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# Noncommutative de Rham cohomology of finite groups

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**Abstract**

We study de Rham cohomology for various differential calculi on finite groups $G$ up to order 8. These include the permutation group $S_3$, the dihedral group $D_4$ and the quaternion group $Q$. Poincaré duality holds in every case, and under some assumptions (essentially the existence of a top form) we find that it must hold in general.

A short review of the bicovariant (noncommutative) differential calculus on finite $G$ is given for selfconsistency. Exterior derivative, exterior product, metric, Hodge dual, connections, torsion, curvature, and biinvariant integration can be defined algebraically. A projector decomposition of the braiding operator is found, and used in constructing the projector on the space of 2-forms. By means of the braiding operator and the metric a knot invariant is defined for any finite group.

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# 1 Introduction

Most differential geometric objects pertaining to smooth manifolds can be generalized in the case of discrete sets. When these sets are related to a group structure (as for example finite group sets $G$), the induced Hopf algebra structure on the functionals $Fun(G)$ gives a canonical way to construct bicovariant calculi on them [1].

Differential geometry plays a basic role in the construction of field theories describing the fundamental interactions in nature: gravity and Yang-Mills actions are rooted in Riemannian and fiber bundle geometry. The idea of translating the concepts of metric, connection, curvature to discrete cases has been explored in the past, one of the first fruitful instances being Regge calculus (for recent reviews and reference lists see for example [2]). The physical motivations of this idea reside in the nonrenormalizability of Einstein gravity (whereas a field theory on discrete spacetime has no ultraviolet divergences) and computational advantages in the study of nonperturbative phenomena in quantum gauge theories by numerical evaluation of path-integrals. Moreover the possibility of a discrete spacetime, with a "granularity" of the order of the Planck length, has emerged also within the framework of string/brane theories.

Another (and related) approach to the "algebrization" of geometry has been pioneered by A. Connes, in the context of noncommutative geometry [3].

Using the general results of Woronowicz [1], (noncommutative) differential calculi have been constructed on $Fun(quantum\ groups)$ (for an introductory review see for ex. [4, 5]) and $Fun(finite\ groups)$ [6, 7, 8, 9, 11, 10, 12, 13], two particular examples of Hopf algebras, of interest for physical applications. The corresponding differential geometric objects and operations can be used to construct actions invariant under the quantum group transformations (see for ex. [14]), or under finite group transformations [15, 8, 9, 11, 10, 12], generalizing the usual gravity and gauge actions.

On finite groups $G$ the noncommutativity is mild, in the sense that functions on $G$ commute between themselves, and only the commutations between functions and differentials, and of differentials between themselves are nontrivial.

For smooth manifolds, de Rham cohomology provides a bridge between differential geometry and topology. It is natural to ask whether this bridge exists also in the case of finite group manifolds. Using integration on finite groups, can one define the analogue of characteristic classes, and relate them to topological properties of finite group spaces ? These spaces are regular graphs (i.e. with each vertex having the same number of incident links) depending on the particular differential calculus defined on them.

In the present paper we begin an investigation of de Rham cohomology of finite group manifolds. A systematic analysis is carried out for finite groups up to order 8.

Table 1 summarizes our findings, and contains the following informations: name of group, labels of independent one-forms, number of independent k-forms, Betti numbers.

The alternating sum of Betti numbers always vanishes. Thus the finite groups up to order 8 have vanishing Euler number (for all the differential calculi we have considered).

An attempt at self-consistency is made in Section 2, with a resumé on the differential geometry of finite groups. A graphical representation of the braiding operator and the metric allows to build a knot invariant for any finite group. Some new results are also presented in Section 3, where a projector decomposition of the braiding operator is found, and used to construct explicitly a projector on the space of 2-forms. The regular graphs corresponding to particular differential calculi on $S_3$, $Q$ and $D_4$ are given in Appendix 1.

Section 4 is devoted to de Rham cohomology of finite groups, and establishes general formulas for the exterior derivative of an arbitrary $k$-form in terms of matrix $M$ whose kernel yields the closed $k$-forms. Hodge decomposition theorem holds, the proof being identical to the one for compact orientable manifolds. Under some assumptions the Laplacian $\Delta = d\delta + \delta d$ commutes with the Hodge operator. Following classical proofs, this implies Poincaré duality.

Section 5 contains some conclusions and open questions.

# 2 Bicovariant calculi on finite groups

Let $G$ be a finite group of order $n$ with generic element $g$ and unit $e$. Consider $Fun(G)$, the set of complex functions on $G$. An element $f$ of $Fun(G)$ is specified by its values $f_g \equiv f(g)$ on the group elements $g$, and can be written as

$$f = \sum_{g \in G} f_g x^g, \quad f_g \in \mathbb{C} \tag{2.1}$$

where the functions $x^g$ are defined by

$$x^g(g') = \delta^g_{g'} \tag{2.2}$$

Thus $Fun(G)$ is a n-dimensional vector space, and the $n$ functions $x^g$ provide a basis. $Fun(G)$ is also a commutative algebra, with the usual pointwise sum and product, and unit $I$ defined by $I(g) = 1, \forall g \in G$. In particular:

$$x^g x^{g'} = \delta_{g,g'} x^g, \quad \sum_{g \in G} x^g = I \tag{2.3}$$

The left and right actions of the group $G$ on itself

$$L_g\, g' = g\, g' = R_{g'}\, g \quad \forall g, g' \in G\ , \tag{2.4}$$

induce the left and right actions (pullbacks) $\mathcal{L}_g$, $\mathcal{R}_g$ on $Fun(G)$

$$[\mathcal{L}_g\, f](g') = f(g\, g') = [\mathcal{R}_{g'}\, f](g) \quad \forall f \in Fun(G)\ . \tag{2.5}$$

For the basis functions we find easily:

$$\mathcal{L}_{g_1} x^g = x^{g_1^{-1} g}, \quad \mathcal{R}_{g_1} x^g = x^{g g_1^{-1}} \tag{2.6}$$

Moreover:

$$\mathcal{L}_{g_1}\mathcal{L}_{g_2} = \mathcal{L}_{g_2 g_1}, \quad \mathcal{R}_{g_1}\mathcal{R}_{g_2} = \mathcal{R}_{g_1 g_2}, \quad \mathcal{L}_{g_1}\mathcal{R}_{g_2} = \mathcal{R}_{g_2}\mathcal{L}_{g_1} \tag{2.7}$$

The $G$ group structure induces a Hopf algebra structure on $Fun(G)$, and this allows the construction of differential calculi on $Fun(G)$, according to the techniques of ref. [1, 4]. We list here the main definitions and properties. A detailed treatment can be found in

[8], and Hopf algebraic formulas, allowing contact with the general method of [1, 4], are listed in the Appendix of [12].

A (first-order) differential calculus on $Fun(G)$ is defined by a linear map $d$: $Fun(G) \rightarrow \Gamma$, satisfying the Leibniz rule $d(ab) = (da)b + a(db), \quad \forall a, b \in Fun(G)$. The "space of 1-forms" $\Gamma$ is an appropriate bimodule on $Fun(G)$, which essentially means that its elements can be multiplied on the left and on the right by elements of $Fun(G)$. From the Leibniz rule $da = d(Ia) = (dI)a + Ida$ we deduce $dI = 0$. Consider the differentials of the basis functions $x^g$. From $0 = dI = d(\sum_{g \in G} x^g) = \sum_{g \in G} dx^g$ we see that in this calculus only $n - 1$ differentials are independent.

A *bicovariant* differential calculus is obtained by requiring that $\mathcal{L}_g$ and $\mathcal{R}_g$ commute with the exterior derivative $d$. This requirement in fact *defines* their action on differentials: $\mathcal{L}_g db \equiv d(\mathcal{L}_g b), \forall b \in Fun(G)$ and similarly for $\mathcal{R}_g db$. More generally:

$$\mathcal{L}_g(adb) \equiv (\mathcal{L}_g a)\mathcal{L}_g db = (\mathcal{L}_g a)d(\mathcal{L}_g b) \tag{2.8}$$

and similarly for $\mathcal{R}_g$.

As in usual Lie group manifolds, we can introduce in $\Gamma$ the left-invariant one-forms $\theta^g$:

$$\theta^g \equiv \sum_{h \in G} x^{hg^{-1}} dx^h = \sum_{h \in G} x^h dx^{hg}, \tag{2.9}$$

It is immediate to check that indeed $\mathcal{L}_k \theta^g = \theta^g$. The right action of $G$ on the elements $\theta^g$ is given by:

$$\mathcal{R}_h \theta^g = \theta^{ad(h)g}, \quad \forall h \in G \tag{2.10}$$

where $ad$ is the adjoint action of $G$ on itself, i.e. $ad(h)g \equiv hgh^{-1}$. Notice that $\theta^e$ is biinvariant, i.e. both left and right invariant.

From $\sum_{g \in G} dx^g = 0$ one finds:

$$\sum_{g \in G} \theta^g = \sum_{g,h \in G} x^h dx^{hg} = \sum_{h \in G} x^h \sum_{g \in G} dx^{hg} = 0 \tag{2.11}$$

Therefore we can take as basis of the cotangent space $\Gamma$ the $n - 1$ linearly independent left-invariant one-forms $\theta^g$ with $g \neq e$. Smaller sets of $\theta^g$ can be consistently chosen as basis, and correspond to different choices of the bimodule $\Gamma$, see later. Using (2.3) the relations (2.9) can be inverted:

$$dx^h = \sum_{g \in G} x^{hg^{-1}} \theta^g = \sum_{g \neq e} (x^{hg^{-1}} - x^h)\theta^g \tag{2.12}$$

Analogous results hold for right invariant one-forms $\zeta^g$:

$$\zeta^g = \sum_{h \in G} x^{g^{-1}h} dx^h \tag{2.13}$$

Using the definition of $\theta^g$ (2.9), the commutations between $x$ and $\theta$ are easily obtained:

$$x^h dx^g = x^h \theta^{h^{-1}g} = \theta^{h^{-1}g} x^g \quad (h \neq g) \quad \Rightarrow \theta^g x^h = x^{hg^{-1}} \theta^g \quad (g \neq e) \tag{2.14}$$

and imply the general commutation rule between functions and left-invariant one-forms:

$$\theta^g f = [\mathcal{R}_g f]\theta^g \qquad (g \neq e) \tag{2.15}$$

Thus functions do commute between themselves (i.e. $Fun(G)$ is a commutative algebra) but do not commute with the basis of one-forms $\theta^g$. In this sense the differential geometry of $Fun(G)$ is noncommutative.

The differential of an arbitrary function $f \in Fun(G)$ can be found with the help of (2.12):

$$\begin{aligned} df = \sum_h f_h\, dx^h &= \sum_{g,h} f_h\, x^{h\,g^{-1}}\, \theta^g = \sum_{g\neq e} (\sum_h f_h\, x^{h\,g^{-1}} - f)\, \theta^g = \\ &= \sum_{g\neq e} ([\mathcal{R}_g\, f] - f)\, \theta^g = \sum_{g\neq e} (t_g\, f)\, \theta^g \; . \end{aligned} \tag{2.16}$$

Here the finite difference operators $t_g = \mathcal{R}_g - 1$ are the analogues of (left-invariant) tangent vectors. They satisfy the Leibniz rule:

$$t_g(ff') = (t_g f)f' + \mathcal{R}_g(f)t_g f' = (t_g f)\mathcal{R}_g f' + f t_g f' \tag{2.17}$$

and close on the fusion algebra:

$$t_g\, t_{g'} = (\mathcal{R}_{gg'} - 1) - (\mathcal{R}_g - 1) - (\mathcal{R}_{g'} - 1) = \sum_h C^h_{g,g'}\, t_h \; , \tag{2.18}$$

where the structure constants $C^h_{g,g'}$ are

$$C^h_{g,g'} = \delta^h_{gg'} - \delta^h_g - \delta^h_{g'} \; , \tag{2.19}$$

The commutation rule (2.15) allows to express the differential of a function $f \in Fun(G)$ as a commutator of $f$ with the biinvariant form $\sum_{g\neq e} \theta^g = -\theta^e$:

$$df = [\sum_{g\neq e} \theta^g,\, f] = -[\theta^e, f] \; . \tag{2.20}$$

An *exterior product*, compatible with the left and right actions of $G$, can be defined as

$$\begin{aligned} \theta^g \wedge \theta^{g'} &= \theta^g \otimes \theta^{g'} - \sum_{k,k'} \Lambda^{g\,g'}{}_{k'\,k} \theta^{k'} \otimes \theta^k = \theta^g \otimes \theta^{g'} - \theta^{gg'g^{-1}} \otimes \theta^g = \\ &= \theta^g \otimes \theta^{g'} - [\mathcal{R}_g \theta^{g'}] \otimes \theta^g \; , \quad (g, g' \neq e) \; , \end{aligned} \tag{2.21}$$

where the tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a\rho'$, $a(\rho \otimes \rho') = (a\rho) \otimes \rho'$ and $(\rho \otimes \rho')a = \rho \otimes (\rho' a)$. The braiding matrix $\Lambda$:

$$\Lambda^{g\,g'}{}_{k'\,k} = \delta^{gg'g^{-1}}_{k'}\, \delta^g_k \; , \qquad \Lambda^{-1\;g\,g'}{}_{k'\,k} = \delta^{g'}_{k'}\, \delta^{g'^{-1}gg'}_k \qquad (g, g' \neq e) \; . \tag{2.22}$$

satisfies the Yang-Baxter equation $\Lambda^{nm}{}_{ij}\Lambda^{js}{}_{kq}\Lambda^{ik}{}_{rp} = \Lambda^{ms}{}_{kj}\Lambda^{nk}{}_{ri}\Lambda^{ij}{}_{pq}$ (or in condensed notation $\Lambda_{12}\Lambda_{23}\Lambda_{12} = \Lambda_{23}\Lambda_{12}\Lambda_{23}$). With this exterior product we find

$$\theta^g \wedge \theta^g = 0 \quad (\forall g) \ , \quad \theta^g \wedge \theta^{g'} = -\theta^{g'} \wedge \theta^g \quad (\forall g, g' : \ \ [g, g'] = 0 \ , \ \ g \neq e) \ . \tag{2.23}$$

Left and right actions on $\Gamma \otimes \Gamma$ are simply defined by:

$$\mathcal{L}_h(\rho \otimes \rho') = \mathcal{L}_h\rho \otimes \mathcal{L}_h\rho', \quad \mathcal{R}_h(\rho \otimes \rho') = \mathcal{R}_h\rho \otimes \mathcal{R}_h\rho' \tag{2.24}$$

Compatibility of the exterior product with $\mathcal{L}$ and $\mathcal{R}$ means that

$$\mathcal{L}(\theta^i \wedge \theta^j) = \mathcal{L}\theta^i \wedge \mathcal{L}\theta^j, \ \ \mathcal{R}(\theta^i \wedge \theta^j) = \mathcal{R}\theta^i \wedge \mathcal{R}\theta^j \tag{2.25}$$

Only the second relation is nontrivial and is verified upon use of the definition (2.21). We can generalize the previous definition to exterior products of $k$ left-invariant one-forms:

$$\theta^{i_1} \wedge ... \wedge \theta^{i_k} \equiv A^{i_1..i_k}_{j_1..j_k} \ \theta^{j_1} \otimes ... \otimes \theta^{j_k} \tag{2.26}$$

or in short-hand notation:

$$\theta^1 \wedge ... \wedge \theta^k = A_{1...k} \ \theta^1 \otimes ... \otimes \theta^k \tag{2.27}$$

The labels $1...k$ in $A$ refer to index couples, and $A_{1,...k}$ is the analogue of the antisymmetrizer of $k$ spaces, defined by the recursion relation

$$A_{1...k} = [1 - \Lambda_{k-1,k} + \Lambda_{k-2,k-1}\Lambda_{k-1,k} - \ldots - (-1)^k\Lambda_{12}\Lambda_{23}\cdots\Lambda_{k-1,k}]A_{1...k-1}, \tag{2.28}$$

where $A_{12} = 1 - \Lambda_{12}$. The space of $k$-forms $\Gamma^{\wedge k}$ is therefore defined as in the usual case but with the new permutation operator $\Lambda$, and can be shown to be a bicovariant bimodule (see for ex. [5]), with left and right action defined as for $\Gamma \otimes ... \otimes \Gamma$ with the tensor product replaced by the wedge product. The graded bimodule $\Omega = \sum_k \Gamma^{\wedge k}$, with $\Gamma^{\wedge k} = \Gamma^{\otimes k}/Ker(A_{1...k})$, is the exterior algebra of forms.

The *exterior derivative* is defined as a linear map $d \ : \ \Gamma^{\wedge k} \longrightarrow \Gamma^{\wedge (k+1)}$ satisfying $d^2 = 0$ and the graded Leibniz rule

$$d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^k \rho \wedge d\rho' \tag{2.29}$$

where $\rho \in \Gamma^{\wedge k}$, $\rho' \in \Gamma^{\wedge k'}$, $\Gamma^{\wedge 0} \equiv Fun(G)$ . Left and right action is defined as usual:

$$\mathcal{L}_g(d\rho) = d\mathcal{L}_g\rho, \quad \mathcal{R}_g(d\rho) = d\mathcal{R}_g\rho \tag{2.30}$$

In view of relation (2.10) the algebra $\Omega$ has natural quotients over the ideals $H_g = \{\theta^{hgh^{-1}}, \forall h\}$, corresponding to the various conjugacy classes of the elements $g$ in $G$. The different bicovariant calculi on $Fun(G)$ are in 1-1 correspondence with different quotients of $\Omega$ by any sum of the ideals $H = \sum H_g$, cf. [6, 7, 8]. In practice one simply sets $\theta^g = 0$ for all $g \neq e$ not belonging to the particular union $G'$ of conjugacy classes characterizing the differential calculus. The dimension of the space of independent 1-forms for each bicovariant calculus on $Fun(G)$ is therefore equal to the dimension of the subspace $\Gamma/H$. If there are $r$ nontrivial conjugacy classes, the number of possible unions $G'$ of these classes is $2^r - 1$. We have then $2^r - 1$ differential calculi.

The *Cartan-Maurer equation* for the differential forms $\theta^g$ (2.9) is obtained by direct calculation, using the definition (2.9), the expression (2.12) of $dx^h$ in terms of $\theta$'s, and the commutations (2.14):

$$d\theta^g = -\sum_{h\neq e,h'\neq e} \delta^g_{hh'}\theta^h \wedge \theta^{h'} + \sum_{k\neq e} \theta^k \wedge \theta^g + \sum_{k\neq e} \theta^g \wedge \theta^k = -\sum_{h\neq e}\sum_{h'\neq e} C^g_{h,h'}\,\theta^h \wedge \theta^{h'} \;, \quad (g \neq e) \tag{2.31}$$

where the structure constants $C^g_{h,h'}$ are given in (2.19). Using the identity:

$$\sum_{h\neq e,h'\neq e} \delta^k_{hh'}\,\theta^h \wedge \theta^{h'} = \sum_{h\neq e,h'\neq e} \delta^k_{hh'}\,\left(\theta^h \otimes \theta^{h'} - \theta^{hh'h^{-1}} \otimes \theta^h\right) = 0 \tag{2.32}$$

the Cartan-Maurer equation can be rewritten by means of the anticommutator of $\theta^g$ with the biinvariant form $\theta^e$:

$$d\theta^g = -\theta^e \wedge \theta^g - \theta^g \wedge \theta^e \tag{2.33}$$

cf. the case of 0-forms (2.20). Considering now a generic element $\rho = a\theta$ of $\Gamma$ it is easy to find that $d\rho = -\theta^e \wedge \rho - \rho \wedge \theta^e$. The general rule is

$$d\rho = [-\theta^e, \rho]_{grad} \equiv -\theta^e \wedge \rho + (-1)^{deg(\rho)}\rho \wedge \theta^e \tag{2.34}$$

valid for any $k$-form, where $[-\theta^e, \rho]_{grad}$ is the graded commutator.

There are two (Hopf algebra) *conjugations* on $Fun(G)$ [6, 10]

$$(x^g)^* = x^g \;, \quad (x^g)^\star = x^{g^{-1}} \tag{2.35}$$

These involutions can be extended to the whole exterior (Hopf) algebra $\Omega$:

$$(\theta^g)^* = -\theta^{g^{-1}} \;, \quad (\theta^g)^\star = \zeta^g \tag{2.36}$$

such that $(\rho \wedge \rho')^* = (-1)^{deg(\rho)deg(\rho')}\rho'^* \wedge \rho^*$ etc. We'll use the $^*$-conjugation in the sequel. Consistency of this conjugation requires that if $\theta^g \neq 0$ then $\theta^{g^{-1}} \neq 0$ as well: we have to include in $\Gamma/H$ at least the two ideals $H_g$ and $H_{g^{-1}}$ (if they do not coincide). We obtain thus a $^*$-differential calculus, i.e. $(df)^* = d(f^*)$.

In fact the conjugations can also be defined directly on the tensor algebra. For example $(\theta_1 \otimes \theta_2)^* = \Lambda(\theta_2^* \otimes \theta_1^*)$, or with explicit indices $(\theta^{i_1} \otimes \theta^{i_2})^* = \theta^{i_2^{-1}i_1^{-1}i_2} \otimes \theta^{i_2^{-1}}$. This rule is consistent with $(\theta_1 \wedge \theta_2)^* = -\theta_2^* \wedge \theta_1^*$ as one proves by recalling that $[\Lambda(\theta_1 \otimes \theta_2)]^* = \Lambda^{-1}[(\theta_1 \otimes \theta_2)^*]$. In general:

$$(\theta^{i_1} \otimes \theta^{i_2} \otimes \cdots \otimes \theta^{i_k})^* = (-1)^k\; \theta^{ad(i_2 \ldots i_k)^{-1} i_1^{-1}} \otimes \theta^{ad(i_3 \ldots i_k)^{-1} i_2^{-1}} \otimes \cdots \otimes \theta^{i_k^{-1}} \tag{2.37}$$

$$(\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k})^* = (-1)^{\frac{k(k+1)}{2}}\; \theta^{i_k^{-1}} \wedge \cdots \wedge \theta^{i_2^{-1}} \wedge \theta^{i_1^{-1}} \tag{2.38}$$

The fact that both $\theta^g$ and $\theta^{g^{-1}}$ are included in the basis of left-invariant 1-forms characterizing the differential calculus also ensures the existence of a unique metric (up to a normalization).

The *metric* is defined as a bimodule pairing, symmetric on left-invariant 1-forms. It maps couples of 1-forms $\rho, \sigma$ into $Fun(G)$, and satisfies the properties

$$< f\rho, \sigma h >= f < \rho, \sigma > h \ , \quad < \rho f, \sigma >=< \rho, f\sigma > \ . \tag{2.39}$$

where $f$ and $h$ are arbitrary functions belonging to $Fun(G)$. Up to a normalization the above properties determine the metric on the left-invariant 1-forms. Indeed from $< \theta^g, f\theta^h >=< \theta^g, \theta^h > \mathcal{R}_{h^{-1}} f = \mathcal{R}_g f < \theta^g, \theta^h >$ one deduces:

$$g^{rs} \equiv < \theta^r, \theta^s > \equiv -\delta^r_{s^{-1}} \tag{2.40}$$

the minus sign being a convenient choice of normalization (so that (2.43), and consequently the positivity property of (2.44) holds). Thus $g^{rs}$ is symmetric and $\theta^r$ has nonzero pairing only with $\theta^{r^{-1}}$. The pairing is compatible with the $^*$-conjugation

$$< \rho, \sigma >^* =< \sigma^*, \rho^* > \tag{2.41}$$

We can generalize $< \ , \ >$ to tensor products of left-invariant 1-forms as follows (as proposed in the second ref. of [12]):

$$< \theta^{i_1} \otimes \cdots \otimes \theta^{i_k}, \theta^{j_1} \otimes \cdots \otimes \theta^{j_k} > \equiv g^{i_k, j_k} g^{i_{k-1}, ad(i_k) j_{k-1}} g^{i_{k-2}, ad(i_{k-1} i_k) j_{k-2}} \ldots \ g^{i_1, ad(i_2 \cdots i_k) j_1} \tag{2.42}$$

Using (2.37) we find the duality relation:

$$< (\theta^{i_1} \otimes \cdots \otimes \theta^{i_k})^*, \theta^{i_1} \otimes \cdots \otimes \theta^{i_k} >= 1 \tag{2.43}$$

The pairing (2.42) is extended to all tensor products by $< f\rho, \sigma h >= f < \rho, \sigma > h$ where now $\rho$ and $\sigma$ are generic tensor products of same order. Then we prove easily that $< \rho f, \sigma >=< \rho, f\sigma >$ for any function $f$, so that $< \ , \ >$ is a bimodule pairing. Moreover $< \rho, \sigma >=< \sigma, \rho >$, i.e. the pairing is symmetric, but only when $\rho$ and $\sigma$ are tensor products of $\theta$'s, as one can prove from the definition (2.42). Another interesting property is

$$< \rho, \rho^* >= N(\rho) \ |f|^2 \tag{2.44}$$

where $\rho$ is a generic $k$-form $\rho = f \ \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}$ and $N(\rho)$ is a real positive constant depending on $\rho$. For example $< \theta^{i_1} \wedge \theta^{i_2}, (\theta^{i_1} \wedge \theta^{i_2})^* >= 2$ (in this case $N(\rho)$ does not depend on $\rho$).

In general for a differential calculus with $m$ independent tangent vectors, there is an integer $p \geq m$ such that the linear space of left-invariant $p$-forms is 1-dimensional, and $(p+1)$- forms vanish identically [1]. This is so far an experimental result, based on the examples we have studied. It implies that every product of $p$ basis one-forms

[1] with the exception of $Z_2$, see ref. [9]

$\theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p}$ is proportional to one of these products, which can be chosen to define the volume form $vol$:

$$\theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} = \epsilon^{g_1,g_2,...g_p} \; vol \tag{2.45}$$

where $\epsilon^{g_1,g_2,...g_p}$ is the proportionality constant. The volume $p$-form is obviously left invariant. It is also right invariant [8] (the proof is based on the $ad(G)$ invariance of the $\epsilon$ tensor: $\epsilon^{ad(g)h_1,...ad(g)h_p} = \epsilon^{h_1,...h_p}$).

Finally, if $vol = \theta^{k_1} \wedge ... \wedge \theta^{k_p}$, then

$$vol^* = (-1)^{\frac{p(p+1)}{2}} \epsilon^{k_p^{-1}...k_1^{-1}} vol \tag{2.46}$$

so that $vol$ is either real or imaginary. If $vol^* = -vol$ we can always multiply it by $i$ and obtain a real volume form. In that case comparing $(\theta^{g_1} \wedge ... \wedge \theta^{g_p})^* = (-1)^{\frac{p(p+1)}{2}} \theta^{g_p^{-1}} \wedge ... \wedge \theta^{g_1^{-1}} = \epsilon^{g_p^{-1}...g_1^{-1}} (-1)^{\frac{p(p+1)}{2}} vol$ with $(\theta^{g_1} \wedge ... \wedge \theta^{g_p})^* = \epsilon^{g_1...g_p} \; (vol)^* = \epsilon^{g_1...g_p} \; vol$ yields

$$\epsilon^{g_p^{-1}...g_1^{-1}} = (-1)^{\frac{p(p+1)}{2}} \; \epsilon^{g_1...g_p} \tag{2.47}$$

The pairing of the volume with itself is simply:

$$< vol, vol >= N(vol) \tag{2.48}$$

Having identified the volume $p$-form it is natural to define the integral of a function on $G$

$$\int f \; vol = \sum_{g \in G} f(g) \tag{2.49}$$

the right-hand side being just the Haar measure of the function $f$.

Due to the biinvariance of the volume form, the integral map $\int : \Gamma^{\wedge p} \mapsto \mathbb{C}$ satisfies the biinvariance conditions:

$$\int \mathcal{L}_g \rho = \int \rho = \int \mathcal{R}_g \rho \tag{2.50}$$

Moreover, under the assumption that $d(\theta^{g_2} \wedge ... \wedge \theta^{g_p}) = 0$, i.e. that any exterior product of $p-1$ left-invariant one-forms $\theta$ is closed, the important property holds:

$$\int df = 0 \tag{2.51}$$

with $f$ any $(p-1)$-form: $f = f_{g_2,...g_p} \, \theta^{g_2} \wedge ... \wedge \theta^{g_p}$. This property, which allows integration by parts, has a simple proof (see ref. [8]). When the volume form belongs to a nontrivial cohomology class, $d(\theta^{g_2} \wedge ... \wedge \theta^{g_p})$ must vanish (otherwise it should be proportional to $vol$, and this contradicts $vol \neq d\rho$) and therefore integration by parts holds.

The *Hodge dual*, an important ingredient for gauge theories, has been defined in [16, 12] as the unique map from $k$-forms $\sigma$ to $(p-k)$-forms $*\sigma$ such that

$$\rho \wedge *\sigma =< \rho, \sigma > vol \qquad \rho, \sigma \; k\text{-forms} \tag{2.52}$$

The Hodge dual is left linear; if $vol$ is central it is also right linear :

$$* (f \rho h) = f(*\rho)h \tag{2.53}$$

with $f, h \in Fun(G)$. Moreover

$$* 1 = 1 \; vol \quad , \quad * vol = N(vol) \tag{2.54}$$

**Conjecture 1:** the definition (2.52) is equivalent to the following explicit expression of the Hodge dual on the exterior products of $\theta$ 1-forms:

$$* (\theta^{i_1} \wedge ... \wedge \theta^{i_k}) = const \cdot \epsilon_{j_{k+1} \cdots j_p}{}^{i_1 \ldots i_k} \;\; \theta^{j_p} \wedge ... \wedge \theta^{j_{k+1}} \tag{2.55}$$

for an appropriate value of $const$, and where the $j$ indices of the epsilon tensor are lowered by means of the metric $g_{i,j}$. We can easily verify a necessary condition for the equivalence: setting $\rho = \theta^{i_1} \wedge ... \wedge \theta^{i_k}$ into

$$\rho^* \wedge *\rho =< \rho^*, \rho > \; vol = N(\rho) \; vol \tag{2.56}$$

is indeed consistent with (2.55) because of (2.47).

**Conjecture 2:** Hodge duality is an involution. It is so when acting on 0-forms and on $vol$: indeed (2.54) imply $* * 1 = N(vol)$, $* * vol = N(vol) \; vol$. When acting on a generic $k$-form, the Hodge duality being an involution is consistent with (2.56) (although it is not clear that it is implied by it). Indeed, the conjugate of (2.56) is:

$$(*\rho)^* \wedge \rho =< \rho^*, \rho > vol \tag{2.57}$$

On the other hand, substituting $\rho \to *\rho$ into (2.56) yields

$$(*\rho)^* \wedge * * \rho =< (*\rho)^*, *\rho > vol \tag{2.58}$$

These two relations are consistent with

$$* *\rho = \frac{< (*\rho)^*, *\rho >}{< \rho^*, \rho >} \rho \tag{2.59}$$

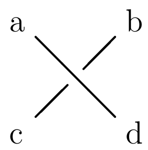
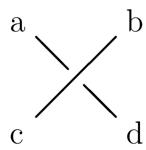
i.e. with the involutive property of $*$.

In the case of the 3-D calculus on $S_3$ a Hodge involution $** = id$ can be defined on the basis $k$-forms as in (2.55) (see also the second ref. in [11]) with $const = 1/\sqrt{3}$.

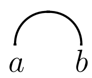
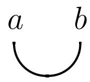
**Note 1:** the "group manifold" of a finite group is simply a collection of points corresponding to the group elements, linked together in various ways, each corresponding to a particular differential calculus on $Fun(G)$ [6, 8]. The links are associated to the tangent vectors $\mathcal{R}_h - 1$ of the differential calculus, or equivalently to the right actions $\mathcal{R}_h$, where $h$ belongs to the union $G'$ of conjugacy classes characterizing the differential calculus. Two points $x^g$ and $x^{g'}$ are linked if $x^{g'} = \mathcal{R}_h x^g$, i.e. if $g' = gh^{-1}$ for some $h$ in $G'$. The link is oriented from $x^g$ to $x^{g'}$ (unless $h = h^{-1}$ in which case the link is unoriented): the resulting "manifold" is an oriented graph. From every point exactly $m$ (= number of independent 1-forms) links originate. Appendix 1 contains the graphs for differential calculi on finite groups up to order eight.

**Note 2: Knot invariants**.

We can represent the braiding operator $\Lambda$ and its inverse $\Lambda^{-1}$ as

$\Lambda^{ab}{}_{cd} =$  $\Lambda^{-1}{}^{ab}{}_{cd} =$ 

The metric $g_{ab}$ is represented as

$g_{ab} =$  $g^{ab} =$ 

The metric $g_{ab}$ allows to close the braids into knots, and the above graphical representations yield a knot invariant for any finite group. This invariant is an integer number KN. The three Reidemeister moves hold because of

i) $g_{ab}\Lambda^{ab}{}_{cd} = g_{cd}$,
ii) the definition of the crossings corresponding to $\Lambda$ and $\Lambda^{-1}$,
iii) the Yang-Baxter equations for $\Lambda$ and the properties:

$$g_{ab}\ \Lambda^{bc}{}_{de} = \Lambda^{-1}{}^{cb}{}_{ad}\ g_{be} \tag{2.60}$$

Thus the unknot has KN equal to $g^{ab}g_{ab} = m$, the dimension of the differential calculus. The KN of the right trefoil is:

$$\Lambda^{-1}{}^{a_1a_2}{}_{a_3a_4}\Lambda^{-1}{}^{b_1b_2}{}_{b_3b_4}\Lambda^{a_4b_3}{}_{c_3c_4}\ g^{c_3a_3}g^{c_4b_4}\ \ g_{a_2b_1}g_{a_1b_2} \tag{2.61}$$

the left trefoil KN being obtained by $\Lambda \leftrightarrow \Lambda^{-1}$. Up to finite groups of order eight, the KN does not distinguish between left and right trefoils, and its values are given in Table 1.

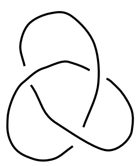


**Fig. 1** : right trefoil

**Note 3: Summary of conventions**.

$n$ : order of the finite group $G$.
$m$ : number of independent 1-forms, depends on the particular differential calculus.
$k$ : generic rank of a form.
$p$ : rank of top forms (volume).

# 3 $k$-forms, components and projector decompositions

The nontriviality of the braiding operator $\Lambda$ entails some complication in the analysis of the space of $k$-forms. Its dimension is usually larger than $\binom{k}{m}$ as when 1-forms simply anticommute. For example the space of two-forms in the case S3 (3D calculus) is four-dimensional, and not three-dimensional as it would be in ordinary differential geometry for a 3D-manifold. The basis $\Theta^I_{(k)}$ of $k$-forms is determined by finding the null eigenvectors of the generalized antisymmetrizer $A^i_j$ defined in (2.26), $i, j$ being here composite indices $i = (i_1 ... i_k)$ etc. Suppose there are $q$ independent null eigenvectors. Then the space of $k$ forms must have dimension $m^k - q$, since $A^i_j$ maps the $m^k$ - dimensional space of $k$ - tensors to the space of $k$-forms. The null eigenvectors lead to a set of independent linear relations between $k$-forms: we can solve these relations in terms of the basis elements $\Theta^I_{(k)}$, $I = 1, ...(m^k - q)$. When their number is not too large, these basis elements are given in Appendix 1.

Any $k$-form $B$ can be expanded on the $\Theta^I_{(k)}$ basis: $B = B_I \; \Theta^I_{(k)}$, where $B_I$ are the components of $B$ on the basis. It may be of some interest to retain explicit information on the $\theta^{i_1} \wedge ... \wedge \theta^{i_k}$ structure of the basis when extracting components. This could be useful, for example, when defining the analogue of Riemann curvature and its contractions, as in ref.s [8]-[13].

Consider the 2-form $B_{ij} \; \theta^i \wedge \theta^j$. What we need is really a *projector* such that

$$\mathcal{A}^{ij}{}_{kl} \; \theta^k \wedge \theta^l = \theta^i \wedge \theta^j \tag{3.1}$$

Then the components of $B$ can be extracted as

$$\mathcal{A}^{ij}{}_{kl} \; B_{ij} \tag{3.2}$$

The generalized antisymmetrizer $A = id - \Lambda$ is a projector only when $\Lambda = \Lambda^{-1}$, i.e. when it is really an antisymmetrizer. To find the projector $\mathcal{A}$ the key observation is that there always exists a power $s$ such that [6]

$$\Lambda^s = id \tag{3.3}$$

In fact this $s$ is given by $s = 2|ad(G)|$, where $|ad(G)|$ denotes the number of elements of $ad(G) := \{ad(g)|g \in G\}$, the group of inner automorphisms of $G$.

We recall the proof of [6] : for any $a \in ad(G)$ let $C(a)$ denote the cyclic subgroup of $ad(G)$ generated by $a$. Since $ad(G)$ is a finite group, $|C(a)|$ is finite and $a^{|C(a)|} = id$. Furthermore, $|C(a)|$ is a divisor of $|ad(G)|$ by Lagrange theorem. Finally, notice that from

$$\Lambda(\theta^g \otimes \theta^h) = \theta^{ad(g)h} \otimes \theta^g \tag{3.4}$$

one finds by induction

$$\Lambda^{2k-1}(\theta^g \otimes \theta^h) = \theta^{ad(gh)^k h} \otimes \theta^{ad(gh)^{k-1} g} \tag{3.5}$$

$$\Lambda^{2k}(\theta^g \otimes \theta^h) = \theta^{ad(gh)^k g} \otimes \theta^{ad(gh)^k h} \tag{3.6}$$

From the last equation the proof follows.

Defining the *order* of $\Lambda$ to be the smallest positive integer $s$ such that $\Lambda^s = id$, the previous proof implies that $s \le 2|ad(G)|$. In general the equality does not hold. For example, the symmetric groups $S_n$ with $n > 3$ (and universal calculus) have $s < 2|ad(G)|$ [6].

Next we notice that $\Lambda^s = id$ means that the eigenvalues of $\Lambda$ are the $s - th$ roots of unity, i.e. $(1, q, q^2, ...q^{s-1})$ with $q = e^{\frac{2\pi i}{s}}$. Then, if we denote by $P_i$ the projector on the eigenspace corresponding to the root $q^i$, the braiding operator has the projector decomposition:

$$\Lambda = P_0 + qP_1 + q^2 P_2 + ...q^{s-1}P_{s-1} \tag{3.7}$$

Using the projector properties $\sum_0^{s-1} P_i = id$ and $P_i P_j = \delta_{ij} P_i$ yields the system of $s$ operator equations:

$$\begin{aligned}
&id = P_0 + P_1 + P_2 + ... + P_{s-1} \\
&\Lambda = P_0 + qP_1 + q^2 P_2 + ... + q^{s-1} P_{s-1} \\
&\Lambda^2 = P_0 + q^2 P_1 + q^4 P_2 + ... + q^{2(s-1)} P_{s-1} \\
&\vdots \\
&\Lambda^{s-1} = \Lambda^{-1} = P_0 + q^{-1} P_1 + q^{-2} P_2 + ... + qP_{s-1}
\end{aligned} \tag{3.8}$$

Using $1 + q + q^2 + ... + q^{s-1} = 0$ these can be easily inverted:

$$P_i = \frac{1}{s} \sum_{j=0}^{s-1} q^{-ij} \Lambda^j \tag{3.9}$$

and one can check directly the projector properties.

The projector $P_0$ satisfies the relation:

$$P_0\ (id - \Lambda) = 0 \quad \Rightarrow P_0\ \theta^i \wedge \theta^j = 0 \tag{3.10}$$

since $P_0 \Lambda = P_0$. On the other hand the complementary projector

$$id - P_0 = id - \frac{1}{s}[id + \Lambda + \Lambda^2 + ...\Lambda^{s-1}] = \frac{1}{s}[(id - \Lambda) + (id - \Lambda^2) + ...(id - \Lambda^{s-1})] \tag{3.11}$$

applied to $\theta^i \wedge \theta^j$ leaves it unvaried. Then $\mathcal{A} = id - P_0$ is the projector on two-forms we were looking for, satisfying (3.1).

Notice that the components $B_{ij}$ as defined by $B = B_{ij} \theta^i \wedge \theta^j$ are ambiguous, since

$$B_{ij} \to B_{ij} + c_{kl}\ (P_0)^{kl}{}_{ij}, \quad c_{kl} \in Fun(G) \tag{3.12}$$

correspond to the same 2-form $B$ (use (3.10). This ambiguity is fixed by projecting with $\mathcal{A}$: the projection removes any piece in $B_{ij}$ proportional to $(P_0)^{kl}{}_{ij}$ due to $\mathcal{A}P_0 = (id - P_0)P_0 = 0$.

# 4 De Rham cohomology

## 4.1 Cohomology classes

Cohomology classes are found by computing the null vectors of the linear mapping $d^{(k)}$ : $\Gamma^{\wedge k} \to \Gamma^{\wedge(k+1)}$ (exterior derivative acting on $k$-forms). These give the closed forms in $\Gamma^{\wedge k}$, and the exact forms in $\Gamma^{\wedge(k+1)}$ (as the image of the space orthogonal to the closed forms in $\Gamma^{\wedge k}$). As usual, the number of independent closed but not exact $k$-forms is simply the difference between $dim[Ker(d^{(k)})]$ and $dim[Im(d^{(k-1)})]$.

These numbers, i.e. the Betti numbers, as well as the explicit list of cohomology representatives, can be computed by finding the null vectors of the matrix $M$ representing $d^{(k)}$.

Let us determine this matrix in terms of quantities related to the differential calculus on $Fun(G)$. A generic $k$-form $B$ can be expanded on the basis of $k$-forms $\Theta^I_{(k)}$: $B = B_I(x)\Theta^I_{(k)}$. Moreover, its components being functions on $G$, can be themselves expanded on the basis $x^g$ defined in (2.2): $B_I(x) = B_{Ig}x^g$. By means of the definitions:

$$d\Theta^I_{(k)} = C^I_J\ \Theta^J_{(k+1)} \tag{4.1}$$

$$\theta^i \wedge \Theta^I_{(k)} = T^{iI}_J\ \Theta^J_{(k+1)} \tag{4.2}$$

the exterior derivative on the generic $k$-form $B$ becomes:

$$\begin{aligned} dB &= (dB_I) \wedge \Theta^I_{(k)} + B_I C^I_J\ \Theta^J_{(k+1)} = [(R_i - 1)B_I]\ \theta^i \wedge \Theta^I_{(k)} + B_I C^I_J\ \Theta^J_{(k+1)} \\ &= [(R_i - 1)B_I\ T^{iI}_J + B_I C^I_J]\ \Theta^J_{(k+1)} = [(R_i - 1)(B_{Ig}x^g)\ T^{iI}_J + B_{Ig}x^g\ C^I_J]\ \Theta^J_{(k+1)} \\ &= [(B_{Ig}x^{gi^{-1}} - B_{Ig}x^g)\ T^{iI}_J + B_{Ig}x^g\ C^I_J]\ \Theta^J_{(k+1)} \end{aligned} \tag{4.3}$$

Projecting on the bases $x^{g'}$ and $\Theta^J_{(k+1)}$ yields finally:

$$[dB]_{Jg'} = {M_{Jg'}}^{Ig}\ B_{Ig} \tag{4.4}$$

with

$${M_{Jg'}}^{Ig} = \sum_i T^{iI}_J\ (\delta^g_{g'i} - \delta^g_{g'}) + C^I_J\ \delta^g_{g'} \tag{4.5}$$

This matrix has $dim(G) \times dim(\Gamma^{\wedge(k+1)})$ rows and $dim(G) \times dim(\Gamma^{\wedge(k)})$ columns. The quantities $T^{iI}_J$ and $C^I_J$ defined in (4.1), (4.2) are easily obtained from the Cartan-Maurer equations (and the Leibniz rule), and the expansion of $k+1$ forms $\theta^{i_1} \wedge ... \wedge \theta^{i_{k+1}}$ on the basis $\Theta^J_{(k+1)}$.

Suppose that ${M_{Jg'}}^{Ig}$ has $q$ null eigenvectors $V^\alpha$, $\alpha = 1, ...q$ with components $V^\alpha_{Ig}$. Then there are $q$ independent closed $k$-forms $C^\alpha_{(k)}$ given by:

$$C^\alpha_{(k)} = V^\alpha_{Ig}\ x^g\ \Theta^I_{(k)}, \quad \alpha = 1, ...q \tag{4.6}$$

This analysis has been carried out for all finite groups up to order 8, and the results are summarized in the following Table.

**Table 1: de Rham cohomology of $S_3$, $Q$, $D_4$, $Z_N$**

| $S_3$ $\theta^a, \theta^b, \theta^c$ | | | | | KN=9 |
|---|---|---|---|---|---|
| order | 0 | 1 | 2 | 3 | 4 |
| $\sharp$ | 1 | 3 | 4 | 3 | 1 |
| $b_k$ | 1 | 1 | 0 | 1 | 1 |

| $S_3$ $\theta^{ab}, \theta^{ba}$ | | | KN=2 |
|---|---|---|---|
| order | 0 | 1 | 2 |
| $\sharp$ | 1 | 2 | 1 |
| $b_k$ | 2 | 4 | 2 |

| $S_3$ $\theta^a, \theta^b, \theta^c, \theta^{ab}, \theta^{ba}$ | | | | | | | | KN=11 |
|---|---|---|---|---|---|---|---|---|
| order | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $\sharp$ | 1 | 5 | 14 | 31 | 58 | 95 | 140 | ... |
| $b_k$ | 1 | 2 | 1 | 2 | 4 | ... | ... | ... |

| $Q$ $\theta^i, \theta^{i^{-1}}, \theta^j, \theta^{j^{-1}}$ | | | | | | | | | KN=4 |
|---|---|---|---|---|---|---|---|---|---|
| order | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\sharp$ | 1 | 4 | 8 | 12 | 14 | 12 | 8 | 4 | 1 |
| $b_k$ | 1 | 2 | 1 | 2 | 4 | 2 | 1 | 2 | 1 |

| $D_4$ $\theta^2, \theta^4, \theta^5, \theta^6$ | | | | | | | | | KN=4 |
|---|---|---|---|---|---|---|---|---|---|
| order | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\sharp$ | 1 | 4 | 8 | 12 | 14 | 12 | 8 | 4 | 1 |
| $b_k$ | 1 | 2 | 1 | 2 | 4 | 2 | 1 | 2 | 1 |

| $Z_N$ $\theta^u, \theta^{u^{-1}}$ | | | KN=2 |
|---|---|---|---|
| order | 0 | 1 | 2 |
| $\sharp$ | 1 | 2 | 1 |
| $b_k$ | 1 | 2 | 1 |

where the order $k$ of independent forms, the number $\sharp$ of independent $k$-forms and the $k$-th Betti number $b_k$ are given for the three nonabelian groups $S_3, Q, D_4$ and for the cyclic groups $Z_N$. We give only partial results for the universal calculus on $S_3$, the volume form being of order at least 12. The independent one-forms characterizing the differential calculus are also indicated (see the Appendix for conventions), together with the knot numbers KN for the trefoils.

## 4.2 Adjoint, Laplacian and Poincaré duality

We first define an inner product between two generic $k$-forms as follows:

$$\langle\langle \rho, \sigma \rangle\rangle \equiv \int_G < \rho^*, \sigma > vol \ = \int_G \rho^* \wedge (*\sigma) \tag{4.7}$$

This product is positive definite because of (2.44). It can be extended to the direct sum $\bigoplus_k \Gamma^{\wedge k}$, requiring the spaces $\Gamma^{\wedge k}$ and $\Gamma^{\wedge k'}$ to be orthogonal if $k \neq k'$. As usual, we define the adjoint of the exterior derivative as the unique mapping $\delta : \Gamma^{\wedge k} \longrightarrow \Gamma^{\wedge (k-1)}$ such that

$$\langle\langle d\alpha, \beta \rangle\rangle = \langle\langle \alpha, \delta\beta \rangle\rangle, \quad \forall \alpha \in \Gamma^{\wedge(k-1)}, \quad \forall \beta \in \Gamma^{\wedge k}. \tag{4.8}$$

*Lemma 1:* if $\int d\rho = 0$, $\forall$ (p-1)-form $\rho$ (see Sect. 2) then:

$$d* = (-1)^k * \delta \tag{4.9}$$

*Proof:* let $\alpha$, $\beta$ be generic $k-1$ and $k$-forms respectively. Then

$$d(\alpha^* \wedge *\beta) = d\alpha^* \wedge *\beta + (-1)^{k-1} \alpha^* \wedge d(*\beta) \tag{4.10}$$

Integrating on the group, using $\int d = 0$ and (4.8) yields

$$\int \alpha^* \wedge *\delta\beta = (-1)^k \int \alpha^* \wedge d(*\beta) \tag{4.11}$$

which implies the theorem, since $\langle\langle \ , \ \rangle\rangle$ is positive definite.

Suppose now that $** = \eta \ id$ (Conjecture 2 of Sect.2), where $\eta$ is a sign. Then

$$\delta = (-1)^k \eta * d * \qquad \Longrightarrow \qquad * \Delta = \Delta * \tag{4.12}$$

where

$$\Delta \equiv d\delta + \delta d \tag{4.13}$$

is the Laplacian. The commutation of the Laplacian with the Hodge operator allows to reproduce the standard proof for Poincaré duality, so that

$$dim(H^k) = dim(H^{p-k}) \tag{4.14}$$

Note that the Hodge decomposition theorem holds in any case, the proof relying on the finiteness of the space of harmonic $k$-forms. Then every cohomology class contains a unique harmonic representative.

# 5 Conclusions and outlook

We have started an investigation on the (de Rham) cohomological properties of finite groups. Most of the classical results for differential manifolds can be translated into this setting, since they are based on algebraic relations holding also for finite groups. A challenging question for future work is how to relate de Rham cohomology of finite groups to the homology of the regular graphs that encode their differential calculi.

Although we have not discussed it in the present paper, a parallel transport commuting with the left and right action of the finite group can be introduced, as well as a torsion and a curvature. This allows the construction of Yang-Mills, Born-Infeld and gravity actions on finite groups, as mentioned in the Introduction. It would be of interest to find how cohomology information (for example the analogue of characteristic classes) reflects itself on the dynamics of these theories.

# A Differential calculi on finite groups of order $\leq 8$

## A.1 The permutation group $S_3$

Elements: $a = (12)$, $b = (23)$, $c = (13)$, $ab = (132)$, $ba = (123)$, $e$.

Multiplication table:

| | e | a | b | c | ab | ba |
|---|---|---|---|---|---|---|
| e | e | a | b | c | ab | ba |
| a | a | e | ab | ba | b | c |
| b | b | ba | e | ab | c | a |
| c | c | ab | ba | e | a | b |
| ab | ab | c | a | b | ba | e |
| ba | ba | b | c | a | e | ab |

Nontrivial conjugation classes: $I = [a, b, c]$, $II = [ab, ba]$.

There are 3 bicovariant calculi $BC_I$, $BC_{II}$, $BC_{I+II}$ corresponding to the possible unions of the conjugation classes. They have respectively dimension 3, 2 and 5.

### A.1.1 $BC_I$ differential calculus

Basis of the 3-dimensional vector space of one-forms:

$$\theta^a, \ \theta^b, \ \theta^c \qquad \text{(A.1)}$$

We'll use the shorthand notation $\{i_1, ...i_k\} = \theta^{i_1} \wedge ...\theta^{i_k}$.
Basis $\Theta_{(2)}$ of the 4-dimensional vector space of two-forms:

$$\{a, b\}, \ \{b, c\}, \ \{a, c\}, \ \{c, b\} \qquad \text{(A.2)}$$

Any other wedge product of two $\theta$'s can be expressed as linear combination of the basis elements:

$$\{b, a\} = -\{a, c\} - \{c, b\}, \quad \{c, a\} = -\{a, b\} - \{b, c\} \qquad \text{(A.3)}$$

Basis $\Theta_{(3)}$ of the 3-dimensional vector space of three-forms:

$$\{a, b, c\}, \ \{a, c, b\}, \ \{b, a, c\} \tag{A.4}$$

and:

$$\begin{aligned}
\{c, b, a\} &= -\{c, a, c\} = -\{a, c, a\} = \{a, b, c\} \\
\{b, c, a\} &= -\{b, a, b\} = -\{a, b, a\} = \{a, c, b\} \\
\{c, a, b\} &= -\{c, b, c\} = -\{b, c, b\} = \{b, a, c\}
\end{aligned} \tag{A.5}$$

Basis $\Theta_{(4)}$ of the 1-dimensional vector space of four-forms:

$$vol = \{a, b, a, c\} \tag{A.6}$$

The $\epsilon$ tensor is defined by:

$$\{g_1, g_2, g_3, g_4\} = \epsilon^{g_1,g_2,g_3,g_4} \ vol \tag{A.7}$$

Its nonvanishing components are:

$$\epsilon^{abac} = \epsilon^{acab} = \epsilon^{cbca} = \epsilon^{cacb} = \epsilon^{babc} = \epsilon^{bcba} = 1 \tag{A.8}$$

$$\epsilon^{baca} = \epsilon^{caba} = \epsilon^{abcb} = \epsilon^{cbab} = \epsilon^{acbc} = \epsilon^{bcac} = -1 \tag{A.9}$$

Note the centrality of $vol$:

$$f \ vol = vol \ f, \quad \forall f \in Fun(G) \tag{A.10}$$

due to $\mathcal{R}_a \mathcal{R}_b \mathcal{R}_a \mathcal{R}_c = \mathcal{R}_{abac} = \mathcal{R}_e = id$

Cartan-Maurer equations:

$$\begin{aligned}
d\theta^a &= -\theta^b \wedge \theta^c - \theta^c \wedge \theta^b \\
d\theta^b &= -\theta^a \wedge \theta^c + \theta^a \wedge \theta^b + \theta^b \wedge \theta^c \\
d\theta^c &= -\theta^a \wedge \theta^b + \theta^a \wedge \theta^c + \theta^c \wedge \theta^b
\end{aligned} \tag{A.11}$$

The exterior derivative on any three-form of the type $\theta \wedge \theta \wedge \theta$ vanishes, as one can easily check by using the Cartan-Maurer equations and the equalities between exterior products given above. Equivalently, the volume form belongs to a nontrivial cohomology class ($H^4$). Then, as discussed in Section 2, integration of a total differential vanishes on the "group manifold" of $S_3$ corresponding to the $BC_I$ bicovariant calculus. This "group manifold" has three independent directions, associated to the cotangent basis $\theta^a$, $\theta^b$, $\theta^c$. Note however that the volume element is of order four in the left-invariant one-forms $\theta$.

De Rham cohomology (generators):

$$H^0 = I, \quad H^1 = X, \quad H^2 = 0, \quad H^3 = (*X), \quad H^4 = \ vol \tag{A.12}$$

where $X = \theta^a + \theta^b + \theta^c$

## A.2 $BC_{II}$ differential calculus

Basis of the 2-dimensional vector space of one-forms:

$$\theta^{ab},\ \theta^{ba} \tag{A.13}$$

Basis of the 1-dimensional vector space of two-forms:

$$vol = \{ab, ba\} = -\{ba, ab\} \tag{A.14}$$

so that:

$$\{g_1, g_2\} = \epsilon^{g_1,g_2} vol \tag{A.15}$$

where the $\epsilon$ tensor is the usual 2-dimensional Levi-Civita tensor. Again $f\ vol = vol\ f$ since $abba = e$.

Cartan-Maurer equations:

$$d\theta^{ab} = 0,\quad d\theta^{ba} = 0 \tag{A.16}$$

Thus the exterior derivative on any one-form $\theta^g$ vanishes and integration of a total differential vanishes on the group manifold of $S_3$ corresponding to the $BC_{II}$ bicovariant calculus. This group manifold has two independent directions, associated to the cotangent basis $\theta^{ab},\ \theta^{ba}$.

De Rham cohomology:

$$H^0 = (x^a + x^b + x^c)\ I,\qquad (x^e + x^{ab} + x^{ba})\ I, \tag{A.17}$$
$$H^1 = (x^a + x^b + x^c)\ \theta^{ab},\quad (x^e + x^{ab} + x^{ba})\ \theta^{ab}, \tag{A.18}$$
$$(x^a + x^b + x^c)\ \theta^{ba},\quad (x^e + x^{ab} + x^{ba})\ \theta^{ba}, \tag{A.19}$$
$$H^2 = (x^a + x^b + x^c)\ vol,\quad (x^e + x^{ab} + x^{ba})\ vol. \tag{A.20}$$

### A.2.1 The $S_3$ group "manifold"

We can draw a picture of the group manifold of $S_3$. It is made out of 6 points, corresponding to the group elements and identified with the functions $x^e, x^a, x^b, x^c, x^{ab}, x^{ba}$.

$BC_I$ - calculus:

From each of the six points $x^g$ one can move in three directions, associated to the tangent vectors $t_a, t_b, t_c$, reaching three other points whose "coordinates" are

$$\mathcal{R}_a x^g = x^{ga},\quad \mathcal{R}_b x^g = x^{gb},\quad \mathcal{R}_c x^g = x^{gc} \tag{A.21}$$

The 6 points and the "moves" along the 3 directions are illustrated in the Fig. 2. The links are not oriented since the three group elements $a, b, c$ coincide with their inverses.

$BC_{II}$ - calculus:

From each of the six points $x^g$ one can move in two directions, associated to the tangent vectors $t_{ab}, t_{ba}$, reaching two other points whose "coordinates" are

$$\mathcal{R}_{ab} x^g = x^{gba}, \quad \mathcal{R}_{ba} x^g = x^{gab} \tag{A.22}$$

The 6 points and the "moves" along the 3 directions are illustrated in Fig. 2. The arrow convention on a link labeled (in italic) by a group element $h$ is as follows: one moves in the direction of the arrow via the action of $\mathcal{R}_h$ on $x^g$. (In this case $h = ab$). To move in the opposite direction just take the inverse of $h$.

Note that the $BC_{II}$ graph has two disconnected pieces. This explains $b_0 \equiv dim(H^0) = 2$.

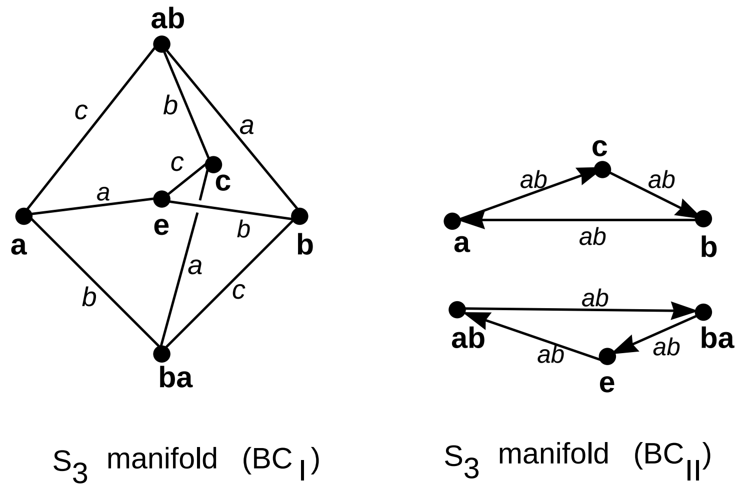



**Fig. 2** : $S_3$ group manifold, and moves of the points under the group action

## A.3 The quaternion group $Q$

Elements of $Q$: $\{e, \ -e, \ i, \ -i, \ j, \ -j, \ k, \ -k\}$

Multiplication table: $ij = k$ and cyclic, $i^2 = -e$ etc.

Nontrivial conjugation classes:
$[-e] = \{-e\}$ ; $[i] = \{i, -i\}$ ; $[j] = \{j, -j\}$ ; $[k] = \{k, -k\}$ ;

There are differential calculi of dimensions 1 up to 7 (universal calculus). Many are isomorphic. The 1D, 2D, 3D differential calculi are rather trivial. We give here details on a 4-dimensional calculus corresponding to the union of the $[i]$ and $[j]$ conjugation classes.

### A.3.1 4D-differential calculus

Basis of 1-forms: $\theta^i, \theta^{i^{-1}}, \theta^j, \theta^{j^{-1}}$.

Basis of 2-forms:
$\{-i, i\}, \{-i, j\}, \{-i, -j\}, \{j, i\}, \{j, -i\}, \{-j, i\}, \{-j, -i\}, \{-j, j\}$.

and
$\{i,-i\} = -\{-i,i\}, \quad \{j,-j\} = -\{-j,j\},$
$\{i,j\} = -\{j,-i\} - \{-i,-j\} - \{-j,i\}, \quad \{i,-j\} = -\{j,i\} - \{-i,j\} - \{-j,-i\}.$

Basis of 3-forms:
$\{-i,j,i\}, \{-i,j,-i\}, \{-i,-j,i\}, \{-i,-j,-i\}, \{-i,-j,j\}, \{j,i,-i\},$
$\{j,-i,-j\}, \{-j,-i,i\}, \{-j,-i,j\}, \{-j,-i,-j\}, \{-j,j,i\}, \{-j,j,-i\}.$

Basis of 4-forms:
$\{-i,j,-i,i\}, \{-i,-j,-i,i\}, \{-i,-j,-i,j\}, \{-i,-j,j,i\}, \{-i,-j,j,-i\},$
$\{j,i,-i,j\}, \{j,-i,-j,-i\}, \{-j,-i,j,i\}, \{-j,-i,j,-i\}, \{-j,-i,-j,i\},$
$\{-j,-i,-j,-i\}, \{-j,-i,-j,j\}, \{-j,j,-i,i\}, \{-j,j,-i,-j\}.$

Basis of 5-forms:
$\{-i,j,-i,-j,i\}, \{-i,-j,-i,j,i\}, \{-i,-j,-i,j,-i\}, \{-i,-j,j,-i,i\},$
$\{-i,-j,j,-i,-j\}, \{-j,-i,j,-i,i\}, \{j,-i,j,-i,-j\}, \{-j,-i,-j,-i,i\},$
$\{-j,-i,-j,-i,j\}, \{-j,-i,-j,j,i\}, \{-j,-i,-j,j,-i\}, \{-j,j,-i,-j,i\}.$

Basis of 6-forms:
$\{-i,-j,-i,j,-i,i\}, \{-i,-j,-i,j,-i,-j\}, \{-i,-j,j,-i,-j,i\},$
$\{-j,-i,j,-i,-j,i\}, \{-j,-i,-j,-i,j,i\}, \{-j,-i,-j,-i,j,-i\},$
$\{-j,-i,-j,j,-i,i\}, \{-j,-i,-j,j,-i,-j\}.$

Basis of 7-forms:
$\{-i,-j,-i,j,-i,-j,i\}, \{-j,-i,-j,-i,j,-i,i\},$
$\{-j,-i,-j,j,-i,-j,i\}, \{-j,j,-i,-j,-i,j,-i\}$

The volume form $vol = \{-j,-i,-j,-i,j,-i,-j,i\}$ is central.

The epsilon tensor has 928 nonvanishing components, with values $1,-1,2,-2$ (mostly $1,-1$).

Cartan-Maurer equations:
$d\theta^i = -\theta^j \wedge \theta^{-i} - \theta^{-i} \wedge \theta^j - \theta^{-i} \wedge \theta^{-j} - \theta^{-j} \wedge \theta^{-i}$
$d\theta^{i^{-1}} = \theta^j \wedge \theta^{-i} + \theta^{-i} \wedge \theta^j + \theta^{-i} \wedge \theta^{-j} + \theta^{-j} \wedge \theta^{-i}$
$d\theta^j = \theta^j \wedge \theta^i + \theta^{-i} \wedge \theta^j - \theta^{-i} \wedge \theta^{-j} - \theta^{-j} \wedge \theta^i$
$d\theta^{j^{-1}} = -\theta^j \wedge \theta^i - \theta^{-i} \wedge \theta^j + \theta^{-i} \wedge \theta^{-j} + \theta^{-j} \wedge \theta^i$

De Rham cohomology:
$H^0: \ I,$
$H^1: \ X^i = \theta^i + \theta^{-i}, \ X^j = \theta^j + \theta^{-j},$
$H^2: \ X^i \wedge X^j,$
$H^3: \ W = \theta^{-i} \wedge \theta^j \wedge \theta^{-i} + \theta^{-i} \wedge \theta^{-j} \wedge \theta^{-i}, \quad Z = \theta^{-i} \wedge \theta^j \wedge \theta^i + \theta^{-j} \wedge \theta^{-i} \wedge \theta^i,$
$H^4: \ X^i \wedge W, \quad X^j \wedge W, \quad X^j \wedge Z, \quad \theta^{-j} \wedge \theta^j \wedge \theta^{-i} \wedge \theta^{-j} + \theta^{-j} \wedge \theta^{-i} \wedge \theta^{-j} \wedge \theta^j$

with $X^i \wedge X^j = -X^j \wedge X^i, X^i \wedge X^i = X^j \wedge X^j = 0.$

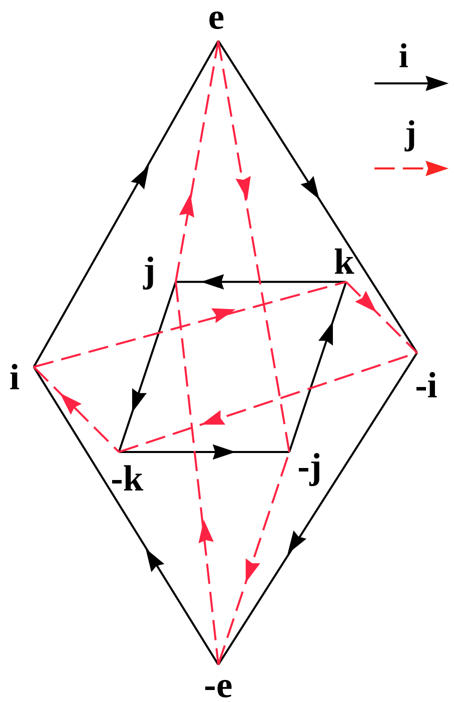



**Fig. 3** : $Q$ group manifold corresponding to the [i,-i,j,-j] differential calculus

## A.4 Dihedral group $D_4$

$D_4$ : group of isometries of the square ABCD.

Elements of $D_4$:

1 = identity $e$
2 = $\frac{\pi}{2}$ clockwise rotation
3 = (diag AC) (diag BD)
4 = $\frac{\pi}{2}$ anticlockwise rotation
5 = horizontal reflection
6 = vertical reflection
7 = (diag BD)
8 = (diag AC)

where (diag AC) and (diag BD) are the reflections on the two diagonals.

Multiplication table:

| . | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 3 | 4 | 1 | 8 | 7 | 5 | 6 |
| 3 | 3 | 4 | 1 | 2 | 6 | 5 | 8 | 7 |
| 4 | 4 | 1 | 2 | 3 | 7 | 8 | 6 | 5 |
| 5 | 5 | 7 | 6 | 8 | 1 | 3 | 2 | 4 |
| 6 | 6 | 8 | 5 | 7 | 3 | 1 | 4 | 2 |
| 7 | 7 | 6 | 8 | 5 | 4 | 2 | 1 | 3 |
| 8 | 8 | 5 | 7 | 6 | 2 | 4 | 3 | 1 |

Nontrivial conjugation classes: $[3], [2,4], [5,6], [7,8]$. There are 15 bicovariant calculi.

### A.4.1 4D-differential calculus

Basis of 1-forms: $\theta^2, \theta^4, \theta^5, \theta^6$.

Basis of 2-forms:
$\{4,2\}, \{4,5\}, \{4,6\}, \{5,2\}, \{5,4\}, \{6,2\}, \{6,4\}, \{6,5\}$

and
$\{2,4\} = -\{4,2\}, \{5,6\} = -\{6,5\}$,
$\{2,5\} = -\{4,6\} - \{5,4\} - \{6,2\}, \{2,6\} = -\{4,5\} - \{5,2\} - \{6,4\}$

Basis of 3-forms:
$\{4,5,2\}, \{4,5,4\}, \{4,6,2\}, \{4,6,4\}, \{5,4,2\}, \{5,4,5\}$
$\{5,4,6\}, \{6,4,2\}, \{6,4,5\}, \{6,4,6\}, \{6,5,2\}, \{6,5,4\}$

Basis of 4-forms:
$\{4,5,4,2\}, \{4,6,4,2\}, \{4,6,4,5\}, \{5,4,5,2\}, \{5,4,5,4\}, \{5,4,6,2\}, \{5,4,6,4\}$
$\{6,4,5,2\}, \{6,4,5,4\}, \{6,4,6,2\}, \{6,4,6,4\}, \{6,5,4,2\}, \{6,5,4,5\}, \{6,5,4,6\}$

Basis of 5-forms:
$\{4,6,4,5,2\}, \{4,6,4,5,4\}, \{5,4,5,4,2\}, \{5,4,6,4,2\}, \{5,4,6,4,5\}, \{6,4,5,4,2\}$
$\{6,4,6,4,2\}, \{6,4,6,4,5\}, \{6,5,4,5,2\}, \{6,5,4,5,4\}, \{6,5,4,6,2\}, \{6,5,4,6,4\}$

Basis of 6-forms:
$\{4,6,4,5,4,2\}, \{5,4,6,4,5,2\}, \{5,4,6,4,5,4\}, \{6,4,6,4,5,2\}$,
$\{6,4,6,4,5,4\}, \{6,5,4,5,4,2\}, \{6,5,4,6,4,2\}, \{6,5,4,6,4,5\}$

Basis of 7-forms:
$\{5,4,6,4,5,4,2\}, \{6,4,6,4,5,4,2\}, \{6,5,4,6,4,5,2\}, \{6,5,4,6,4,5,4\}$

The volume form $\{6,5,4,6,4,5,4,2\}$ is central.

The epsilon tensor has 928 nonvanishing components, with values $1, -1, 2, -2$ (mostly $1, -1$). Note the perfect similarity with the quaternion case.

Cartan-Maurer equations:

$$d\theta^2 = -\theta^4 \wedge \theta^5 - \theta^4 \wedge \theta^6 - \theta^5 \wedge \theta^4 - \theta^6 \wedge \theta^4$$
$$d\theta^4 = \theta^4 \wedge \theta^5 + \theta^4 \wedge \theta^6 + \theta^5 \wedge \theta^4 + \theta^6 \wedge \theta^4$$
$$d\theta^5 = \theta^4 \wedge \theta^5 - \theta^4 \wedge \theta^6 + \theta^5 \wedge \theta^2 - \theta^6 \wedge \theta^2$$
$$d\theta^6 = -\theta^4 \wedge \theta^5 + \theta^4 \wedge \theta^6 - \theta^5 \wedge \theta^2 + \theta^6 \wedge \theta^2$$

De Rham cohomology:

$H^0: \ I,$
$H^1: \ X = \theta^2 + \theta^4, \ Y = \theta^5 + \theta^6,$
$H^2 = X \wedge Y,$
$H^3: \ W = \{4,5,4\} + \{4,6,4\}, \quad Z = \{4,5,2\} + \{6,4,2\} + \{4,6,2\} + \{5,4,2\},$
$H^4 = X \wedge W, \quad Y \wedge W, \quad Y \wedge Z, \quad Z \wedge X$

with $X \wedge Y = -Y \wedge X, \ X \wedge X = Y \wedge Y = 0.$

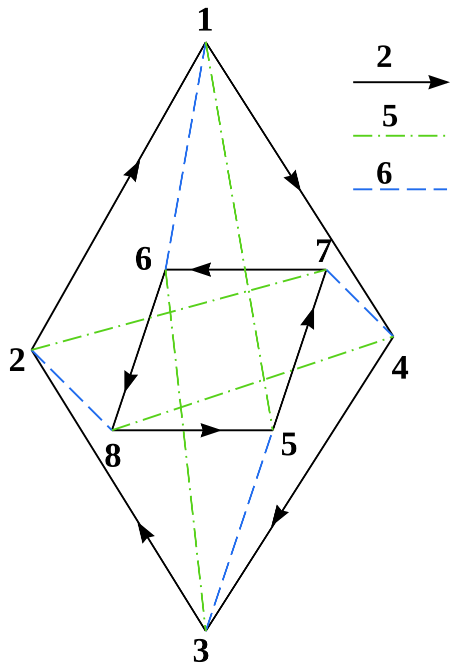



**Fig. 4** : $D_4$ group manifold,corresponding to the [2,4,5,6] differential calculus

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