since the derivatives act effectively as negative forms and the delta functions carry zero form degree. We recall the following properties:

$$d\theta^\alpha \delta(d\theta^\alpha) = 0, \ d\delta^{(b)}(d\theta^\alpha) = 0, \ d\theta^\alpha \delta^{(b)}(d\theta^\alpha) = -b\delta^{(b-1)}(d\theta^\alpha), \ b > 0. \tag{3.2}$$

The index α is not summed. The indices $a_1 \ldots a_r$ and $\beta_1 \ldots \beta_q$ are anti-symmetrized, the indices $\alpha_1 \ldots \alpha_s$ are symmetrized because of the rules of the graded wedge product:

$$dx^a dx^b = -dx^b dx^a, \ dx^a d\theta^\alpha = d\theta^\alpha dx^a, \ d\theta^\alpha d\theta^\beta = d\theta^\beta d\theta^\alpha, \tag{3.3}$$

$$\delta(d\theta^\alpha)\delta(d\theta^\beta) = -\delta(d\theta^\beta)\delta(d\theta^\alpha), \tag{3.4}$$

$$dx^a \delta(d\theta^\alpha) = -\delta(d\theta^\alpha)dx^a, \ d\theta^\alpha \delta(d\theta^\beta) = \delta(d\theta^\beta)d\theta^\alpha. \tag{3.5}$$

As usual the module of $(p|q)$ pseudoforms is denoted by $\Omega^{(p|q)}$; if $q = 0$ or $q = m$ it is finitely generated.

It is possible to define the integral over the superspace $\mathbb{R}^{(n|m)}$ of an *integral top* form $\omega^{(n|m)}$ that can be written locally as:

$$\omega^{(n|m)} = f(x, \theta)dx^1 \ldots dx^n \delta(d\theta^1) \ldots \delta(d\theta^m) \tag{3.6}$$

where $f(x, \theta)$ is a superfield. By changing the 1-forms $dx^a, d\theta^\alpha$ as $dx^a \to E^a = E_m^a dx^m + E_\mu^a d\theta^\mu$ and $d\theta^\alpha \to E^\alpha = E_m^\alpha dx^m + E_\mu^\alpha d\theta^\mu$, we get

$$\omega \to \mathrm{sdet}(E) f(x, \theta)dx^1 \ldots dx^n \delta(d\theta^1) \ldots \delta(d\theta^m) \tag{3.7}$$

where $\mathrm{sdet}(E)$ is the superdeterminant of the supervielbein (E^a, E^a).

The integral form $\omega^{(n|m)}$ can be also viewed as a superfunction $\omega(x, \theta, dx, d\theta)$ on the *odd* dual[4] $T^*(\mathbb{R}^{(n|m)})$ acting superlinearly on the parity reversed tangent bundle $\Pi T(\mathbb{R}^{(n|m)})$, and its integral is defined as follows:

$$I[\omega] \equiv \int_{\mathbb{R}^{(n|m)}} \omega^{(n|m)} \equiv \int_{T^*(\mathbb{R}^{(n|m)})=\mathbb{R}^{(n+m|m+n)}} \omega(x, \theta, dx, d\theta)[dx d\theta \, d(dx)d(d\theta)] \tag{3.8}$$

where the order of the integration variables is kept fixed. The symbol $[dx d\theta \, d(dx)d(d\theta)]$ denotes the Berezin integration "measure" and it is invariant under any coordinate transformation on $\mathbb{R}^{(n|m)}$. It is a section of the *Berezinian bundle* of $T^*(\mathbb{R}^{(n|m)})$ (a super line bundle that generalizes the determinant bundle of a purely bosonic manifold). The sections of the determinant bundle transform with the determinant of the jacobian and the sections of the Berezinian with the superdeterminant of the super-Jacobian.

---

[4]In order to make contact with the standard physics literature we adopt the conventions that d is an odd operator and dx (an odd form) is dual to the even vector $\frac{\partial}{\partial x}$. The same holds for the even form $d\theta$ dual to the odd vector $\frac{\partial}{\partial \theta}$. As clearly explained for example in the appendix of the paper [27] if one introduces also the natural concept of even differential (in order to make contact with the standard definition of cotangent bundle of a manifold) our cotangent bundle (that we consider as the bundle of one-forms) should, more appropriately, be denoted by ΠT^*.

The integrations over the fermionic variables θ and dx are Berezin integrals[5], and those over the bosonic variables x and $d\theta$ are Lebesgue integrals (we assume that $\omega(x, \theta, dx, d\theta)$ has compact support in the variables x and it is a product of Dirac's delta distributions in the $d\theta$ variables). A similar approach for a superform would not be possible because the polynomial dependence on the $d\theta$ leads to a divergent integral.

As usual, this definition can be extended to supermanifolds $\mathcal{M}^{(n|m)}$ by using bosonic partitions of unity.

Note that this definition of integration is a simple generalization of the integration of differential forms. For example, if $\omega = f(x)dx$ is an integrable one form, its integral over $\mathbb{R}$ can be interpreted as a Berezin integral of the superfunction $\omega(x, dx) = f(x)dx$ on $T^*(\mathbb{R})$ (considered as a supermanifold) with respect to the bosonic variable x and the fermionic variable dx :

$$I[\omega] \equiv \int_{\mathbb{R}} \omega \equiv \int_{\mathbb{R}} f(x)dx = \int_{T^*(\mathbb{R}^{(1|0)})=\mathbb{R}^{(1|1)}} \omega(x, dx)[dxd(dx)] \qquad (3.9)$$

The symbol $[dxd(dx)]$ denotes the integration "measure" and it is invariant under any coordinate transformations on $\mathbb{R}$: for a change of coordinates in $\mathbb{R}$ given by $x = x(y)$ the super-Jacobian matrix is $\begin{pmatrix} \frac{\partial x}{\partial y} & 0 \\ 0 & \frac{\partial dx}{\partial dy} \end{pmatrix}$ whose superdeterminant is 1 $\left(\text{using sdet} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \frac{\det A}{\det B} \right)$.

See again Witten [23] for a more detailed discussion on the symbol $[dxd\theta d(dx)d(d\theta)]$ and many other important aspects of the integration theory of integral forms.

According to the previous discussion, if a superform $\omega^{(n|0)}$ with form degree n (equal to the bosonic dimension of the reduced bosonic submanifold[6] $\mathcal{M}^{(n)} \hookrightarrow \mathcal{M}^{(n|m)}$) and picture number zero is multiplied by a $(0|m)$ integral form $\gamma^{(0|m)}$, we can define the integral on the supermanifold of the product:

$$\int_{\mathcal{M}^{(n|m)}} \omega^{(n|0)} \wedge \gamma^{(0|m)}. \qquad (3.10)$$

---

[5]In the following, for a given set $\{\xi^i\}_{i=1}^n$ of Grassmann variables, our definition of the Berezin integral is $\int \xi^1...\xi^n \, [d^n\xi] = 1$ and not $\int \xi^1...\xi^n \, [d^n\xi] = (-1)^{\frac{n(n-1)}{2}}$. Moreover, if α is a monomial expression of some anticommuting variables α^k not depending on the ξ^i, we define: $\int \alpha\xi^1...\xi^n \, [d^n\xi] = \alpha$, where the product between α and the ξ^i is the usual $\mathbb{Z}_2-$graded wedge product in the superalgebra generated by the graded tensor product of the Grassmann algebra generated by the ξ^i and that generated by the α^k : if $\mathcal{A}$ and $\mathcal{B}$ are two $\mathbb{Z}_2$-graded algebras with products $\cdot_{\mathcal{A}}$ and $\cdot_{\mathcal{B}}$, the $\mathbb{Z}_2$-graded tensor product $\mathcal{A} \otimes \mathcal{B}$ is a $\mathbb{Z}_2$-graded algebra with the product (for homogeneous elements) given by :

$$(a \otimes b) \cdot_{\mathcal{A}\otimes\mathcal{B}} (a' \otimes b') = (-1)^{|a'||b|} a \cdot_{\mathcal{A}} a' \otimes b \cdot_{\mathcal{B}} b'$$

In our case the algebras are Grassmann algebras and the products $\cdot$ are wedge products. The symbols $\otimes$ and $\wedge$ will be, in general, omitted.
[6]see sec.2.

This type of integrals can be given a geometrical interpretation in terms of the reduced bosonic submanifold $\mathcal{M}^{(n)}$ of the supermanifold and the corresponding Poincaré dual.

We start with a submanifold $\mathcal{S}$ of dimension s of a differentiable manifold $\mathcal{M}$ of dimension n. We take an embedding i:

$$i : \mathcal{S} \to \mathcal{M}$$

and a compact support $s-$form $\omega \in \Omega^s(\mathcal{M})$. The **Poincaré dual** of $\mathcal{S}$ is a *closed* form $\eta_{\mathcal{S}} \in \Omega^{n-s}(\mathcal{M})$ such that:

$$I[\omega, \mathcal{S}] = \int_{\mathcal{S}} i^*\omega = \int_{\mathcal{M}} \omega \wedge \eta_{\mathcal{S}} \tag{3.11}$$

where i^* is the pull-back of forms.

If we suppose that the submanifold $\mathcal{S}$ is described locally by the vanishing of $n - s$ coordinates $t^1, \ldots, t^{n-s}$, its Poincaré dual can also be described as a *singular closed localization form*:

$$\eta_{\mathcal{S}} = \delta(t^1)...\delta(t^{n-s})dt^1 \wedge ... \wedge dt^{n-s} \tag{3.12}$$

This distribution-valued form is clearly closed (from the properties of the delta distributions $d\,\delta(t) = \delta'(t)dt$ and from $dt^i \wedge dt^i = 0$). This form belongs to $\Omega^{n-s}(\mathcal{M})$ and is constructed in such a way that it projects on the submanifold $t^1 = \cdots = t^{n-s} = 0$. Thus, by multiplying a given form $\omega \in \Omega^s(\mathcal{M})$ by $\eta_{\mathcal{S}}$, the former is restricted to those components which are not proportional to the differentials dt^i.

Observing that the Dirac δ-function of an odd variable (dt is odd if t is even) coincides with the variable itself, we rewrite $\eta_{\mathcal{S}}$ in a form that will turn out to be useful for the generalization to supermanifolds (omitting as usual wedge symbols):

$$\eta_{\mathcal{S}} = \delta(t^1)...\delta(t^{n-s})\delta(dt^1)...\delta(dt^{n-s}) \tag{3.13}$$

which heuristically corresponds to the localisation to $t^1 = \cdots = t^{n-s} = 0$ and $dt^1 = \cdots = dt^{n-s} = 0$. Note that if a submanifold $\mathcal{S}$ is described by the vanishing of $n - s$ functions $f^1(t) = \cdots = f^{n-s}(t) = 0$ the corresponding Poincaré dual $\eta_{\mathcal{S}}$ is:

$$\eta_{\mathcal{S}} = \delta(f^1)...\delta(f^{n-s})\delta(df^1)...\delta(df^{n-s}) \tag{3.14}$$

If we change (in the same homology class) the submanifold $\mathcal{S}$ to $\mathcal{S}'$, which is equivalent to change continously the embedding, the corresponding Poincaré duals $\eta_{\mathcal{S}}$ and $\eta_{\mathcal{S}'}$ differ by an *exact* form. This can be easily proved by recalling that the Poincaré duals are closed $d\eta_{\mathcal{S}} = 0$ and any variation (denoted by Δ) of $\eta_{\mathcal{S}}$ is exact:

$$\Delta\eta_{\mathcal{S}} = d\Big(\Delta f \delta(f)\Big) \tag{3.15}$$

Given the explicit expression of $\eta_{\mathcal{S}}$, it is easy to check eq. (3.15) by expanding both members assuming that the derivation Δ follows the Leibniz rule, and using also the commutation relation $d\Delta = \Delta d$. For example, in the simple case $\eta_{\mathcal{S}} = \delta(f)df$ of a single bosonic function f, we have

$\Delta\left[\delta(f)df\right] = \delta'(f)\Delta f df + \delta(f)\Delta df$, which is also equal to $d\left(\Delta f \delta(f)\right) = \Delta df \delta(f) + \Delta f \delta'(f)df$.

Using this property we can show that, if $d\omega = 0$ (in $\mathcal{M}$ since $d_{\mathcal{S}}\left(i^*\omega\right) = 0$ trivially in $\mathcal{S}$), then the integral does not depend on the embedding of the submanifold. Indeed varying the embedding amounts to vary the Poincaré dual, so that the variation of the integral reads

$$\Delta I[\omega,\mathcal{S}] = I[\omega,\Delta\mathcal{S}] = \int_{\mathcal{M}}\omega\wedge\Delta\eta_{\mathcal{S}} = \int_{\mathcal{M}}\omega\wedge d\xi_{\mathcal{S}} = (-)^s\int_{\mathcal{M}}d\omega\wedge\xi_{\mathcal{S}} \quad (3.16)$$

where $\Delta\eta_{\mathcal{S}} = d\xi_{\mathcal{S}}$.

The same arguments apply in the case of supermanifolds. Consider a submanifold $\mathcal{S}^{(s|q)}$ of a supermanifold $\mathcal{M}^{(n|m)}$. We take an embedding i :

$$i : \mathcal{S}^{(s|q)} \to \mathcal{M}^{(n|m)}$$

and an integral form $\omega \in \Omega^{s|q}(\mathcal{M}^{(n|m)})$ (integrable in the sense of superinte-gration when pulled back on $\mathcal{S}^{(s|q)}$). The **Poincaré dual** of $\mathcal{S}^{(s|q)}$ is a d-*closed* form $\eta_{\mathcal{S}} \in \Omega^{n-s|m-q}(\mathcal{M}^{(n|m)})$ such that:

$$\int_{\mathcal{S}^{(s|q)}}i^*\omega = \int_{\mathcal{M}^{(n|m)}}\omega\wedge\eta_{\mathcal{S}} \quad (3.17)$$

Again we can write:

$$\eta_{s} = \delta(f^1)...\delta(f^{\cdots})\delta(df^1)...\delta(df^{\cdots}) \quad (3.18)$$

where the f's are the functions defining (at least locally) the submanifold $\mathcal{S}^{(s|q)}$. Here some of them are even functions and some of them are odd functions. The Poincaré dual is a closed integral form that, in general, if written explicitly in the coordinates (x,θ), contains delta forms and their derivatives.[7]

Again it is easy to check, for example in the simple case of a single f fermionic, that any variation of η_{S} is d-exact:

$$\Delta\eta_S = d\left((\Delta f)f\delta'(df)\right) \quad (3.19)$$

Note that the two formulae (3.15) and (3.19) for the variation of η_S can be combined in a formula that holds true in both cases:

$$\Delta\eta_S = d\left(\Delta f\delta(f)\delta'(df)\right) \quad (3.20)$$

Indeed, one has $\delta'(df) = 1$ or $\delta(f) = f$ when f is respectively bosonic or fermionic.

---

[7]We recall that the modules of integral forms are constructed in terms of compact-support distributions of $d\theta$'s and its derivatives. Therefore a PCO could in principle contain also the derivatives of Dirac delta forms $\delta(d\theta)$. In the forthcoming sections, we will illustrate this point with explicit examples of PCO's built with derivatives of delta forms. Note that, instead, the Heaviside (step) function $\Theta(d\theta)$ is not an admissible distribution for an integral form.

If we take now an embedding i of the reduced bosonic submanifold $\mathcal{M}^{(n)} \overset{i}{\hookrightarrow} \mathcal{M}^{(n|m)}$ and a representative of its Poincaré dual $\mathbb{Y}^{(0|m)}$, we have:

$$\int_{\mathcal{M}^{(n|m)}} \omega^{(n|0)} \wedge \mathbb{Y}^{(0|m)} = \int_{\mathcal{M}^{(n)}} i^* \omega^{(n|0)} \tag{3.21}$$

The "standard" embedding is given by $\theta^\alpha = 0$ for all α. The corresponding standard Poincaré dual is $\mathbb{Y}_{st}^{(0|m)} = \theta^1 ... \theta^m \delta(d\theta^1) ... \delta(d\theta^m)$.

If $\omega^{(n|0)}$ is a *closed* form, this integral is unchanged if we modify the embedding by adding an exact term to $\mathbb{Y}^{(0|m)}$. This fact will be used to change the standard Poincaré dual into another with manifest supersymmetry.

For rigid supersymmetric models, the closed form $\omega^{(n|0)}$ is the Lagrangian of the model $\mathcal{L}^{(n|0)}(\Phi, V, \psi)$ built using the rheonomic rules (see [29]) and it contains the dynamical fields Φ (each dynamical field is promoted to a superfield) and the rigid supervielbeins $V^a = dx^a + \theta \gamma^a d\theta$, $\psi^a = d\theta^a$ satisfying the Maurer-Cartan equations

$$dV^a = \psi \gamma^a \psi, \quad d\psi^a = 0. \tag{3.22}$$

In the present formula, we have used real Majorana spinors.

On the other side the Poincaré dual forms $\mathbb{Y}^{(0|m)}$, called *Picture Changing Operators* (PCO's) in string theory literature (see [22] for details), contains only geometric data (for instance the supervielbeins or the coordinates themselves).

For rigid supersymmetric models we have

$$S_{rig} = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)}(\Phi, V, \psi) \wedge \mathbb{Y}^{(0|m)}(V, \psi) \tag{3.23}$$

with $d\mathcal{L}^{(n|0)}(\Phi, V, \psi) = 0$ in order to be able to freely change the PCO by exact terms, without changing S_{rig}.

In the case of supergravity, the supervielbeins V^a and ψ^α are promoted to dynamical fields (E^a, E^α) and therefore the action becomes

$$S_{sugra} = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)}(\Phi, E) \wedge \mathbb{Y}^{(0|m)}(E). \tag{3.24}$$

The closure of the action and of the PCO's implies the conventional constraints for supergravity, reducing the independent fields to the physical fields.

4. PCO's and their Algebraic Properties.

In this section we recall a few definitions and useful computations about the PCO's in our notations. For more details see [22] and [26].

We start with the *Picture Lowering Operators* that map cohomology classes in picture q to cohomology classes in picture $r < q$.

Given an integral form, we can obtain a superform by acting on it with operators decreasing the picture number. Consider the following operator:

$$\delta(\iota_D) = \int_{-\infty}^{\infty} \exp\left(it\iota_D\right) dt \tag{4.1}$$

where D is an odd vector field on $T(\mathcal{SM})$ with $\{D, D\} \neq 0$ [8] and ι_D is the contraction along the vector D. The contraction ι_D is an even operator.

For example, if we decompose D on a basis $D = D^\alpha \partial_{\theta^\alpha}$, where the D^α are even coefficients and $\{\partial_{\theta^\alpha}\}$ is a basis of the odd vector fields, and take $\omega = \omega_\beta d\theta^\beta \in \Omega^{(1|0)}$, we have

$$\iota_D \omega = D^\alpha \omega_\alpha = D^\alpha \frac{\partial \omega}{\partial d\theta^\alpha} \in \Omega^{(0|0)} . \tag{4.2}$$

In addition, due to $\{D, D\} \neq 0$, we have also that $\iota_D^2 \neq 0$. The differential operator $\delta(\iota_\alpha) \equiv \delta(\iota_D)$ – with $D = \partial_{\theta^\alpha}$ – acts on the space of integral forms as follows (we neglect the possible introduction of derivatives of delta forms, but that generalization can be easily done):

$$\delta(\iota_\alpha) \prod_{\beta=1}^{m} \delta(d\theta^\beta) = \pm \int_{-\infty}^{\infty} \exp\left(it\iota_\alpha\right) \delta(d\theta^\alpha) \prod_{\beta=1\neq\alpha}^{m} \delta(d\theta^\beta) dt \tag{4.3}$$

$$= \pm \int_{-\infty}^{\infty} \delta(d\theta^\alpha + it) \prod_{\beta=1\neq\alpha}^{m} \delta(d\theta^\beta) dt = \mp i \prod_{\beta=1\neq\alpha}^{m} \delta(d\theta^\beta)$$

where the sign $\pm$ is due to the anticommutativity of the delta forms and it depends on the index α. We have used also the fact that $\exp\left(it\iota_\alpha\right)$ represents a finite translation of $d\theta^\alpha$. The result contains $m-1$ delta forms, and therefore it has picture $m - 1$. It follows that $\delta(\iota_\alpha)$ is an odd operator.

We can define also the Heaviside step operator $\Theta\left(\iota_D\right)$:

$$\Theta\left(\iota_D\right) = \lim_{\epsilon \to 0^+} -i \int_{-\infty}^{\infty} \frac{1}{t - i\epsilon} \exp\left(it\iota_D\right) dt \tag{4.4}$$

The operators $\delta\left(\iota_D\right)$ and $\Theta\left(\iota_D\right)$ have the usual formal distributional properties: $\iota_D \delta(\iota_D) = 0$, $\iota_D \delta'(\iota_D) = -\delta(\iota_D)$ and $\iota_D \Theta\left(\iota_D\right) = \delta(\iota_D)$.

In order to map cohomology classes into cohomology classes decreasing the picture number, we introduce the operator (see [22]):

$$Z_D = [d, \Theta\left(\iota_D\right)] \tag{4.5}$$

In the simplest case $D = \partial_{\theta^\alpha}$ we have:

$$Z_{\partial_{\theta^\alpha}} = i\delta(\iota_\alpha)\partial_{\theta^\alpha} \equiv Z_\alpha \tag{4.6}$$

The operator Z_α is the composition of two operators acting on different quantities: ∂_{θ^α} acts only on functions, and $\delta(\iota_\alpha)$ acts only on delta forms.

In order to further reduce the picture we simply iterate operators of type Z.

In the simple case of $\mathbb{R}^{(1|1)}$ the operator $Z_1 \equiv Z$ acts on the spaces $\Omega^{(0|1)}$ and $\Omega^{(1|1)}$ producing elements of $\Omega^{(0|0)}$ and $\Omega^{(1|0)}$ respectively.

A generic form $\omega \in \Omega^{(0|1)} \oplus \Omega^{(1|1)}$ but $\notin \ker Z$ can be written as:

$$\omega(x, dx, \theta, d\theta) = f(x)\theta\delta(d\theta) + g(x)\theta dx\delta(d\theta). \tag{4.7}$$

because $Z\Big(\delta(d\theta)\Big) = Z\Big(dx\delta'(d\theta)\Big) = Z\Big(\theta dx\delta'(d\theta)\Big) = 0$

---

[8] Here and in the following $\{,\}$ is the anticommutator (i.e. the graded commutator).

The action of the operator Z is:

$$Z(\omega) = f(x) - g(x)dx \in \Omega^{(0|0)} \oplus \Omega^{(1|0)} \tag{4.8}$$

As explained in [23], the operator Z can be defined also in terms of "integration along the fibers". Intuitively, to remove a Dirac delta of a given $d\theta$ from an integral form, changing its picture number, it is sufficient to integrate along that coordinate.

For example, again in the simple case of $\mathbb{R}^{(1|1)}$, the transformation of coordinates generated by the vector ∂_θ is given by:

$$x \to x \tag{4.9}$$

$$\theta \to \theta + \epsilon \tag{4.10}$$

where ϵ is an auxiliary fermionic parameter (see sec. 2 for a rigorous treatment in terms of the functor of points) .

This change of coordinates maps ω to

$$\omega^* = f(x)\,(\theta + \epsilon)\,\delta(d\theta + d\epsilon) + g(x)\,(\theta + \epsilon)\,dx\delta(d\theta + d\epsilon) \tag{4.11}$$

The picture changing can be obtained integrating with respect to the variables $d\epsilon$ and ϵ :

$$Z(\omega) = \int \omega^*\,[d\,(d\epsilon)\,d\epsilon] \tag{4.12}$$

A similar description in terms of the Voronov integral transform can be found in [26].

The Z operator is in general not invertible but it is possible to find a non unique operator Y such that $Z \circ Y$ is an isomorphism in the cohomology. These operators are the called *Picture Raising Operators*. The operators of type Y are non trivial elements of the de Rham cohomology.

We apply a PCO of type Y on a given form by taking the graded wedge product; given ω in $\Omega^{(p|q)}$, we have:

$$\omega \xrightarrow{Y} \omega \wedge Y \in \Omega^{(p|q+1)}\ , \tag{4.13}$$

Notice that if $q = m$, then $\omega \wedge Y = 0$. In addition, if $d\omega = 0$ then $d(\omega \wedge Y) = 0$ (by applying the Leibniz rule), and if $\omega \neq dK$ then it follows that also $\omega \wedge Y \neq dU$ where U is a form in $\Omega^{(p-1|q+1)}$. So, given an element of the cohomogy $H^{(p|q)}$, the new form $\omega \wedge Y$ is an element of $H^{(p|q+1)}$.

For a simple example in $\mathbb{R}^{(1|1)}$ we can consider the PCO $Y = \theta\delta(d\theta)$, corresponding to the vector ∂_θ; we have $Z \circ Y = Y \circ Z = 1$

More general forms for Z and Y can be constructed, for example starting with the vector $Q = \partial_\theta + \theta\partial_x$.

The corresponding PCO of type Z can be computed observing that the transformation of coordinates generated by the vector $Q = \partial_\theta + \theta\partial_x$ is:

$$x \to x + \epsilon\theta \tag{4.14}$$

$$\theta \to \theta + \epsilon \tag{4.15}$$

If, as usual, we want to consider $\delta_\epsilon\theta = \epsilon$ as a translation in the (unique) fermionic direction θ we must conclude that $\epsilon\theta = 0$. So, if we want to give

the geometrical meaning of a translation to the transformation $\delta_\epsilon x = \epsilon\theta$ we should introduce an auxiliary Grassmann algebra with two nilpotents generators ϵ_1 and ϵ_2. In this way ϵ and θ are both interpreted, using the functor of points, as linear combinations of ϵ_1 and ϵ_2, and hence ϵ and θ are as usual anticommuting and nilpotent; moreover $\epsilon\theta$ is not a real number but it is bosonic and different from zero.

This change of coordinates maps the generic integral form $\varphi = g(x)\theta dx\delta(d\theta) \in \Omega^{(1|1)}$ to

$$\varphi^* = g(x + \epsilon\theta)\,(\theta + \epsilon)\,(dx + d\epsilon\theta - \epsilon d\theta)\,\delta(d\theta + d\epsilon) \qquad (4.16)$$

The picture changing operation can be obtained integrating over the variables $d\epsilon$ and ϵ :

$$Z_Q(\varphi) = \int \varphi^*\,[d\,(d\epsilon)\,d\epsilon] = -g(x)dx \qquad (4.17)$$

The explicit computation using instead the formula $Z = [d, \Theta(\iota_Q)]$ is:

$$Z_Q[\varphi] = d[\Theta(\iota_Q)\varphi] = d\Big[\Theta(\iota_Q)g(x)\theta dx\delta(d\theta)\Big] \qquad (4.18)$$

$$= d\Big[\lim_{\epsilon\to0^+} -i\int_{-\infty}^{\infty} \frac{1}{t - i\epsilon}g(x)\theta dx\delta(d\theta + it)dt\Big] =$$

$$= d\Big[-\frac{g(x)\theta dx}{d\theta}\Big] = -g(x)dx\,.$$

The last expression is clearly closed. Note that in the above computations (4.18) we have introduced formally the inverse of the (commuting) superform $d\theta$. Using a terminology borrowed from superstring theory we can say that, even though in a computation we need an object that lives in the *Large Hilbert Space*, the result is still in the *Small Hilbert Space*.

Note that the negative powers of the superform $d\theta$ are well defined only in the complexes of superforms (i.e. in picture 0). In this case the inverse of the $d\theta$ and its powers are closed and exact and behave with respect to the graded wedge product as *negative degree* superforms of picture 0. In picture $\neq 0$ negative powers are not defined because of the distributional relation $d\theta\delta\,(d\theta) = 0$.

An "inverse" PCO of type Y invariant under the rigid supersymmetry transformations (generated by the vector Q) $\delta_\epsilon x = \epsilon\theta$ and $\delta_\epsilon\theta = \epsilon$ is, for example, given by:

$$Y_Q = (dx + \theta d\theta)\delta'(d\theta) \qquad (4.19)$$

We have:

$$Y_Q Z_Q[\varphi] = -g(x)dx \wedge (dx + \theta d\theta)\delta'(d\theta) = g(x)\theta dx\delta(d\theta) = \varphi\,. \qquad (4.20)$$

5. Super-Quantum Mechanics.

5.1. $D = 1, N = 1$.

In the present section, we present a very special model in the lowest possible dimension $D = 1$ and $N = 1$, namely $N = 1$ super quantum mechanics[9]. This model is very useful to understand several details in more complicate theories and provides a simple and calculable example. We list all ingredients and we discuss some implications.

1. The local coordinates of the superspace $\mathbb{R}^{(1|1)}$ are denoted by (x, θ), the flat supervielbeins are given by $V = dx + \theta d\theta$, $\psi = d\theta$, satisfying the usual Maurer-Cartan algebra

$$dV = \psi^2 \,, \ d\psi = 0 \,. \tag{5.1}$$

The covariant derivative D and the supersymmetry generator Q are

$$D = \partial_\theta - \theta \partial_x \,, \ Q = \partial_\theta + \theta \partial_x \,, \tag{5.2}$$

with the algebra

$$D^2 = \frac{1}{2}\{D, D\} = -\partial_x \,, \ Q^2 = \frac{1}{2}\{Q, Q\} = \partial_x \,, \ \{Q, D\} = 0 \,. \tag{5.3}$$

2. To construct a Lagrangian we need superfields. The supermultiplet in this simplified framework is composed by a single boson and a single fermion. Then, we can easily arrange them into a single scalar superfield Φ. If we denote by (ϕ, λ) its component fields, we have

$$\Phi(x, \theta) = \phi(x) + \theta \lambda(x) \,, \tag{5.4}$$
$$W(x, \theta) = D\Phi = \lambda(x) - \theta \partial_x \phi(x) \,, \tag{5.5}$$
$$F(x, \theta) = D\,W(x, \theta) = D^2\Phi = -\partial_x \Phi = -\partial_x \phi(x) - \theta \partial_x \lambda(x) \,. \tag{5.6}$$

Together with the superfield Φ we have also some derived superfields such as W and F. W plays the rôle of a fermionic superfield (the superfield whose first component is the physical fermion λ).

3. Supersymmetry. If we denote by ϵ the constant anticommuting parameter of supersymmetry, we have

$$\delta_\epsilon \Phi = \epsilon Q\Phi = \epsilon(\lambda + \theta \partial_x \phi) \,, \tag{5.7}$$

from which we deduce the supersymmetry variations of the component fields

$$\delta_\epsilon \phi = \epsilon \lambda \,, \ \delta_\epsilon \lambda = -\partial_x \phi \epsilon \,. \tag{5.8}$$

These transformations will be used to check the invariance of the action and the covariance of the equations of motion.

---

[9]See sec.2.1 for a mathematical introduction to the "supersymmentric point particle" and the related formalism and concepts.

4. Supersymmetric action and the equations of motion. We write the action[10] in the superspace version and then we compute the component action explicitly (the integrals are usual Berezin integrals):

$$S = \frac{1}{2} \int \partial_x \Phi D\Phi [dx\, d\theta] = \frac{1}{2} \int \Big((\partial_x \phi + \theta \partial_x \lambda)(\lambda - \theta \partial_x \phi) \Big) [dx\, d\theta] \qquad (5.9)$$

$$= \frac{1}{2} \int \Big(\partial_x \phi \lambda + \theta(\partial_x \lambda \lambda - (\partial_x \phi)^2) \Big) [dx\, d\theta] = -\frac{1}{2} \int_{\mathbb{R}} \Big[\lambda \partial_x \lambda + (\partial_x \phi)^2 \Big] dx\,.$$

and the equations of motion are:

$$\partial_x D\Phi = 0\,, \implies \partial_x^2 \phi = 0\,,\ \partial_x \lambda = 0\,, \qquad (5.10)$$

i.e. the Klein-Gordon and the Dirac equation. The model is very simple and there is a complete match of degrees of freedom both off-shell and on-shell. Thus we do not need auxiliary fields.

5. Let us move to the geometrical construction. For that, we extend all physical fields to superfields. In our case the field ϕ is promoted to Φ and the fermion λ is promoted to W (we adopt the same letters as above because it will turn out that they do coincide). Promoting the fields to superfields introduces more degrees of freedom, and we reduce them by imposing the differential conditions

$$d\Phi = V\partial_x \Phi + \psi W\,,\ dW = V\partial_x W - \psi \partial_x \Phi\,, \qquad (5.11)$$

Being d nilpotent, the equation for W is obtained by imposing the Bianchi identities. From these equations we immediately see that

$$D\Phi = W\,,\ DW = D^2\Phi = -\partial_x \Phi\,. \qquad (5.12)$$

6. Now, we are in a position to construct the geometric Lagrangian $\mathcal{L}^{(1|0)}$. It is built in terms of superforms, their differentials and geometrical data of the supermanifold parametrized by V, ψ. Using the Hodge dual in supermanifolds one could construct a supersymmentric action in the supermanifold using a Lagrangian in picture 1 of type $\mathcal{L}^{(1|1)}$ (see section 3.2.2 of the paper [9]). For constructing instead a geometrical Lagrangian in picture 0, we need an additional $(0|0)$-form superfield ξ (first order formalism) and we have

$$\mathcal{L}^{(1|0)} = \frac{1}{2}\xi^2 V + \xi(d\Phi - \psi W) + \frac{1}{2}W dW\,, \qquad (5.13)$$

from which we compute the equations of motion

$$\xi V + d\Phi - \psi W = 0\,,\ d\xi = 0\,,\ -\xi\psi + dW = 0\,. \qquad (5.14)$$

The rheonomic action is built according to the rules presented in [29]: inspired by the kinetic terms of the component action, promoting all fields to superfields and adding those terms allowed by scaling dimensions, form degree and other quantum numbers. Then, imposing the d-closure one fixes the coefficients. The equations stemming from that

---

[10] The action $S = \frac{1}{2}\int \partial_x \Phi D\Phi [dx\, d\theta]$ is manifestly invariant under the supersymmetry transformation 4.14 and 4.15.

action should reproduce both the differential conditions (5.11) and the equations of motion (5.10).

By expanding $d\Phi = V\partial_x\Phi + \psi D\Phi$ and $dW = V\partial_x W + \psi D\partial_x W$, we have

$$\xi = -\partial_x\Phi\,,\ \ D\Phi = W\,,\ \ \partial_x\xi = 0\,,\ \ \partial_x W = 0\,,\ \ DW = \xi\,. \tag{5.15}$$

so that by consistency

$$\partial_x^2\Phi = 0\,,\ \ DW = -\partial_x\Phi\,.$$

These equations are the complete set of conditions and equations of motion. For the convenience of reader, we also decompose the action into the V-dependent part and the ψ-dependent part

$$\mathcal{L}^{(1|0)} = \left(\frac{1}{2}\xi^2 + \xi\partial_x\Phi - \frac{1}{2}W\partial_x W\right)V + \left(\xi(D\Phi - W) + \frac{1}{2}WDW\right)\psi\,. \tag{5.16}$$

7. Closure of the action. As discussed in sec.3, in order to be able to choose freely the appropriate PCO, the action must be closed under d. Notice that in a supermanifold this statement is not trivial. We have

$$d\mathcal{L}^{(1|0)} = d\xi \wedge (\xi V + d\Phi - \psi W) + \frac{1}{2}\xi^2\psi \wedge \psi - \xi dW \wedge \psi + \frac{1}{2}dW \wedge dW \tag{5.17}$$

$$= d\xi \wedge (\xi V + d\Phi - \psi W) + \frac{1}{2}\left(\xi\psi - dW\right) \wedge \left(\xi\psi - dW\right)\,.$$

Using the first equation of motion, namely $\xi V + d\Phi - \psi W = 0$, we get that the first term vanishes. To prove the vanishing of the second term, one needs the last equation in (5.14). Since these equations of motion are algebraic, we can use them at the level of the action. This proves that the action is closed without using the auxiliary fields.

8. PCO's. As described in sec. 3, we construct the action integral for the supermanifold by the formula

$$S_{geo} = \int_{\mathbb{R}^{(1|1)}} \mathcal{L}^{(1|0)} \wedge \mathbb{Y}^{(0|1)}\,, \tag{5.18}$$

where $\mathbb{Y}^{(0|1)}$ is a PCO. We consider here two possible choices (where st means standard and ss means supersymmetric)

$$\mathbb{Y}_{st}^{(0|1)} = \theta\delta(\psi)\,,\ \ \mathbb{Y}_{ss}^{(0|1)} = -V\delta'(\psi)\,, \tag{5.19}$$

which are both closed and not exact. The first one is not manifestly supersymmetric, but its variation under a supersymmetry transformation is d-exact. The second one is supersymmetric invariant. The two PCO's are cohomologous:

$$\mathbb{Y}_{st}^{(0|1)} - \mathbb{Y}_{ss}^{(0|1)} = d\left(x\delta'(\psi)\right) \tag{5.20}$$

9. Component action. Choosing $\mathbb{Y}_{st}^{(0|1)}$ we have:

$$S = \int_{\mathbb{R}^{(1|1)}} \mathcal{L}^{(1|0)} \wedge \mathbb{Y}_{st}^{(0|1)} \tag{5.21}$$

$$= \int_{T^*\mathbb{R}^{(1|1)}} \left(\frac{1}{2}\xi^2 + \xi\partial_x\Phi - \frac{1}{2}W\partial_x W \right) V\theta\delta(\psi)[dxd\theta d(dx)d\psi]$$

$$= \int_{T^*\mathbb{R}^{(1|1)}} \left(\frac{1}{2}\xi_0^2 + \xi_0\partial_x\phi - \frac{1}{2}\lambda\partial_x\lambda \right) dx\theta\delta(\psi)[dxd\theta d(dx)d\psi]$$

where $\xi_0 = \xi(x,0)$ (namely the first component of superfield ξ). Now, we can integrate over θ, dx and ψ to get the final component action (5.9).

10. Superspace action. Choosing $\mathbb{Y}_{ss}^{(0|1)}$ we have to pick up the second term in (5.16), because of the derivative of the Dirac delta function. Then, we have

$$S = \int_{\mathbb{R}^{(1|1)}} \mathcal{L}^{(1|0)} \wedge \mathbb{Y}_{ss}^{(0|1)} \tag{5.22}$$

$$= -\int_{T^*\mathbb{R}^{(1|1)}} \left(\xi(D\Phi - W) + \frac{1}{2}WDW \right) \psi V\delta'(\psi)[dxd\theta d(dx)d\psi]$$

$$= \int_{T^*\mathbb{R}^{(1|1)}} \left(\xi(D\Phi - W) + \frac{1}{2}WDW \right) V\delta(\psi)[dxd\theta d(dx)d\psi] .$$

The equation of motion for ξ implies that $W = D\Phi$ and, being an algebraic equation, we can insert it back into the action and obtain

$$S = \frac{1}{2} \int_{\mathbb{R}^{(1|1)}} \partial_x\Phi D\Phi[dxd\theta] . \tag{5.23}$$

after Berezin integration over the variables dx and ψ.

We thus retrieve the superspace action.

11. Picture Lowering Operator. We use the PCO $Z = [d, \Theta(\iota_\psi)]$ where $\Theta(\iota_\psi)$ is the Heavisde function (step function) of the contraction operator ι_ψ. Notice that $[d, \iota_\psi]f = \mathcal{L}_\psi f = Df$. Then, we can write Z as $Z = D\delta(\iota_\psi)$. Let us check its action on $\mathbb{Y}_{st}^{(0|1)}$, $\mathbb{Y}_{ss}^{(0|1)}$ and on the volume form $\text{Vol}^{(1|1)} = dx\delta(\psi)$:

$$Z\left(\mathbb{Y}_{st}^{(0|1)} \right) = [d, \Theta(\iota_\psi)]\mathbb{Y}_{st}^{(0|1)} = d\left(\Theta(\iota_\psi)\theta\delta(\psi) \right) = d\left(\frac{\theta}{\psi} \right) = 1 \tag{5.24}$$

$$Z\left(\mathbb{Y}_{ss}^{(0|1)} \right) = [d, \Theta(\iota_\psi)]\mathbb{Y}_{ss}^{(0|1)} = -d\left(\Theta(\iota_\psi)V\delta'(\psi) \right) = -d\left(\frac{V}{\psi^2} \right) = -1 \tag{5.25}$$

$$Z\left(\text{Vol}^{(1|1)} \right) = [d, \Theta(\iota_\psi)]V\delta(\psi)] = d\left(\Theta(\iota_\psi)V\delta(\psi) \right) = d\left(\frac{V}{\psi} \right) = \psi \tag{5.26}$$

Notice that since the two PCO differ by an exact term, that difference drops out acting with Z. Notice also that, again, even though we need an object living in the *Large Hilbert Space*, the result of the computation still is in the *Small Hilbert Space*. In addition, using the definition of the

super Hodge dual $\star$ proposed in the paper [9], it can be easily seen that $\star\psi = \mathbb{Y}_{ss}^{(1|0)}$, or, equivalently:

$$\psi \wedge -V\delta'(\psi) = \text{Vol}^{(1|1)} . \qquad (5.27)$$

5.2. $D = 1, N = 1$ Curved.

We consider a curved $N = 1$ supermanifold $\mathcal{M}^{(1|1)}$ locally parametrized by the coordinates (x, θ). Its geometry is described by the supervielbein

$$E^v = E_x^v V + E_\theta^v \psi, \ E^\psi = E_x^\psi V + E_\theta^\psi \psi \qquad (5.28)$$

where V and ψ are the flat superspace supervielbeins. We impose the supergravity constraints

$$dE^v = E^\psi \wedge E^\psi , \ dE^\psi = 0 . \qquad (5.29)$$

which are solved by

$$E^v = (E_\theta^\psi)^2 V , \ E^\psi = E_\theta^\psi \psi + DE_\theta^\psi V . \qquad (5.30)$$

The geometrical properties of the supermanifold are encoded into the superfield $E_\theta^\psi(x, \theta)$. Since a curved supermanifold $\mathcal{M}^{(1|1)}$ is rather simple, the torsion constraints (5.29) concentrate the entire geometrical information into a single superfield [30]. This will happen also in the $\mathcal{M}^{(1|2)}$ case as will be shown later.

Because of the equations (5.29), it is easy to show that

$$\mathbb{Y}^{(0|1)} \equiv -E^v \delta'(E^\psi) \qquad (5.31)$$

is a closed integral form. Using the solution (5.30), we have

$$\mathbb{Y}^{(0|1)} = -(E_\theta^\psi)^2 V \delta'\left(E_\theta^\psi \psi + DE_\theta^\psi V\right) = -(E_\theta^\psi)^2 V \frac{1}{(E_\theta^\psi)^2} \delta'(\psi) = -V\delta'(\psi) \qquad (5.32)$$

which coincides with the flat supersymmetric PCO $\mathbb{Y}_{ss}^{(0|1)} = -V\delta'(\psi)$.

The volume form is in this case

$$\text{Vol}^{(1|1)} = E^v \delta(E^\psi) = (E_\theta^\psi)^2 V \delta\left(E_\theta^\psi \psi + DE_\theta^\psi V\right)$$

$$= E_\theta^\psi V \delta(\psi) = E_\theta^\psi \text{Vol}_{flat}^{(1|1)} = \text{Sdet}(E) \text{Vol}_{flat}^{(1|1)} \qquad (5.33)$$

where $\text{Vol}_{flat}^{(1|1)} = V\delta(\psi)$ is the flat volume form. Note that:

$$E^\psi \wedge \mathbb{Y}^{(0|1)} = -E^\psi \wedge E^v \delta'(E^\psi) = \text{Vol}^{(1|1)} . \qquad (5.34)$$

As a check, we have also $\left(E_\theta^\psi \psi + DE_\theta^\psi V\right) \wedge -V\delta'(\psi) = E_\theta^\psi \text{Vol}_{flat}^{(1|1)} = \text{Vol}^{(1|1)}$.

A PCO for a generic (non constrained) vielbein is given by:

$$\mathbb{Y}^{(0|1)} = \left(E_x^v V + E_\theta^v \psi\right) \delta'(E_x^\psi V + E_\theta^\psi \psi)$$

$$= \left(E_x^v V + E_\theta^v (E_\theta^\psi)^{-1}(\psi - E_x^\psi V)\right) \frac{1}{(E_\theta^\psi)^2} \delta'(\psi) \qquad (5.35)$$

$$= \frac{1}{(E_\theta^\psi)^2}\left(E_x^v - E_\theta^v (E_\theta^\psi)^{-1} E_x^\psi\right) V\delta'(\psi) - E_\theta^v \frac{1}{(E_\theta^\psi)^3}\delta(\psi)$$

If we set $\left(E_x^v - E_\theta^v(E_\theta^\psi)^{-1}E_x^\psi\right) = 0$, $E_\theta^v = \theta$ and $E_\theta^\psi = 1$, we get the PCO $\theta\delta(\psi)$. If we set $\left(E_x^v - E_\theta^v(E_\theta^\psi)^{-1}E_x^\psi\right) = -1$ and $E_\theta^v = 0$, we get instead the supersymmetric PCO $-V\delta'(\psi)$.

6. Super Quantum Mechanics $N = 2$.

Here we formulate the SQM $N = 2$ in the language of integral forms. We follow the same strategy as in the previous section. We first discuss the superfield for the multiplet (in the present case, we need also the auxiliary field F to close the algebra). Then we describe the action in superspace and the equations of motion. We also give the action in components and then we study the rheonomic (geometric) action.

To describe the $N = 2$ model we recall that we have a scalar field ϕ, two fermions λ and $\bar\lambda$ and an auxiliary field f. Both on-shell and off-shell we get a matching of the fermionic and bosonic degrees of freedom. The superspace is described by a bosonic coordinate x and two fermionic coordinates θ and $\bar\theta$. The supersymmetric vielbeins are $V = dx + i(\theta d\bar\theta + d\theta\bar\theta)$, $\psi = d\theta$ and $\bar\psi = d\bar\theta$ and they satisfy the MC equations

$$dV = 2i\psi \wedge \bar\psi, \ d\psi = 0, \ d\bar\psi = 0. \tag{6.1}$$

Correspondingly, the superderivatives are defined as

$$D = \partial_\theta + i\bar\theta\partial_x, \ \bar D = \partial_{\bar\theta} + i\theta\partial_x, \tag{6.2}$$

with the algebra

$$\{D, D\} = 0, \ \{\bar D, \bar D\} = 0, \ \{D, \bar D\} = 2i\partial_x. \tag{6.3}$$

The physical degrees of freedom are collectively encoded into a $N = 2$ superfield Φ and its derivatives:

$$\Phi(x, \theta, \bar\theta) = \phi(x) + \lambda(x)\bar\theta + \bar\lambda(x)\theta + f(x)\theta\bar\theta, \tag{6.4}$$

$$\overline W(x, \theta, \bar\theta) = D\Phi = -\lambda + \theta(i\partial_x\phi - f) - i\theta\bar\theta\partial_x\lambda, \tag{6.5}$$

$$W(x, \theta, \bar\theta) = -\bar D\Phi = \bar\lambda + \bar\theta(i\partial_x\phi + f) - i\theta\bar\theta\partial_x\bar\lambda. \tag{6.6}$$

From these equations, we can compute the product of $D\Phi\bar D\Phi$ and collecting the term proportional to $\theta\bar\theta$, we get

$$D\Phi\bar D\Phi = \cdots + \theta\bar\theta\left[(\partial_x\phi)^2 + i(\bar\lambda\partial_x\lambda - \partial_x\bar\lambda\lambda) + f^2\right] \tag{6.7}$$

which implies that the superspace action and the component action are given by

$$S_{N=2,kin} = \frac{1}{2}\int D\Phi\bar D\Phi[dxd\theta d\bar\theta] = \int \left[\frac{1}{2}(\partial_x\phi)^2 + i\bar\lambda\partial_x\lambda + \frac{f^2}{2}\right]dx \tag{6.8}$$

To include the interaction terms, we consider the function $\mathcal{W}(\Phi)$ and we add the action

$$S_{N=2,int} = \int \mathcal{W}(\Phi)[dxd\theta d\bar{\theta}] \tag{6.9}$$

$$= \int \bar{D}\Big(\mathcal{W}'(\Phi)D\Phi\Big)dx = \int \Big(\mathcal{W}''\bar{D}\Phi D\Phi + \mathcal{W}'\bar{D}D\Phi\Big)dx$$

$$= \int \Big(2\mathcal{W}'(\phi)f - \mathcal{W}''\lambda\bar{\lambda}\Big)dx \,.$$

Now, we consider the geometric approach. For that we start from the differential of Φ:

$$d\Phi = V\partial_x\Phi + \psi\overline{W} - \bar{\psi}W \,, \tag{6.10}$$

where $W = -\bar{D}\Phi$ and $\overline{W} = D\Phi$. Then, we can apply the differential d on both sides of (6.10) to derive the equations for W and $\overline{W}$. We get

$$d\overline{W} = V\partial_x\overline{W} - i\bar{\psi}\partial_x\Phi + \bar{\psi}F \,, \tag{6.11}$$

$$dW = V\partial_x W + i\psi\partial_x\Phi + \psi F \,, \tag{6.12}$$

where we have introduced the auxiliary superfield F to solve the consistency condition for the first Bianchi identity. Then, applying again the differential, we derive the condition on F:

$$dF = V\partial_x F - i(\psi\partial_x\overline{W} + \bar{\psi}\partial_x W) \,. \tag{6.13}$$

No additional superfield is needed to close the algebra. From these relations, we can easily derive the equations relating the four superfields $\Phi, W, \overline{W}, F$:

$$W = -\bar{D}\Phi \,, \quad \overline{W} = D\Phi \,, \tag{6.14}$$

$$D\overline{W} = 0 \,, \quad \bar{D}W = 0 \,, \tag{6.15}$$

$$\overline{DW} = F - i\partial_x\Phi \,, \quad DW = F + i\partial_x\Phi \,. \tag{6.16}$$

Again it is easy, by using the algebra of superderivatives, to check that all the above equations are consistent. We can now construct the geometric Lagrangian $\mathcal{L}_{N=2} = \mathcal{L}_{N=2,kin} + \mathcal{L}_{N=2,int}$ as follows

$$\mathcal{L}_{N=2,kin} = \xi\Big(d\Phi - \psi\overline{W} + \bar{\psi}W\Big) - \frac{1}{2}(\xi^2 + F^2)V + \frac{i}{2}\Big(\overline{W}dW + d\overline{W}W\Big) \,, \tag{6.17}$$

$$\mathcal{L}_{N=2,int} = \Big(\mathcal{W}'F - \mathcal{W}''W\overline{W}\Big)V - i\mathcal{W}'\Big(\psi\overline{W} + W\bar{\psi}\Big) \tag{6.18}$$

$\mathcal{L}_{N=2}$ is a closed $(1|0)$ form (see equation 6.24 below). Notice that, again, the Hodge dual operator has not been used and an additional superfield ξ is needed in order to write the action in first order formalism. Note also that there are only three quantities that carry the 1-form degree, namely V, ψ and $\bar{\psi}$, and that the action can be expanded into powers of them. In the present case it is rather easy since the action is linear in V, ψ and $\bar{\psi}$.

From the Lagrangian we can easily compute the equations of motion which read (by setting the superpotential $\mathcal{W} = 0$ to simplify the discussion)

$$d\Phi - \psi\overline{W} + \bar{\psi}W - \xi V = 0 \,, \tag{6.19}$$

$$d\xi = 0 \,, \tag{6.20}$$

$$F = 0 \,, \tag{6.21}$$

$$d\overline{W} = -i\bar{\psi}\xi \,, \tag{6.22}$$

$$dW = i\xi\psi \,. \tag{6.23}$$

It is an easy exercise to check the consistency of this set of equations. Then, by computing the differential of $\mathcal{L}_{N=2,kin}$ we get:

$$
\begin{aligned}
d\mathcal{L}_{N=2,kin} = {}& d\xi \wedge (d\Phi - \psi\overline{W} + \bar{\psi}W - \xi V) - F dF \wedge V \\
& + i(d\overline{W} - i\bar{\psi}\xi) \wedge (dW + i\xi\bar{\psi}) = 0
\end{aligned}
\tag{6.24}
$$

The differential vanishes because of the algebraic equations of motion (6.19). In the same way one can show that also the interaction term $\mathcal{L}_{N=2,int}$ is closed. The kinetic terms and the interaction terms are independent, hence the closure property must be shown by taking the differential of $\mathcal{L}_{N=2,kin}$ and of $\mathcal{L}_{N=2,int}$ separately.[11] This is consistent with what is observed in the component formalism: in the case $N = 2$, one can freely add a superpotential to the action and its parameters are independent coupling constants. At the level of the rheonomic action we can add a combination of superfields with the correct scaling dimensions, form degree and other quantum numbers, which is independent of the kinetic terms.

Before constructing the PCO, we expand the action in powers of the gravitinos ψ and $\bar{\psi}$. We find:

$$
\begin{aligned}
\mathcal{L}_{N=2,kin} = {}& \left(\xi\partial_x\Phi - \frac{1}{2}(\xi^2 + F^2) + \frac{i}{2}(\overline{W}\partial_x W - W\partial_x\overline{W})\right)V \\
& + \left(\xi(D\Phi - \overline{W}) - \frac{i}{2}(D\overline{W}W - \overline{W}DW)\right)\psi \\
& + \left(\xi(\bar{D}\Phi + W) - \frac{i}{2}(\bar{D}\overline{W}W - \overline{W}\bar{D}W)\right)\bar{\psi} \,.
\end{aligned}
\tag{6.25}
$$

$$\mathcal{L}_{N=2,int} = (\mathcal{W}'F - \mathcal{W}''W\overline{W})V - i(\mathcal{W}'\overline{W})\psi - i(\mathcal{W}'W)\bar{\psi} \,, \tag{6.26}$$

Each piece of this expansion encodes all information regarding the equations of motion of the theory. Therefore, by choosing a suitable PCO, one can derive various equivalent forms of the action with different amounts of manifest supersymmetries. This would be interesting for applications where only some partial supersymmetries can be manifestly realized (such as in $D = 4$ $N = 4$ SYM).

---

[11] One has to use the equations of motion with the interaction terms.

Now, we are in a position to construct the PCO's. The PCO producing the component action is the simplest (standard) choice:

$$\mathbb{Y}_{st}^{(0|2)} = \theta\bar{\theta}\delta(\psi)\delta(\bar{\psi})\,. \tag{6.27}$$

Then the component action is obtained as

$$S_{N=2} = \int_{\mathcal{M}^{(1|2)}} \mathcal{L}_{N=2}^{(1|0)} \wedge \mathbb{Y}_{st}^{(0|2)} \tag{6.28}$$

$$= \int \left(\xi\partial_x\Phi - \frac{1}{2}(\xi^2 + F^2) + \frac{i}{2}(\overline{W}\partial_x W - W\partial_x\overline{W})\right)\theta\bar{\theta}[dxd\theta d\bar{\theta}]$$

$$+ \int \left(\mathcal{W}'F - \mathcal{W}''W\overline{W}\right)\theta\bar{\theta}[dxd\theta d\bar{\theta}]\,.$$

The presence of the $\theta\bar{\theta}$ factor projects all superfields to their first components and then to the component action.

To reproduce the superspace action, we need another PCO. For that we see that the following expression

$$\mathbb{Y}^{(0|2)} = -\frac{1}{2}iV \wedge (\theta\iota - \bar{\theta}\bar{\iota})\delta(\psi)\delta(\bar{\psi})\,, \tag{6.29}$$

has the correct properties. The symbols ι and $\bar{\iota}$ denote the derivative with respect to ψ and $\bar{\psi}$. Let us compute its differential

$$d\mathbb{Y}^{(0|2)} = i\psi\bar{\psi} \wedge (\theta\iota - \bar{\theta}\bar{\iota})\delta(\psi)\delta(\bar{\psi}) - iV \wedge (\psi\iota - \bar{\psi}\bar{\iota})\delta(\psi)\delta(\bar{\psi}) = 0\,. \tag{6.30}$$

The first term vanishes because one of the two gravitinos (ψ and $\bar{\psi}$) goes through the derivatives ι and $\bar{\iota}$ until it hits the corresponding Dirac delta. On the other side the two terms in the second piece are not vanishing separately: we have to perform an integration by parts for ι and $\bar{\iota}$ yielding two identical terms which cancel each other. To check that $\mathbb{Y}^{(0|2)}$ is not exact, we use the formula

$$d\frac{1}{2}\left[iV\theta\bar{\theta}\iota\bar{\iota}\delta(\psi)\delta(\bar{\psi})\right] = -i\frac{1}{2}V\wedge(\theta\iota-\bar{\theta}\bar{\iota})\delta(\psi)\delta(\bar{\psi})-\theta\bar{\theta}\delta(\psi)\delta(\bar{\psi}) = \mathbb{Y}^{(0|2)}-\mathbb{Y}_{st}^{(0|2)} \tag{6.31}$$

which shows that the two PCO's in (6.27) and (6.29) are cohomologous. The presence of θ and $\bar{\theta}$ in the expression is crucial to get the correct superspace action:

$$S_{N=2} = \int_{\mathcal{M}^{(1|2)}} \mathcal{L}_{N=2}^{(1|0)} \wedge \mathbb{Y}^{(0|2)}\,. \tag{6.32}$$

The contribution to the superspace action comes from the two terms proportional to the gravitinos ψ and $\bar{\psi}$. The structure of the PCO (6.29) resembles that in higher dimensions[12]. The presence of the superspace coordinates θ and $\bar{\theta}$ prevents it from being manifestly supersymmetric. However, as for $\mathbb{Y}_{st}^{(0|2)}$, its supersymmetry variation is d-exact.

---

[12]The applications of the formalism of integral forms to theories in higher dimensions will be the subject of a forthcoming paper. The case $D = 3$ $N = 1$ supergravity was analyzed in [28].

Some final remarks are in order. In the previous section, we have seen the rôle of the PCO's of type Z to reduce the picture of a given integral form. Here we would like to apply the same technique to the $N = 2$ case.

We start with the simplest volume form:

$$\omega^{(1|2)} = V\delta(\psi)\delta(\bar\psi)\,. \tag{6.33}$$

It is an integral form, it is closed since it belongs to $\Omega^{(1|2)}$, but is also exact since it can be expressed as the differential of a $(0|2)$-form:

$$\omega^{(1|2)} = d\left[\frac{1}{2}V(\theta\iota + \bar\theta\bar\iota)\delta(\psi)\delta(\bar\psi)\right] \tag{6.34}$$

where the sign in the square bracket is opposite w.r.t. the sign of the PCO in (6.29).[13] One can verify that the integral of $\omega^{(1|2)}$ on the supermanifold $\mathcal{M}^{(1|2)}$ vanishes. To avoid this problem, we need to construct a different volume form and the easiest is

$$\mathrm{Vol}^{(1|2)} = V\theta\bar\theta\delta(\psi)\delta(\bar\psi)\,. \tag{6.35}$$

which is closed, but it is not exact. It is not manifestly supersymmetric, but its supersymmetry variation is d-exact. Let us now apply a PCO of type Z to decrease the picture of the volume form:

$$
\begin{aligned}
Z_\psi\left(\mathrm{Vol}^{(1|2)}\right) &\equiv [d, \Theta(\iota_D)]\left(V\theta\bar\theta\delta(\psi)\delta(\bar\psi)\right) = d\left[\Theta(\iota_D)\left(V\theta\bar\theta\delta(\psi)\delta(\bar\psi)\right)\right]\\
&= d\left[V\theta\bar\theta\frac{1}{\psi}\delta(\bar\psi)\right] = \left[2i\psi\bar\psi\theta\bar\theta\frac{1}{\psi}\delta(\bar\psi) - V\bar\theta\delta(\bar\psi)\right] = -V\bar\theta\delta(\bar\psi)\,.
\end{aligned}
\tag{6.36}
$$

where ι_D is the contraction operator along the odd vector field $D = \frac{\partial}{\partial\theta} + i\bar\theta\partial_x$. The resulting pseudoform is closed, not exact, and it belongs to the space $\Omega^{(1|1)}$. Then, we act with a PCO denoted by $Z_{\bar\psi}$ (putting now $D = \frac{\partial}{\partial\bar\theta}$) on the result of (6.36) and we obtain:

$$
\begin{aligned}
Z_{\bar\psi}Z_\psi\left(\mathrm{Vol}^{(1|2)}\right) &= Z_{\bar\psi}\left(-V\bar\theta\delta(\bar\psi)\right) = d\left[\Theta(\iota_{\bar D})\left(-V\bar\theta\delta(\bar\psi)\right)\right]\\
&= d\left[-V\bar\theta\frac{1}{\bar\psi}\right] = -2i\psi\bar\psi\bar\theta\frac{1}{\bar\psi} + V\bar\psi\frac{1}{\bar\psi} = V - 2i\psi\bar\theta = \widetilde{V}^{(1|0)}
\end{aligned}
\tag{6.37}
$$

where $\widetilde{V}^{(1|0)}$ is a closed superform in $\Omega^{(1|0)}$. We can finally check that this superform is the dual of the PCO $\mathbb{Y}^{(0|2)}$:

$$
\begin{aligned}
\widetilde{V}^{(1|0)} \wedge \mathbb{Y}^{(0|2)} &= \left(V - 2i\psi\bar\theta\right) \wedge \left[-\frac{1}{2}iV \wedge (\theta\iota - \bar\theta\bar\iota)\delta(\psi)\delta(\bar\psi)\right]\\
&= V\theta\bar\theta\delta(\psi)\delta(\bar\psi) = \mathrm{Vol}^{(1|2)}\,.
\end{aligned}
\tag{6.38}
$$

---

[13]both terms in the r.h.s are necessary for ω to be real.

6.1. Coupling to gauge fields.

In the case on $N = 2$ theory, there is an additional symmetry (R-symmetry) rotating the fermions

$$W' = e^{i\alpha}W\,, \quad \overline{W}' = e^{-i\alpha}\overline{W}\,, \tag{6.39}$$

and the gravitinos

$$\psi' = e^{i\alpha}\psi\,, \quad \bar\psi' = e^{-i\alpha}\bar\psi\,, \tag{6.40}$$

To gauge this symmetry, we replace the differential d in the action $\mathcal{L}_{N=2}^{(1|0)}$ by the covariant differential ∇ such that

$$\nabla W = dW + iAW\,, \quad \nabla\overline{W} = d\overline{W} - iA\overline{W}\,, \tag{6.41}$$

The modifications appear only in the kinetic terms (in the present section, we neglect the interaction terms) and we get

$$\frac{i}{2}\left(\overline{W}\nabla W + \nabla\overline{W}W\right) = \frac{i}{2}\left(\overline{W}dW + d\overline{W}W\right) - AW\overline{W} \tag{6.42}$$

from which we derive the current $J = W\overline{W}V$ by taking the derivative with respect the bosonic component of the gauge field.

We recall that a gauge field in the supermanifold $\mathcal{M}^{(1|2)}$ is defined in terms of a $(1|0)$-connection

$$A = A_x V + A_\psi \psi + A_{\bar\psi}\bar\psi\,. \tag{6.43}$$

As always, the components of the $(1|0)$-connection exceed the physical components, therefore we impose some constraints as follows. First, we compute the field strength

$$F = dA = (DA_x - \partial_x A_\psi)V\psi + (\bar{D}A_x - \partial_x \bar{A}_\psi)V\bar\psi + DA_\psi \psi^2 \tag{6.44}$$

$$+ \left(DA_{\bar\psi} + \bar{D}A_\psi + 2iA_x\right)\psi\bar\psi + \bar{D}A_{\bar\psi}\bar\psi^2\,.$$

Then we set

$$\left(DA_{\bar\psi} + \bar{D}A_\psi + 2iA_x\right) = 0\,, \quad \bar{D}A_{\bar\psi} = 0\,, \quad DA_\psi = 0\,. \tag{6.45}$$

The first equation can be easily solved in terms of A_x to get

$$A_x = \frac{i}{2}\left(DA_{\bar\psi} + \bar{D}A_\psi\right) \tag{6.46}$$

with the condition that A_ψ is anti-chiral and $A_{\bar\psi}$ is chiral. As a consequence, by computing the combinations $DA_x - \partial A_\psi$ and $(\bar{D}A_x - \partial A_{\bar\psi})$ we find that they vanish. This implies that the full field strength F vanishes. The connection is given by

$$A = \frac{i}{2}\left(DA_{\bar\psi} + \bar{D}A_\psi\right)V + A_\psi\psi + A_{\bar\psi}\bar\psi\,. \tag{6.47}$$

The gauge symmetry is $\delta A_\psi = D\Lambda$ and $\delta A_{\bar\psi} = \bar{D}\Lambda$ where Λ is the superfield gauge parameter. As a consequence, $\delta A_x = \partial_x \Lambda$. Given the supermanifold $\mathcal{M}^{(1|2)}$, we can integrate the gauge field directly. For that we need a $(1|2)$

integral form and therefore we multiply the gauge connection A by the PCO's discussed above. We have

$$A^{(1|2)}_{st} = A^{(1|0)} \wedge \mathbb{Y}^{(0|2)}_{st} = \frac{i}{2}\Big(DA_{\bar\psi} + \bar D A_\psi\Big)V\theta\bar\theta\delta(\psi)\delta(\bar\psi)\,, \qquad (6.48)$$

$$A^{(1|2)} = A^{(1|0)} \wedge \mathbb{Y}^{(0|2)} = -\frac{i}{2}\Big(A_\psi\psi + A_{\bar\psi}\bar\psi\Big)V(\theta\iota - \bar\theta\bar\iota)\delta(\psi)\delta(\bar\psi)\,.$$

and one can show that:

$$\int_{\mathcal{M}^{(1|2)}} A^{(1|2)}_{st} = \int_{\mathcal{M}^{(1|2)}} A^{(1|2)} = \frac{i}{2}\int dx\, (DA_{\bar\psi} + \bar D A_\psi)\big|_{\theta=\bar\theta=0} \qquad (6.49)$$

which is manifestly supersymmetric and gauge invariant if the supermanifold has no boundary.

This approach can be followed to define a supersymmetric Wilson loop if instead we choose a supermanifold $\mathcal{M}^{(1|2)}$ whose reduced manifold is the circle S^1.

Notice that given a generic gauge connection $A^{(1|0)}$, there is no reason for the two expressions $\int A^{(1|2)}_{st}$ and $\int A^{(1|2)}$ to match. Indeed, as discussed above the choice of the PCO is arbitrary when the superform $\mathcal{O}^{(1|0)}$ to which it is applied is d-closed. Otherwise, it turns out that:

$$\mathcal{O}^{(1|0)} \wedge \mathbb{Y}^{(0|2)}_{st} = \mathcal{O}^{(1|0)} \wedge \mathbb{Y}^{(0|2)} + d\Big(\mathcal{O}^{(1|0)} \wedge \eta\Big) - d\mathcal{O}^{(1|0)} \wedge \eta \qquad (6.50)$$

where $\eta = iV\theta\bar\theta\iota\bar\iota\delta(\psi)\delta(\bar\psi)$ was computed in (6.31). Thus, if we integrate both members of (6.50), the second term on r.h.s. drops out, but the third remains. If $\mathcal{O}^{(1|0)}$ is a connection form the above equation can be written as:

$$A^{(1|0)} \wedge \mathbb{Y}^{(0|2)}_{st} = A^{(1|0)} \wedge \mathbb{Y}^{(0|2)} + d\Big(A^{(1|0)} \wedge \eta\Big) - F^{(2|0)} \wedge \eta \qquad (6.51)$$

where we see that the last term vanishes if the field strength vanishes.

The Lagrangian is finally given by

$$\mathcal{L}_{N=2,kin\ gauge} = \xi\Big(d\Phi - \psi\overline{W} + \bar\psi W\Big) - \frac{1}{2}(\xi^2 + F^2)V + \frac{i}{2}\Big(\overline{W}\nabla W + \nabla\overline{W}W\Big) \qquad (6.52)$$

which is gauge invariant. The equations of motion are

$$d\Phi - \psi\overline{W} + \bar\psi W - \xi V = 0\,, \qquad (6.53)$$

$$\nabla W = -\xi\psi\,, \qquad (6.54)$$

$$\nabla\overline{W} = i\xi\bar\psi\,, \qquad (6.55)$$

$$d\xi = 0\,, \qquad (6.56)$$

$$F = 0\,. \qquad (6.57)$$

To check consistency of the equations, we act with ∇ on the r.h.s. on the fermionic equations and we get

$$iFW = -id\xi \wedge \psi - i\xi\nabla\psi\,. \qquad (6.58)$$

The r.h.s. vanishes because $d\xi = 0$ and $\nabla\psi = 0$. The second equation is the generalisation of $d\psi = 0$ to the gauged version. It follows from (6.58) that

the field strength F vanishes. This is consistent with the derivation outlined above.

Finally, if we consider the expression $J_0 = -W\overline{W}$ (the function appearing in the current for the R-symmetry) and we compute its differential we obtain:

$$dJ_0 = -i\xi(\psi\overline{W} + W\bar\psi). \tag{6.59}$$

The expression for J_0 is given in terms of superfields and belongs to a supermultiplet. The supersymmetry variations can be computed directly from (6.59).

6.2. $D = 1, N = 2$ Curved.

To conclude this section, we analyze the curved manifold case. We replace the flat supervielbein $V, \psi, \bar\psi$ with the curved ones $E^v, E^\psi, E^{\bar\psi}$ and we require that they satisfy the constraints

$$dE^V = 2iE^\psi \wedge E^{\bar\psi}, \ dE^\psi = 0, \ dE^{\bar\psi} = 0.$$

To solve these constraints we have to expand the superviebeins on a basis

$$E^V = E_x^V V + E_\theta^V \psi + E_{\bar\theta}^V \bar\psi \tag{6.60}$$
$$E^\psi = E_x^\psi V + E_\theta^\psi \psi + E_{\bar\theta}^\psi \bar\psi$$
$$E^{\bar\psi} = E_x^{\bar\psi} V + E_\theta^{\bar\psi} \psi + E_{\bar\theta}^{\bar\psi} \bar\psi.$$

The various components can be cast into a supermatrix. If we insert them into the constraints, we find the final result

$$E^V = E_\theta^\psi E_{\bar\theta}^{\bar\psi} V, \ E^\psi = E_\theta^\psi \psi + \bar D E_\theta^\psi V, \ E^{\bar\psi} = E_{\bar\theta}^{\bar\psi} \bar\psi + D E_\theta^\psi V. \tag{6.61}$$

that resembles the $N = 1$ case. All the equations can be solved in terms of the two superfields E_θ^ψ and $E_{\bar\theta}^{\bar\psi}$, that are anti-chiral and chiral respectively:

$$DE_\theta^\psi = 0, \ \bar D E_{\bar\theta}^{\bar\psi} = 0. \tag{6.62}$$

As in the $N = 1$ case we have:

$$E^V \wedge \delta(E^\psi)\delta(E^{\bar\psi}) = E_\theta^\psi E_{\bar\theta}^{\bar\psi} V \wedge \delta\left(E_\theta^\psi \psi + \bar D E_{\bar\theta}^{\bar\psi} V\right) \wedge \delta\left(E_{\bar\theta}^{\bar\psi} \bar\psi + D E_\theta^\psi V\right)$$

$$\tag{6.63}$$

$$= E_\theta^\psi E_{\bar\theta}^{\bar\psi} V \wedge \frac{1}{E_\theta^\psi}\delta\left(\psi + \frac{1}{E_\theta^\psi}\bar D E_\theta^\psi V\right) \wedge \frac{1}{E_{\bar\theta}^{\bar\psi}}\delta\left(E_{\bar\theta}^{\bar\psi}\bar\psi + D E_\theta^\psi V\right)$$

$$= V \wedge \delta\left(\psi + \frac{1}{E_\theta^\psi}\bar D E_\theta^\psi V\right) \wedge \delta\left(E_{\bar\theta}^{\bar\psi}\bar\psi + D E_\theta^\psi V\right)$$

$$= V\delta(\psi)\delta(\bar\psi)$$

Therefore also in the $N = 2$ case the volume form is not modified going from a flat to a curved supermanifold.

Let us analyse the PCO $\mathbb{Y}^{(0|2)} = -\frac{1}{2}iV(\theta\iota - \bar{\theta}\bar{\iota})\delta(\psi)\delta(\bar{\psi})$. We propose the following curved version:

$$\mathbb{Y}^{(0|2)} = -\frac{i}{2}E^V(F\iota - \overline{F}\bar{\iota})\delta(E^\psi)\delta(E^{\bar{\psi}}) \qquad (6.64)$$

where F and $\overline{F}$ are two scalar superfields. We impose that this integral form is closed

$$d\mathbb{Y}^{(0|2)} = -\frac{i}{2}E^V(\nabla F - \bar{\nabla}\overline{F})\delta(E^\psi)\delta(E^{\bar{\psi}}) = 0 \qquad (6.65)$$

which implies that $\nabla F = \bar{\nabla}\overline{F}$. To solve this equation, we note that we can use the same procedure as for the volume form in (6.63) to get:

$$\mathbb{Y}^{(0|2)} = -\frac{i}{2}E^\psi_\theta E^{\bar{\psi}}_{\bar{\theta}}V \wedge \left(\frac{F}{(E^\psi_\theta)^2 E^{\bar{\psi}}_{\bar{\theta}}}\iota\delta(\psi)\delta(\bar{\psi}) - \frac{\overline{F}}{E^\psi_\theta (E^{\bar{\psi}}_{\bar{\theta}})^2}\delta(\psi)\bar{\iota}\delta(\bar{\psi})\right) \quad (6.66)$$

$$= -\frac{i}{2}V\left(\frac{F}{E^\psi_\theta}\iota\delta(\psi)\delta(\bar{\psi}) - \frac{\overline{F}}{E^{\bar{\psi}}_{\bar{\theta}}}\delta(\psi)\bar{\iota}\delta(\bar{\psi})\right)$$

from which it follows that:

$$F = \theta E^\psi_\theta\,, \ \ \overline{F} = \bar{\theta}E^{\bar{\psi}}_{\bar{\theta}} \qquad (6.67)$$

They satisfy the condition $DF = E^\psi_\theta$ and $\bar{D}\overline{F} = E^{\bar{\psi}}_{\bar{\theta}}$. Therefore the PCO in the curved case is the same as in the flat case.

7. Quantization.

Quantization for these simple systems can be obtained very easily. We consider here the case $N = 2$ with no superpotential ($\mathcal{W} = 0$) for simplicity. Furthermore, we promote the superfields Φ^I to be the components of a multiplet $I = 1, \ldots, n$ describing a map

$$\Phi^I(x, \theta, \bar{\theta}) : \mathcal{M}^{(1|2)} \longrightarrow \mathcal{M}^{(n)}$$

of the supermanifold into a n-dimensional Riemannian manifold $\mathcal{M}^{(n)}$.

From the equations of motion (6.53-6.57), we find that the solution is given by the following zero modes

$$\Phi^I = \phi^I_0 + ixp^I_0 + \lambda^I_0\bar{\theta} + \bar{\lambda}^I_0\theta\,, \ \ W^I = \lambda^I_0 + p^I_0\theta\,,$$

$$\overline{W}^I = -\bar{\lambda}^I - p^I_0\bar{\lambda}_0\,, \ \ \xi^I = p^I_0\,, \ \ F^I = 0\,. \qquad (7.1)$$

where the zero modes $\phi^I_0, p^I_0, \lambda^I_0, \bar{\lambda}^I_0$ satisfy the commutation relations

$$[\phi^I_0, p^J_0] = i\hbar\,\eta^{IJ}\,, \ \ \{\lambda^I_0, \bar{\lambda}^J_0\} = \hbar\,\eta^{IJ}\,. \qquad (7.2)$$

and the Hilbert space $\mathcal{H}$ is constructed as follows:

$$p^I_0|0\rangle = 0\,, \ \ \lambda^I_0|0\rangle = 0\,, \ \ \forall I = 1, \ldots, n$$

and a generic state is given by:

$$|\chi\rangle = \sum_{p=0}^{n} |\chi, p\rangle = \sum_{p=0}^{n} \chi_{[I_1 \dots I_n]}(\phi_0)\bar{\lambda}_0^{I_1} \dots \bar{\lambda}_0^{I_n}|0\rangle \tag{7.3}$$

where the functions $\chi_{[I_1 \dots I_n]}(\phi_0)$ are $L^2(\mathcal{M}^{(n)})$-integrable functions. The indices $I_1 \dots I_n$ are anti-symmetrized because of the Grassman variables $\bar{\lambda}_0^I$.

Let us project the Maurer-Cartan equations (6.10-6.12) on the ground state.

$$d\Phi^I|0\rangle = (iV p_0^I - \lambda_0^I \bar{\psi} - \bar{\lambda}_0^I \psi)|0\rangle = -\psi \bar{\lambda}_0^I|0\rangle = \psi \overline{W}^I|0\rangle$$

$$d\overline{W}^I|0\rangle = -\bar{\psi} p_0^I|0\rangle = 0\,, \tag{7.4}$$

(by consistency $dW^I|0\rangle = \psi p_0^I|0\rangle = 0$).

Let us consider now a differential form of $\Omega^\bullet(\mathcal{M}^{(n)})$, written in local coordinates, applied to the ground state $|0\rangle$:

$$\omega^{(p)}|0\rangle = \omega_{I_1 \dots I_p}(\Phi)d\Phi^{I_1} \wedge \cdots \wedge d\Phi^{I_p}|0\rangle$$

$$= \psi^p \omega_{I_1 \dots I_p}(\phi_0)\bar{\lambda}_0^{I_1} \dots \bar{\lambda}_0^{I_p}|0\rangle = \psi^p|\omega, p\rangle \tag{7.5}$$

We obtain a map between the exterior bundle $\Omega^\bullet(\mathcal{M}^{(n)})$ and the Hilbert space. The powers of the gravitinos (ψ^p) parametrize each state at a given fermion number. The right hand side of (7.5) must be interpreted as a superform on $\mathcal{M}^{(1|2)}$. That would be impossible in the case $\mathcal{M}^{(1)} \longrightarrow \mathcal{M}^{(n)}$ of a pure bosonic 1-dimensional manifold because the pullback of any differential form on $\mathcal{M}^{(n)}$ gives always a 1-form on $\mathcal{M}^{(1)}$.

In particular, we have a map:

$$\Omega^{(p)}(\mathcal{M}^{(n)}) \longrightarrow \Omega^{(p|0)}(\mathcal{M}^{(1|2)})$$

Let us compute the action of the differential d on a p-form:

$$d\omega^{(p)}|0\rangle = \partial_K \omega_{I_1 \dots I_p}(\Phi)d\Phi^K \wedge d\Phi^{I_1} \wedge \cdots \wedge d\Phi^{I_p}|0\rangle \tag{7.6}$$

$$= \psi^p \bar{\psi}\, \partial^K \omega_{[K I_2 \dots I_p]}(\phi_0)\bar{\lambda}_0^{I_2} \dots \bar{\lambda}_0^{I_p}|0\rangle$$

$$+ \psi^{p+1} \partial_{[K}\omega_{I_1 \dots I_p]}(\phi_0)\bar{\lambda}_0^K \bar{\lambda}_0^{I_1} \dots \bar{\lambda}_0^{I_p}|0\rangle$$

The closure of ω^p gives the equations:

$$\partial^K \omega_{[K I_2 \dots I_p]}(\phi_0) = 0\,, \quad \partial_{[K}\omega_{I_1 \dots I_p]}(\phi_0) = 0\,, \tag{7.7}$$

(where $\partial_K = \partial/\partial\phi_0^K$) that imply also $\partial^2 \omega_{[I_1 \dots I_p]} = 0$. Therefore, the states of the present theory are represented by on-shell p-forms of $\mathcal{M}^{(p)}$.

The closure and the co-closure of the differential form implies that $\omega^{(p)}$ is an harmonic form. We refer to [31] for further comments on this point.

We conclude the present section by studying two operators that share some characteristics with the PCO's and have an action on the Hilbert space. We consider the following two $(0|n)$-pseudoforms

$$\mathcal{Y}^{(0|n)} = \prod_{I=1}^{n} W^I \delta(dW^I)\,, \quad \overline{\mathcal{Y}}^{(0|n)} = \prod_{I=1}^{n} \overline{W}^I \delta(d\overline{W}^I)\,, \tag{7.8}$$

They are non trivial elements of the cohomology $H_d^{(0|n)}$. We analyze them from the quantum point of view and we act with a single pseudoform $W^I\delta(dW^I)$ on the generic state $|\chi, q\rangle$ as given in (7.3)

$$
\begin{aligned}
\mathcal{Y}^{(0|n)}|\chi, q\rangle &= W_0^I\delta(dW_0^I)|\chi, q\rangle = W_0^I\delta(p_0^I\psi)|\chi, q\rangle \\
&= \lambda_0^I\delta(p_0^I\psi)\Big[\chi_{I_1\ldots I_q}(\phi_0)\bar{\lambda}_0^{I_1}\ldots\bar{\lambda}_0^{I_q}\Big]|0\rangle \\
&= q\frac{1}{\psi}\delta(p_0^I)\chi_{II_2\ldots I_q}(\phi_0)\bar{\lambda}_0^{I_2}\ldots\bar{\lambda}_0^{I_q}\Big]|0\rangle
\end{aligned}
\tag{7.9}
$$

The action of $\delta(p_0^I)$ on the wave function $\chi_{I_1\ldots I_q}(\phi_0)$ is computed using the integral representation of the Dirac delta function:

$$
\delta(p_0^I)\chi_{I_1\ldots I_q}(\phi_0) = \int_{-\infty}^{\infty} du\, e^{iup_0^I}\chi_{I_1\ldots I_q}(\phi_0) = \int_{-\infty}^{\infty} du\, \chi_{I_1\ldots I_q}(\phi_0^K + \delta_I^K u)
$$

since the exponential operator $e^{iup_0^I}$ acts as a finite translation on the coordinate ϕ_0^I. If the expression is integrable, we are left with a wave function with a variable less. The operator $\delta(p_0^I)$ projects the quantum state into a zero-momentum state along the direction I. Notice the appearance of the inverse of ψ. Note also that acting with this operator on (7.5), the inverse of ψ reduces the power of ψ appearing in (7.5). In the same way, acting with $\overline{\mathcal{Y}}^{(0|n)}$ increases the power of $\bar{\lambda}_0^I$ (namely the degree form).

8. A note on Observables.

Since we are dealing with a quantum mechanical system, we are interested to study the observables of the theory.

Let us suppose that the observables are identified by means of a nilpotent charge Q anticommuting with the differential d. For example, one can consider one of the two supercharges Q or $\bar{Q}$ (associated to each supercharge there is a unit of the R-charge discussed in sec. 5.1, and we refer to that unit charge (positive for Q and negative for $\bar{Q}$) as ghost number, since usually the form number and this number are identified in the literature (see [31]) with the bigrading of the BRST complex .

So, given $\mathcal{O}_1^{(0)}$ a ghost number 1 form, we have the sequence of descent equations:

$$
\Big[Q, \mathcal{O}_1^{(0)}\Big] = 0\,, \quad \Big[Q, \mathcal{O}_0^{(1)}\Big] = d\mathcal{O}_1^{(0)}
\tag{8.1}
$$

where $\mathcal{O}_0^{(1)}$ has zero ghost-number and 1 form degree, while $\mathcal{O}_1^{(0)}$ has ghost number 1 and zero form degree. The operators $\mathcal{O}_1^{(0)}$ and $\mathcal{O}_0^{(1)}$ are written in term of the superfields Φ and W of the supersymmetric model with $N = 1$. We notice that the integral of $\mathcal{O}_0^{(1)}$

$$
\mathcal{T} = \int_{\mathcal{M}^{(1|1)}} \mathcal{O}_0^{(1)} \wedge \mathbb{Y}^{(0|1)},
\tag{8.2}
$$

is invariant under the action of Q. Here we have introduced the PCO $\mathbb{Y}^{(0|1)}$ to convert the observable $\mathcal{O}_0^{(1)}$ into an integral form of type $(1|1)$. The observable $\mathcal{O}_0^{(1)}$ is closed (using the descent equations), and therefore one can suitably change the PCO to a different (but cohomologous) one. Consequently, by changing the PCO by an exact term we have:

$$\mathcal{O}^{(1)} \wedge \left(\mathbb{Y}^{(0|1)} + d\eta\right) = \mathcal{O}^{(1)} \wedge \mathbb{Y}^{(0|1)} + d(\mathcal{O}^{(1)} \wedge \eta) \qquad (8.3)$$

which shows that a redefinition of the PCO amounts to a shift by exact terms of the observables and this drops out from the integral in (8.2). Acting with the PCO on the descent equations (8.1) we have:

$$\left[Q, \mathcal{O}_1^{(0)}\right] \wedge \mathbb{Y}^{(0|1)} = \left[Q, \left(\mathcal{O}_1^{(0)} \wedge \mathbb{Y}^{(0|1)}\right)\right] = 0\,, \qquad (8.4)$$

$$\left[Q, \left(\mathcal{O}_0^{(1)} \wedge \mathbb{Y}^{(0|1)}\right)\right] = d\left(\mathcal{O}_1^{(0)} \wedge \mathbb{Y}^{(0|1)}\right) \qquad (8.5)$$

where we assumed that $[Q, \mathbb{Y}^{(0|1)}] = 0$ (which implies that the PCO is super-symmetric invariant). Notice that if the PCO is shifted by a d-exact term we have

$$\mathcal{O}_1^{(0)} \wedge (\mathbb{Y}^{(0|1)} + d\eta) = \mathcal{O}_1^{(0)} \wedge \mathbb{Y}^{(0|1)} + d\left(\mathcal{O}_1^{(0)} \wedge \eta\right) - (d\mathcal{O}_1^{(0)}) \wedge \eta \qquad (8.6)$$

$$= \mathcal{O}_1^{(0)} \wedge \mathbb{Y}^{(0|1)} + d\left(\mathcal{O}_1^{(0)} \wedge \eta\right) - \left[Q, \mathcal{O}_0^{(1)}\right] \wedge \eta$$

$$= \mathcal{O}_1^{(0)} \wedge \mathbb{Y}^{(0|1)} + d\left(\mathcal{O}_1^{(0)} \wedge \eta\right) - \left[Q, \mathcal{O}_0^{(1)} \wedge \eta\right]$$

which shows that the variation of the PCO results into a d-exact term plus a Q-exact term.

Associated to the complex $0 \longrightarrow \Omega_1^{(0)} \longrightarrow \Omega_0^{(1)} \to 0$, (here $\Omega_q^{(p)}$ denotes the space of the observables with quantum numbers p and q) we have the complex of *integral* observables:

$$0 \longrightarrow \Omega_1^{(0|1)} \equiv \Omega_1^{(0)} \wedge \mathbb{Y}^{(0|1)} \longrightarrow \Omega_0^{(1|2)} \equiv \Omega_0^{(1)} \wedge \mathbb{Y}^{(0|1)} \longrightarrow 0$$

The choice of the PCO allows us to choose the representation most useful to compute the correlators. The choice of a non-supersymmetric one reduces the observable to the component fields, otherwise the choice of a supersymmetric PCO produces observables in superspace.

References

[1] V. S. Varadarajan, *Supersymmetry for mathematicians: An introduction* Courant Lectures Notes, American Mathematical Society, 2004

[2] C. Bartocci, U. Bruzzo, and D. Hernandez-Ruiperez, *The Geometry of Super-manifolds* Ed. Kluvers Academic Publishers (1991).

[3] D. S. Freed, *Five Lectures on Supersymmetry.* American Mathematical Society, 1999

[4] R. Fioresi, M. A. Lledo and V. S. Varadarajan, *The Minkowski and conformal superspaces*, J. Math. Phys. **48**, 113505 (2007) [arXiv:math/0609813].

[5] T. Voronov, *Geometric integration theory on supermanifolds.* Soviet Scientific Review, Section C: Mathematical Physics, 9, Part 1. Harwood Academic Publisher, Chur. 1991 (Second edition 2014).

[6] R. Catenacci, M. Debernardi, P. A. Grassi and D. Matessi, *Čech and de Rham Cohomology of Integral Forms,*" J. Geom. Phys. **62** (2012) 890 [arXiv:1003.2506 [math-ph]].

[7] L. Castellani, R. Catenacci and P. A. Grassi, *The Geometry of Supermanifolds and new Supersymmetric Actions*, Nucl. Phys. B **899**, 112 (2015) [arXiv:1503.07886 [hep-th]].

[8] L. Castellani, R. Catenacci and P. A. Grassi, *Supergravity Actions with Integral Forms,* Nucl. Phys. B **889**, 419 (2014) [arXiv:1409.0192 [hep-th]].

[9] L. Castellani, R. Catenacci and P. A. Grassi, *Hodge Dualities on Supermanifolds*, Nucl. Phys. B **899**, 570 (2015) [arXiv:1507.01421 [hep-th]].

[10] I. L. Buchbinder and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity: Or a walk through superspace,* Bristol, UK: IOP (1998) 656 p

[11] P. S. Howe and R. W. Tucker, *A locally supersymmetric and reparametrization invariant action for a spinning membrane*, J. Phys. A **10**, L155 (1977); *Local supersymmetry in (2+1) dimensions. 1. Supergravity and differential forms,* J. Math. Phys. **19**, 869 (1978); *Local supersymmetry in (2+1) dimensions. 2. An action for a spinning membrane*, J. Math. Phys. **19**, 981 (1978).

[12] M. Brown and S. J. Gates Jr., *Superspace Bianchi identities and the supercovariant derivative*, Annals Phys. **122**, 443 (1979).

[13] S. J. Gates, Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace, or One Thousand and One Lessons in Supersymmetry*, Front. Phys. **58**, 1 (1983) [arXiv:hep-th/0108200].

[14] F. Ruiz Ruiz and P. van Nieuwenhuizen, *Lectures on supersymmetry and supergravity in (2+1)-dimensions and regularization of supersymmetric gauge theories*, Published in Tlaxcala 1996, Recent developments in gravitation and mathematical Mexican School on Gravitation and Mathematical Physics, Tlaxcala, Mexico, 1 (1996) 7. physics-2

[15] S. M. Kuzenko, U. Lindstrom and G. Tartaglino-Mazzucchelli, *Off-shell supergravity-matter couplings in three dimensions*, JHEP **1103** (2011) 120 [arXiv:1101.4013 [hep-th]].

[16] S. M. Kuzenko, U. Lindstrom, M. Rocek, I. Sachs and G. Tartaglino-Mazzucchelli, *Three-dimensional N=2 supergravity theories: From superspace to components*, arXiv:1312.4267 [hep-th].

[17] P. van Nieuwenhuizen, *D = 3 conformal supergravity and Chern-Simons terms*, Phys. Rev. D **32**, 872 (1985).

[18] B. M. Zupnik and D. G. Pak, *Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities*, Theor. Math. Phys. **77** (1988) 1070 [Teor. Mat. Fiz. **77** (1988) 97].

[19] T. Uematsu, *Structure of N=1 conformal and Poincaré supergravity in (1+1)-dimensions and (2+1)-dimensions*, Z. Phys. C **29**, 143 (1985); *Constraints and actions in two-dimensional and three-dimensional N=1 conformal supergravity*, Z. Phys. C **32**, 33 (1986).

[20] M. Becker, D. Constantin, S. J. Gates Jr., W. D. Linch III, W. Merrell and J. Phillips, *M-theory on Spin(7) manifolds, fluxes and 3D, N = 1 supergravity*, Nucl. Phys. B **683**, 67 (2004) [arXiv:hep-th/0312040].

[21] J. Wess, J. Bagger, *Supersymmetry and Supergravity*, Princeton Univ. Press; 2 Revised edition (1992).

[22] A. Belopolsky, *Picture changing operators in supergeometry and superstring theory*, [arXiv:hep-th/9706033].

[23] E. Witten, *Notes on Supermanifolds and Integration,* [arXiv:1209.2199 [hep-th]].

[24] E. Witten, *Superstring Perturbation Theory Revisited*, arXiv:1209.5461 [hep-th].

[25] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, JHEP **0409**, 047 (2004) [hep-th/0406055].

[26] L. Castellani, R. Catenacci, P.A. Grassi, *Integral Representations on Supermanifolds: super Hodge duals, PCOs and Liouville forms*, Lett. Math. Phys. (2016). [arXiv:1603.01092 [hep-th]]

[27] T. Voronov, *On Volumes of Classical Supermanifolds* [arXiv:1503.06542v1 [math.DG]]

[28] L. Castellani, R. Catenacci, P.A. Grassi, JHEP10 (2016) 049, *The integral form of supergravity,* [arXiv:1607.05193 [hep-th]]

[29] L. Castellani, R. D'Auria and P. Fré, *Supergravity and superstrings: A Geometric perspective.* in 3 vol.s, Singapore, Singapore: World Scientific (1991)

[30] D. P. Sorokin, *Superbranes and superembeddings,* Phys. Rept. **329**, 1 (2000) [hep-th/9906142].

[31] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *Mirror symmetry,*, Clay Mathematics Monographs, American Mathematical Society (August 19, 2003).

L. Castellani
Dipartimento di Scienze e Innovazione Tecnologica,
Università del Piemonte Orientale, Viale T. Michel, 11, 15121 Alessandria, Italy
INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy
Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy
e-mail: `leonardo.castellani@uniupo.it`

R. Catenacci
Dipartimento di Scienze e Innovazione Tecnologica,
Università del Piemonte Orientale, Viale T. Michel, 11, 15121 Alessandria, Italy
Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy
Gruppo Nazionale di Fisica Matematica, INdAM, P.le Aldo Moro 5, 00185 Roma, Italy
e-mail: `roberto.catenacci@uniupo.it`

P.A. Grassi
Dipartimento di Scienze e Innovazione Tecnologica,
Università del Piemonte Orientale, Viale T. Michel, 11, 15121 Alessandria, Italy
INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy
Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy
Center for Gravitational Physics, Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan
e-mail: `pietro.grassi@uniupo.it`