Physical Applications of Group Cohomology

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1 - Introduction

The *algebraic* group cohomology (see e.g. [1]), as opposed to the *topological* cohomology, has been recently applied to some aspect of field theory. The topological three dimensional gauge theories are perhaps the most important examples [2]. Beside this application, many problems in which a group or a group action play an important role, can be described using the methods of group cohomology.

The problem of anomalies in field theories is one of them, and will be the main subject of this talk. The origin of the anomalies problem is the lacking of gauge invariance of the effective action; local anomalies arise when considering "infinitesimal" gauge transformation, while global anomalies are connected with "large" transformation (i.e. not in the connected component of the identity).

The general framework for testing the possible occurrence of anomalies in field theories can be constructed in terms of the theory of group actions on line bundles (see e.g.[3] and its references). In this talk we revisit this topological construction through the application of methods of group cohomology.

The starting point is the concept of $\mathcal{G}$-line bundle over a principal $\mathcal{G}$-bundle $P \stackrel{\pi}{\to} M$. In physical applications P is the configuration space, while $\mathcal{G}$ is the invariance group of the theory and the effective action $\mathcal{Z}(p)$ is a section of this $\mathcal{G}$-line bundle.

In gauge theories P is the space of connections and $\mathcal{G}$ is the "pointed" gauge group, while for non linear supersymmetric sigma models P is the space of maps from a space X to a homogeneous space K/H and $\mathcal{G}$ is the group of maps from X to a subgroup H' of K acting freely on K/H. For the bosonic string theory, P is the space of isometry-free hyperbolic metrics on a Riemann surface of genus greater than two, and $\mathcal{G}$ is the semidirect product of the diffeomorphisms and the Weyl (rescalings) group.

The relevant cohomology group in which the anomalies live is interpreted as the kernel of the map induced in (integer) cohomology by the projection map of the principal fibration P. Local and global anomalies are split via the quotient fibration $P/\mathcal{G}_0$ (where $\mathcal{G}_0$ is the identity connected component of $\mathcal{G}$) : $P \stackrel{l}{\to} P' = P/\mathcal{G}_0 \stackrel{g}{\to} M = P'/\pi_0(\mathcal{G})$. Then some spectral sequences analysis applies to describe the "anomalies groups" and the "anomalies sequence".

2 - Group Cohomology

If $\mathcal{G}$ is a group and $\mathcal{M}$ a right $\mathcal{G}$-module, the group cohomology with coefficients in $\mathcal{M}$, $H^*(\mathcal{G},\mathcal{M})$, is the cohomology of the complex $C^*(\mathcal{G},\mathcal{M})$, where $C^n(\mathcal{G},\mathcal{M})$, the module of n-cochains, is the abelian group of maps from $\mathcal{G}\times\mathcal{G}\times ... \times\mathcal{G}$ to $\mathcal{M}$. The coboundary operator, $d^n : C^n(\mathcal{G},\mathcal{M}) \to C^{n+1}(\mathcal{G},\mathcal{M})$, is:

$$d^n F(g_1...g_{n+1}) = F(g_2...g_{n+1})g_1 + \sum_1^n (-1)^i F(g_1...g_i g_{i+1}...g_{n+1}) + (-1)^{n+1} F(g_1...g_n)$$

An interesting fact, which was one of the starting point of the theory, relies group cohomology to the usual singular cohomology; the cohomology of an *aspherical* space depends only on its fundamental group. Stated in another way, if Y is an Eilenberg-MacLane space of type $K(\mathcal{G},1)$, $H^*(\mathcal{G},\mathbf{Z}) = H^*(Y,\mathbf{Z})$, where, in the left-hand side, $\mathbf{Z}$ is considered as a trivial $\mathcal{G}$ module. For *finite* groups, $K(\mathcal{G},1)$ is homotopy equivalent to $B\mathcal{G}$, the classifying space. The isomorphism $H^*(\mathcal{G},\mathbf{Z}) = H^*(B\mathcal{G},\mathbf{Z})$ is the heart of Witten's construction of a lattice gauge theory description of three dimensional topological gauge theory for finite gauge groups [2].

The module of n-cochains $C^n(\mathcal{G},\mathcal{M})$ is a $\mathcal{G}$-module with a $\mathcal{G}$-action given by:

$$(Fg)(g_1...g_n) = F(g^{-1}g_1 g...g^{-1}g_n g)g$$

This action is trivial on $H^*(\mathcal{G},\mathcal{M})$.

When $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$, one has an exact sequence:

$$0 \to H^1(\mathcal{G}/\mathcal{H},\mathcal{M}^{\mathcal{H}}) \stackrel{inf}{\to} H^1(\mathcal{G},\mathcal{M}) \stackrel{res}{\to} H^1(\mathcal{H},\mathcal{M})^{\mathcal{G}} \stackrel{T}{\to} H^2(\mathcal{G}/\mathcal{H},\mathcal{M}^{\mathcal{H}}) \stackrel{inf}{\to} H^2(\mathcal{G},\mathcal{M})$$

where, for a $\mathcal{G}$-module $\mathcal{N}$, we have denoted by $\mathcal{N}^{\mathcal{G}}$ the $\mathcal{G}$-invariant elements. The homomorphisms res and inf are, respectively, the *restriction* to $\mathcal{H}$ of the cocycles of $\mathcal{G}$, and the *inflation*, i.e. the composition of the cocycles of $\mathcal{G}/\mathcal{H}$ with the projection $p : \mathcal{G} \to \mathcal{G}/\mathcal{H}$. The homomorphism T is the *transgression*.

When $\mathcal{H}$ is a *finite index* normal subgroup of $\mathcal{G}$, there exists a homomorphism, called *corestriction*, going in the opposite direction: $cor : H^*(\mathcal{H}, \mathcal{M}) \to H^*(\mathcal{G}, \mathcal{M})$. It is the homomorphism defined, in dimension zero (where $cor : \mathcal{M}^{\mathcal{H}} \to \mathcal{M}^{\mathcal{G}}$), by:

$$cor(m) = \sum_{c \in \mathcal{G}/\mathcal{H}} m\bar{c}$$

where, for each coset $c \in \mathcal{G}/\mathcal{H}$ choose, once and for all, a representative $\bar{c}$ requiring that for $c = \mathcal{H}$, $\bar{c} = 1$. Note that, for $g \in \mathcal{G}$, $\bar{c}g$ and $\overline{cg}$ are such that $\bar{c}g\overline{cg}^{-1} \in \mathcal{H}$. The definition of cor in dimension one is:

$$(cor u)(g) = \sum_{c \in \mathcal{G}/\mathcal{H}} u(\bar{c}g\overline{cg}^{-1})\bar{c}$$

An important property of cor is that the two compositions $cor \cdot res$ and $res \cdot cor$ are both the multiplication by $n = (\mathcal{G} : \mathcal{H})$. It follows that, for any m different from zero, $na = 0$ for any $a \in H^m(\mathcal{G}/\mathcal{H}, \mathcal{M}^{\mathcal{H}})$.

3 - The Anomalies Sequence

The elementary properties of group cohomology briefly recalled in the previous section turn out to be well suited for a description of anomalies. The only point is to find the "relevant" $\mathcal{G}$-module $\mathcal{M}$.

The interpretation of the effective action $\mathcal{Z}$ as a section of a $\mathcal{G}$-line bundle over P, gives naturally that $\mathcal{M} = \mathbf{C}^*(P)$, the right $\mathcal{G}$-module of the non vanishing functions from P to $\mathbf{C}$, with the natural action $(fg)(p) = f(pg)$. As usual, we switch to a multiplicative notation for $\mathbf{C}^*$ and, therefore, for the cohomology.

A one-cochain $F : \mathcal{G} \to \mathbf{C}^*(P)$, putting $f(p, g) = F(g)(p)$, gives a map $f : P \times \mathcal{G} \to \mathbf{C}^*$, and the cocycle condition become $f(p, g_1 g_2) = f(pg_1, g_2)f(p, g_1)$. The cohomology group $H^1(\mathcal{G}, \mathbf{C}^*(P))$ represents, geometrically, the group of $\mathcal{G}$-isomorphism classes of *topologically trivial* $\mathcal{G}$-lines bundles over P, that is the *anomalies*. This can be seen from the exact sequence:

$$1 \to H^1(\mathcal{G}, \mathbf{C}^*(P)) \to H^2(M, \mathbf{Z}) \stackrel{\pi^*}{\to} H^2(P, \mathbf{Z})$$

where the first (injective) arrow is given by $f \to P \times_f \mathbf{C}$ (i.e. we identify (p, c) and $(pg, f(p, g)c)$).

These $\mathcal{G}$-isomorphism classes represent the anomalies in the sense that in perturbative field theory one first defines the effective action $\mathcal{Z}(p)$; the obstruction to extending this functional to the whole $\mathcal{G}$-orbit is given by the non triviality of f. In fact the action of $\mathcal{G}$ on $\mathcal{Z}$ is represented by $\mathcal{Z}(pg) = f(p,g)\mathcal{Z}(p)$.

When the group $\mathcal{G}$ is not connected, one could find "global anomalies", that is trivial $\mathcal{G}_0$-cocycles that extends non trivially to $\mathcal{G}$. The non trivial (and $\mathcal{G}$-invariant) $\mathcal{G}_0$-cocycles are called "local anomalies".

Putting $\mathcal{H} = \mathcal{G}_0$ and observing that $\mathbf{C}^*(P)^{\mathcal{G}_0} = \mathbf{C}^*(P')$, the exact sequence of the previous section gives the "anomalies sequence":

$$1 \to H^1(\pi_0(\mathcal{G}), \mathbf{C}^*(P')) \stackrel{inf}{\to} H^1(\mathcal{G}, \mathbf{C}^*(P)) \stackrel{res}{\to} H^1(\mathcal{G}_0, \mathbf{C}^*(P))^{\mathcal{G}} \stackrel{T}{\to} H^2(\pi_0(\mathcal{G}), \mathbf{C}^*(P'))$$

The geometrical interpretation in terms of line bundles over P, applied to the factorisation $P \stackrel{l}{\to} P' = P/\mathcal{G}_0 \stackrel{g}{\to} M = P'/\pi_0(\mathcal{G})$, gives:

$$1 \to H^1(\mathcal{G}_0, \mathbf{C}^*(P)) \to H^2(P', \mathbf{Z}) \stackrel{l^*}{\to} H^2(P, \mathbf{Z})$$

and

$$1 \to H^1(\pi_0(\mathcal{G}), \mathbf{C}^*(P')) \to H^2(M, \mathbf{Z}) \stackrel{g^*}{\to} H^2(P', \mathbf{Z})$$

These sequences identify $H^1(\pi_0(\mathcal{G}), \mathbf{C}^*(P'))$ with the global anomalies and the $\mathcal{G}$-invariant elements of $H^1(\mathcal{G}_0, \mathbf{C}^*(P))$ with the local anomalies.

In the case of $SU(N)$ gauge theories ($N \geq 3$), $\pi_0(\mathcal{G})$ is finite when, for example, the first Betti number of the four-manifold X representing the (euclidean) compact space-time is zero (i.e. $H^3(X, \mathbf{Z})$ is finite). In this case the existence of the *cor* map says that all the global anomalies are *torsion elements*. This means that, if local anomalies are absent, and if $n = (\mathcal{G} : \mathcal{G}_0)$, the n-th power of the effective action is gauge invariant.

Note that, if $\pi_0(\mathcal{G})$ is finite, the anomalies sequence implies that, if $H^2(\pi_0(\mathcal{G}), \mathbf{C}^*(P'))$ is trivial, the map *res* is surjective. Moreover *cor* is injective. This means that the only torsion elements in the anomaly group are the global anomalies. All local anomalies, in this case, can be detected *via* the family index theorem and represented by functionals on the space P.

The topological interpretation of the first cohomology group of $\mathcal{G}$ with values in the $\mathcal{G}$-module $\mathbf{C}^*(P)$ gives a more explicit description of anomalies in terms of the topology of $\mathcal{G}$ and P.

For the local anomalies, $H^1(\mathcal{G}_0, \mathbf{C}^*(P))$, one can apply the low dimensional exact cohomology sequence of the bundle $P \to P'$. In this case (recall that $\mathcal{G}_0$ is, by definition, a connected group) the Leray spectral sequence gives, in absence of monodromy, an exact sequence. The result is:

$$0 \to H^1(P', \mathbf{Z}) \to H^1(P, \mathbf{Z}) \to H^1(\mathcal{G}_0, \mathbf{Z}) \to H^2(P', \mathbf{Z}) \stackrel{l^*}{\to} H^2(P, \mathbf{Z})$$

From this,

$$H^1(\mathcal{G}_0, \mathbf{C}^*(P)) = kerl^* = H^1(\mathcal{G}_0, \mathbf{Z})/H^1(P, \mathbf{Z})$$

In the case of gauge theories, where P is the space of connections , all the cohomology of P is trivial and we find:

$$H^1(\mathcal{G}_0, \mathbf{C}^*(P)) = H^1(\mathcal{G}_0, \mathbf{Z}) = H^2(P', \mathbf{Z})$$

The last group is the group of lines bundles over P'; the effective action is a section of the determinant line bundle of the Dirac chiral operator over P. This line bundle is $\mathcal{G}_0$ trivial if the corresponding bundle over P' has vanishing Chern class. The local anomaly cancellation is controlled by the family index theorem that computes precisely this class.

In the case of the bosonic string theory, the situation is completely different: local anomalies are absent and global anomalies are of *free* type (see [3]).

One final remark is in order: even if it is not suited for practical computations, the group cohomological analysis of anomalies that we have described in this talk shows its effectiveness in giving a *a priori* description of the objects we are dealing with. For example, it is not always true that global anomalies are of torsion type and local anomalies are of free type; this depends on the topology of both the "gauge group" and the "configuration space". A detailed analysis of concrete examples could reveal anomalies of a completely unexpected nature.

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