Supergravity Actions with Integral Forms

L. Castellani [a,b,*], R. Catenacci [a,†], and P.A. Grassi [a,b,‡]

[(a)] *Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale Viale T. Michel, 11, 15121 Alessandria, Italy*

[(b)] *INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino*

Abstract

Integral forms provide a natural and powerful tool for the construction of supergravity actions. They are generalizations of usual differential forms and are needed for a consistent theory of integration on supermanifolds. The group geometrical approach to supergravity and its variational principle are reformulated and clarified in this language. Central in our analysis is the Poincaré dual of a bosonic manifold embedded into a supermanifold. Finally, using integral forms we provide a proof of Gates' so-called "Ectoplasmic Integration Theorem", relating superfield actions to component actions.

November 5, 2014

[*] leonardo.castellani@mfn.unipmn.it
[†] roberto.catenacci@mfn.unipmn.it
[‡] pietro.grassi@mfn.unipmn.it

Contents

1 Introduction

In the study of quantum field theories, of string theory and several other modern theoretical models the action is a fundamental bookkeeping device for all needed constraints, equations of motion and quantum corrections. In many cases having an action has tremendous advantages over the only knowledge of the equations of motion or other auxiliary constraints. In particular, the action encodes both the dynamics of the theory and the symmetries of the model (by means of Noether theorem) in a very compact formulation. Nonetheless, there are several situations where the construction of an action does not seem possible or out of the reach by present means. For example, it is not known whether a manifestly supersymmetric $N = 4, D = 4$ super-Yang-Mills action exists in superspace (which would guarantee the well-known renormalisation theorems), and this is due to the self-duality constraints and to the lacking of an off-shell superspace formulation. Again, no standard superspace action for type IIB $D = 10$ supergravity theory exists, due to the self-duality constraints on RR fields. For the same reasons, no superspace formulation of $N = 2, D = 6$ supergravity is known.

Furthermore, even when the superspace formulation exists, it is difficult to extract the component action. This happens mainly for supergravity theories, where the superdeterminant of the supervielbein is needed for the construction of the action. In many cases, that computation is very cumbersome. On the other side in the work of Gates et al. [1, 2, 3, 4] a new method is been provided to extract the component action from the superspace formulation. This is based on a formula which relates the superfield action to the component action via a density projection operator acting on a closed superform. This procedure incorporates the integration over the fermionic coordinates and the contributions due to the gravitons. We show here that the origin of that formula can be understood by interpreting the superfield action as an integral form. The relation between the density projection operator and the component action is achieved by partial integration using picture changing operators.

Three decades ago a group-based geometric approach to supergravity was put forward, known as group manifold approach [5, 6], intermediate between the superfield and the component approaches. This framework provides a systematic algorithm to construct supergravities in any dimension. The starting point is a supergroup, and the fields of the theory are identified with the vielbein one-forms of (a manifold diffeomorphic to) the supergroup manifold. For example in D=4, N=1 supergravity the dynamical fields are the vierbein, the spin connection and the gravitino one-forms, dual respectively to the translation, Lorentz rotations and supersymmetry tangent vectors. Thus supermultiplets come out of supergroups, rather than from a superfield depending on bosonic and fermionic coordinates. Actions in D dimensions are constructed by considering integrals of D-form Lagrangians L on D-dimensional submanifolds of the supergroup manifold. The action depends in general also on how the submanifold is chosen inside the supergroup manifold, and the action principle includes also variations in the submanifold embedding functions. The resulting field equations are $(D-1)$-form equations holding on the whole supergroup manifold. The way to relate these actions and their field equations to those of the "ordinary" D-dimensional supergravities is exhaustively illustrated by many examples in ref. [5]. One of the advantages of this approach is that it yields the self-duality constraints of the $D = 6$ and $D = 10$ supergravities mentioned above as *part of the equations of motion*, besides allowing to construct the corresponding actions [7, 8].

We show here how the variational principle of the group manifold approach can be reformulated and clarified by using integral forms and the Poincaré dual of the submanifold. In particular we derive the condition for the embedding independence of the submanifold. This coincides with the condition for local supersymmetry invariance of the spacetime action, and reduces to the vanishing of the contraction of dL along tangent vectors orthogonal to the submanifold.

The paper has the following organisation. In sec. 2, the integration on supermanifolds is briefly discussed and presented both from a mathematical point of view, and from a more intuitive/physical point of view. The integration on curved supermanifolds is also discussed. In sec. 3, we describe, also for the case of supermanifolds, a simple and explicit form of the Poincaré dual as a singular localization form. The integration on a submanifold and the independence of the embedding is discussed. The construction of the actions in the group-geometric approach is presented and the variational principle is explained. Finally, in sec. 4 we consider the relation between the integral of superforms in the ectoplasmic integration formalism and integral forms. The "ethereal conjecture" of Gates et al. [1, 2, 3, 4] is proved using integral forms. Appendix A contains some additional material ancillary to the main text.

2 Integration on Supermanifolds

In this section we give a short introduction to the theory of integration on supermanifolds (see for example the review by Witten [9]). The translation of the *picture changing operators* into supergeometry has been explored in [10, 11]. More recently, the application to target space supersymmetry and Chern-Simons theories have been discussed in [12]. Thom classes for supermanifolds have been constructed in [13]. The picture changing operators have been introduced in string theory in [14], from world sheet point of view, and in [15], from target space point of view.

We start, as usual, from the case of the real superspace $\mathbb{R}^{n|m}$ with n bosonic ($x^i, i = 1, \dots, n$) and m fermionic ($\theta^\alpha, \alpha = 1, \dots, m$) coordinates. We take a function $f(x, \theta)$ in $\mathbb{R}^{n|m}$ with values in the real algebra generated by 1 and by the anticommuting variables, and we expand f as a polynomial in the variables θ :

$$f(x, \theta) = f_0(x) + ... + f_m(x)\theta^1 ... \theta^m$$

If the real function $f_m(x)$ is integrable in some sense in $\mathbb{R}^n$, the Berezin integral of $f(x, \theta)$ is defined as:

$$\int_{\mathbb{R}^{n|m}} f(x, \theta)[d^n x d^m \theta] = \int_{\mathbb{R}^n} f_m(x) d^n x$$

Note that $d^n x \equiv dx^1 \wedge ... \wedge dx^n$ is a volume form (a top form) in $\mathbb{R}^n$, but $[d^n x\, d^m \theta]$ is just a formal symbol that has nothing to do neither with "exterior products", nor with "top forms" mainly because if θ is a fermionic quantity, $d\theta$ is bosonic ($d\theta \wedge d\theta \neq 0$).

An important property of $[d^n x\, d^m \theta]$ is elucidated by the following simple example: consider in $\mathbb{R}^{1|1}$ the function $f(x, \theta) = g(x)\theta$ (with $g(x)$ integrable function in $\mathbb{R}$). We have:

$$\int_{\mathbb{R}^{1|1}} f(x, \theta)[dx d\theta] = \int_{-\infty}^{+\infty} g(x) dx\,.$$

If we rescale $\theta \to \lambda\theta$ ($\lambda \in \mathbb{R}$) we find $f(x,\theta) \to \lambda f(x,\theta)$. For the integral to be invariant under coordinate changes, the "measure" $[dxd\theta]$ must rescale as $[dxd\theta] \to \frac{1}{\lambda}[dxd\theta]$ and not as $[dxd\theta] \to \lambda[dxd\theta]$.

Generalizing this fact it is known that under general coordinate transformations in superspace the symbol $[d^n x\, d^m\theta]$ transforms with the "Berezinian", a.k.a. the superdeterminant, while $d^n x$ transforms in $\mathbb{R}^n$ with the Jacobian determinant. This fact is very important, because supermanifolds are obtained by gluing together open sets[1] homeomorphic to $\mathbb{R}^{n|m}$. The transformation properties (i.e. transition functions) allow to define integration on supermanifolds. The concept of **integral forms** arises also for giving a definite meaning to the symbols $[d^n x d^m\theta]$ by specifying what kind of object is "integrated".

A brief review of the formal properties of integral forms [10, 12] is given in Appendix A. Here we elaborate on their definition and on the computation of integrals.

The usual integration theory of differential forms for bosonic manifolds can be conveniently rephrased to shed light on its relations with Berezin integration.

We start again with a simple example: consider in $\mathbb{R}$ the integrable 1-form $\omega = g(x)dx$ (with $g(x)$ integrable function in $\mathbb{R}$). We have:

$$\int_{\mathbb{R}} \omega = \int_{-\infty}^{+\infty} g(x)dx\,.$$

Observing that dx is an anticommuting quantity, and denoting it by ψ, we could think of ω as a function on $\mathbb{R}^{1|1}$:

$$\omega = g(x)dx = f(x,\psi) = g(x)\psi \tag{2.1}$$

This function can be integrated *à la* Berezin reproducing the usual definition:

$$\int_{\mathbb{R}^{1|1}} f(x,\psi)[dxd\psi] = \int_{-\infty}^{+\infty} g(x)dx \; = \int_{\mathbb{R}} \omega$$

Note that (as above) the symbol $[dxd\psi]$ is written so as to emphasize that we are integrating on the **two** variables x and ψ, hence the dx inside $[dxd\psi]$ is *not* identified with ψ.

[1] The most natural topology in $\mathbb{R}^{n|m}$ is the topology in which the open sets are the **complete** cylinders over open sets in $\mathbb{R}^n$. This "coarse" topology is then transferred to the supermanifold.

This can be generalized as follows. Denoting by M, a bosonic differentiable manifold with dimension n, we define the exterior bundle $\Omega^\bullet(M) = \sum_{p=0}^{n} \bigwedge^p(M)$ as the direct sum of $\bigwedge^p(M)$ (sometimes denoted also by $\Omega^p(M)$). A section ω of $\Omega^\bullet(M)$ can be written locally as

$$\omega = \sum_{p=0}^{n} \omega_{[i_1 \dots i_p]}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p} \tag{2.2}$$

where the coefficients $\omega_{[i_1 \dots i_p]}(x)$ are functions on M and the indices $[i_1, \dots, i_p]$ are antisymmetrized. The integral of ω is defined as:

$$I[\omega] = \int_M \omega = \int_M \epsilon^{i_1 \dots i_n} \omega_{[i_1 \dots i_n]}(x)\, d^n x\,, \tag{2.3}$$

At first sight this might seem a bit strange, but we are actually saying that in the definition of the integral only the "part of top degree" of ω is involved. This opens the way to the relations between the integration theory of forms and the Berezin integral, that can be exploited by substituting every 1-form dx^i with a corresponding Grassmann variable ψ^i. A section ω of $\Omega^\bullet(M)$ is viewed locally as a function on a supermanifold $\mathcal{M}$ with coordinates (x^i, ψ^i)

$$\omega(x, \psi) = \sum_{p=0}^{n} \omega_{[i_1 \dots i_p]}(x) \psi^{i_1} \dots \psi^{i_p}\,; \tag{2.4}$$

such functions are polynomials in ψ's. Supposing now that the form ω is integrable we have as above that the Berezin integral "selects" the top degree component of the form:

$$\int_{\mathcal{M}} \omega(x, \psi)[d^n x d^n \psi] = \int_M \omega \tag{2.5}$$

If the manifold is equipped with a metric g (that for the moment we assume globally defined), we can expand a generic form ω on the basis of forms $\psi^a = e^a_i dx^i$ $(a = 1, \dots, n)$ such that $g = \psi^a \otimes \psi^b \eta_{ab}$ where η^{ab} is the flat metric on the tangent space $T(M)$ and we have that

$$I[\omega, g] = \int_{\mathcal{M}} \omega(x, e)[d^n x d^m \psi] = \int_M e\, \epsilon^{i_1 \dots i_n} \omega_{[i_1 \dots i_n]}(x) d^n x = \int_M \sqrt{g}\, \epsilon^{i_1 \dots i_n} \omega_{[i_1 \dots i_n]}(x)\, d^n x\,, \tag{2.6}$$

where $e = \det(e^a_i)$, and $g = \det(g_{ij})$. Again, we use the Berezin integral to select the top degree component of the form. Notice that the last integral can be computed if suitable convergence conditions are satisfied according to Riemann or Lebesgue integration theory.

In the following we will need also distributions, and therefore we consider expressions that factorize $\epsilon^{i_1 \cdots i_n}\omega_{[i_1 \ldots i_n]}(x)$ into a distributional part $\frac{1}{\sqrt{g}}\prod_{i=1}^n \delta(x^i)$ (the additional $\frac{1}{\sqrt{g}}$ is added for covariance under diffeomorphisms) and into a test function $\tilde{\omega}(x)$ (for example, belonging to the space of fast decreasing functions). In this case:

$$\int_M \omega = \int_M \tilde{\omega}(x) \left[\prod_{i=1}^n \delta(x^i)\right] d^n x = \tilde{\omega}(0)\,, \tag{2.7}$$

where in the last term we evaluate the expression at $x^i = 0$. In this case, the compactness of the space or other convergence conditions do not matter, since the measure is concentrated in the point $x^i = 0$. The points x_i where the integral is localised can be moved by suitable diffeomorphisms.

We denote now by $\mathcal{M}$ a supermanifold with coordinates (x^i, θ^α) (with $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$) and we consider the "exterior" bundle $\Omega^\bullet(\mathcal{M})$ as the direct sum of bundles of fixed degree forms. The local coordinates in the total space of this bundle are $(x^i, d\theta^\alpha, dx^j, \theta^\beta)$, where $(x^i, d\theta^\alpha)$ are bosonic and $\left(dx^j, \theta^\beta\right)$ fermionic. In contrast to the pure bosonic case, a top form does not exist because the $1-$ forms of the type $d\theta^\alpha$ commute among themselves $d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha$. Then we can consider forms of any degree (wedge products are omitted in the following)

$$\omega = \sum_{p=0}^{n}\sum_{l=0}^{\infty} \omega_{[i_1 \ldots i_p](\alpha_1 \ldots \alpha_l)}(x,\theta) dx^{i_1} ... dx^{i_p} d\theta^{\alpha_1} \ldots d\theta^{\alpha_l} \tag{2.8}$$

where the coefficients $\omega_{[i_1 \ldots i_p](\alpha_1 \ldots \alpha_l)}(x,\theta)$ are functions on the supermanifold $\mathcal{M}$ with the first $1 \ldots p$ indices antisymmetrized and the last $1 \ldots l$ symmetrized.

The component functions $\omega_{[i_1 \ldots i_p](\alpha_1 \ldots \alpha_l)}(x,\theta)$ are polynomial expressions in the θ^α and their coefficients are functions of x^i only. However, we can adopt a different point of view: instead of simply expanding formally a generic form $\omega(x,\theta,dx,d\theta)$ in $d\theta$, we can consider analytic functions of the bosonic variables $d\theta$ and in addition we will admit also distributions acting on the space of test functions of $d\theta$. In this way, the exterior bundle in the $d\theta$ directions is a conventional bosonic manifold with coordinates $d\theta^\alpha$ and the superforms become distribution-valued on that space. In particular, we introduce the distributions $\delta(d\theta^\alpha)$ that have most (but not all!) of the usual properties of the Dirac delta function $\delta(x)$. As explained in Appendix A, one must have:

$$\delta(d\theta^\alpha)\delta(d\theta^\beta) = -\delta(d\theta^\beta)\delta(d\theta^\alpha) \tag{2.9}$$

Therefore, the product of all Dirac's delta functions $\delta^m(d\theta) \equiv \prod_{\alpha=1}^m \delta(d\theta^\alpha)$ serves as a "top form".

An **integral** form $\omega^{(p|q)}$ belonging to $\Omega^{(p|q)}(\mathcal{M})$ is characterised by two indices $(p|q)$: the first index is the usual form degree and the second one is the *picture* number which counts the number of delta's. For a top form, that number must be equal to the fermionic dimension of the space. Consequently, an *integral* form reads:

$$\omega^{(p|q)} = \sum_{r=1}^{p} \omega_{[i_1\dots i_r](\alpha_{r+1}\dots\alpha_p)[\beta_1\dots\beta_q]} dx^{i_1}\dots dx^{i_r} d\theta^{\alpha_{r+1}}\dots d\theta^{\alpha_p}\delta(d\theta^{\beta_1})\dots\delta(d\theta^{\beta_q}) \tag{2.10}$$

with $\omega_{[i_1\dots i_r](\alpha_{r+1}\dots\alpha_p)[\beta_1\dots\beta_q]}(x,\theta)$ superfields.

The $d\theta^\alpha$ appearing in the product and those appearing in the delta functions are reorganised respecting the rule $d\theta^\alpha\delta(d\theta^\beta) = 0$ if $\alpha = \beta$. We see that if the number of delta's is equal to the fermionic dimension of the space no $d\theta$ can appear; if moreover the number of the dx is equal to the bosonic dimension the form (of type $\omega^{(n|m)}$) is a **top form.**

Notice that $\omega^{(p|q)}$ as written above is not the most generic form, since we could have added the derivatives of delta functions (and they indeed turn out to be unavoidable and will play an important role). They act by reducing the form degree according to the rule $d\theta^\alpha\delta'(d\theta^\alpha) = -\delta(d\theta^\alpha)$, where $\delta'(x)$ is the first derivative of the delta function with respect to its variable. (We denote also by $\delta^{(p)}(x)$ the p-derivative).

We also define as a *superform* a 0-picture integral form $\Omega^{(p|0)}$

$$\begin{aligned}\omega^{(p|0)} &= \sum_{r=1}^{p} \omega_{[i_1\dots i_r](\alpha_{r+1}\dots\alpha_p)} dx^{i_1}\dots dx^{i_r} d\theta^{\alpha_{r+1}}\dots d\theta^{\alpha_p} \\ &= \omega_{M_1\dots M_p}(Z) dZ^{M_1}\dots dZ^{M_p}\end{aligned} \tag{2.11}$$

where the first indices are antisymmetrized while the spinorial indices $\alpha_1\dots\alpha_s$ are symmetrized. In the last line, we have collectively denoted by Z^M the superspace coordinates and the indices $M_1\dots M_p$ of the superform are graded symmetric.

Integral top forms (with maximal form degree in the bosonic variables and maximal number of delta forms) are **the only objects we can hope to integrate** on supermanifolds.

In general, if ω is a form in $\Omega^{\bullet}(\mathcal{M})$, its integral on the supermanifold is defined as follows: (in analogy with the Berezin integral for bosonic forms):

$$\int_{\mathcal{M}} \omega \equiv \int_{\mathcal{M}} \epsilon^{i_1 \ldots i_n} \epsilon^{\beta_1 \ldots \beta_m} \omega_{[i_1 \ldots i_n][\beta_1 \ldots \beta_m]}(x, \theta)[d^n x \, d^m \theta] \tag{2.12}$$

where the last integral over $\mathcal{M}$ is the usual Riemann-Lebesgue integral over the coordinates x^i (if it is exists) and the Berezin integral over the coordinates θ^α. The expressions $\omega_{[i_1 \ldots i_n][\beta_1 \ldots \beta_m]}(x, \theta)$ denote those components of (2.10) with no symmetric indices.

Note that under the rescaling $\theta \to \lambda\theta$ ($\lambda \in \mathbb{R}$) the measure $[dx \, d\theta d\,(dx\,)\, d\,(d\theta)]$ is an **invariant** quantity, in fact it is locally a "product measure", and we know that $[dxd\theta] \to \frac{1}{\lambda}[dxd\theta]$ and $[d\,(dx\,)\, d\,(d\theta)] \to \lambda\, [d\,(dx\,)\, d\,(d\theta)]$. This can be extended to general coordinate transformations, and the outcome is that $[d^n x \, d^m \theta d^n\,(dx\,)\, d^m\,(d\theta)]$ is an **invariant** measure.

It is clear now that **we cannot integrate a generic** $\omega\,(x, \theta, dx, d\theta)$. Suppose that the Riemann-Lebesgue integrability conditions are satisfied with respect to the x variables; the integrals over dx and θ (being Berezin integrals) pose no further problem but, if $\omega\,(x, \theta, dx, d\theta)$ has a polynomial dependence in the (bosonic) variables $d\theta$, the integral diverges unless $\omega\,(x, \theta, dx, d\theta)$ depends on the $d\theta$ only through the product of **all** the "distributions" $\delta\,(d\theta^\alpha)$.[2] This solves the problem of the divergences for all the $d\theta^\alpha$ variables because

$$\int \delta\,(d\theta^\alpha)\,[d\,(d\theta^\alpha)] = 1 \tag{2.13}$$

Summing up we can integrate only integral forms ω, the integral selecting only forms contained in ω with top degree in bosonic variables and top picture number, namely the so-called **integral top forms**.

In order to shorten notations, when the "variables of integration" are evident, we will omit in the integrals all the "integration measures symbols" such as $[d^n x \, d^m \theta d^n\,(dx\,)\, d^m\,(d\theta)]$ or $[d^n x \, d^m \theta]$.

[2] We could (as explained above) also admit a more general $d\theta$ dependence i.e. in the form of a test function in the $d\theta^\alpha$ multiplied by the product of all distributions $\delta\,(d\theta^\alpha)$, but this generalization is not needed here.

In the case of curved supermanifolds, by expressing the 1-forms dx^i and $d\theta^\alpha$ in terms of the supervielbeins $E^A_M \equiv (e^i_M, e^\alpha_M)$ (where A runs over the flat indices i and α, and M runs over the curved indices), we have

$$\int_{\Omega^\bullet(\mathcal{M})} \omega(x,\theta,e^i,e^\alpha) = \int_{\mathcal{M}} E\, \epsilon^{i_1\dots i_n}\epsilon^{\alpha_1\dots\alpha_m}\, \omega_{[i_1\dots i_n][\alpha_1\dots\alpha_m]}(x,\theta) \tag{2.14}$$

where $E = \mathrm{sdet}(E^A_M)$ is the superdeterminant (the Berezinian) of the supermatix $E^A_M(x,\theta)$. As usual this definition is invariant under (orientation preserving super) diffeomorphisms.

3 Poincaré duals and Variational Principles on Submanifolds

As discussed in the introduction, we consider a submanifold $\mathcal{S}$ of a bigger space $\mathcal{M}$ – that could be also a supermanifold – and we give a recipe to construct an action I on that submanifold. The next step is to derive the equations of motion from a variational principle varying both the Lagrangian L and the embedding of the submanifold into $\mathcal{M}$. This can be achieved by extending the integral of the Lagrangian L to an integral over the entire bigger space $\mathcal{M}$. For that we need the notion of the Poincaré dual of the submanifold $\mathcal{S}$ into $\mathcal{M}$. The result is an extended Lagrangian, depending dynamically on the fields and on the embedding functions, integrated over a fixed manifold $\mathcal{M}$.

3.1 Poincaré Duals

We start with a submanifold $\mathcal{S}$ of dimension s of a differentiable manifold $\mathcal{M}$ of dimension n. We take an embedding i :

$$i : \mathcal{S} \to \mathcal{M}$$

and a compact support form $L \in \Omega^s(\mathcal{M})$. The **Poincaré dual** of $\mathcal{S}$ is a *closed* form $\eta_{\mathcal{S}} \in \Omega^{n-s}(\mathcal{M})$ such that $\forall L$:

$$I[L,\mathcal{S}] = \int_{\mathcal{S}} i^* L = \int_{\mathcal{M}} L \wedge \eta_{\mathcal{S}} \tag{3.1}$$

where i^* is the pull-back of forms. We are not interested here in a rigorous mathematical treatment (see [16]) and we take a heuristic approach well-adapted for the generalization to the supermanifold case. In the symbol $I[L, \mathcal{S}]$, we have recalled the dependence upon the embedding of $\mathcal{S}$ into $\mathcal{M}$.

If we suppose that the submanifold $\mathcal{S}$ is described locally by the vanishing of $n-s$ coordinates $t^1, \dots, t^{n-s}$, its Poincaré dual can also be described as a *singular closed localization form* (the correct mathematics is the de Rham current theory [17]):

$$\eta_{\mathcal{S}} = \delta(t^1)...\delta(t^{n-s})dt^1 \wedge ... \wedge dt^{n-s} \tag{3.2}$$

This distribution-valued form is clearly closed (from the properties of the delta distributions $d\,\delta(t) = \delta'(t)dt$ and from $dt^i \wedge dt^i = 0$). This form belongs to $\Omega^{n-s}(\mathcal{M})$ and is constructed in such a way that it projects on the submanifold $t^1 = \cdots = t^{n-s} = 0$ and orthogonally to $dt^1 \wedge \ldots \wedge dt^{n-s}$. Thus, by multiplying a given form $L \in \Omega^s(\mathcal{M})$ by η_S, the former is restricted to those components which are not proportional to the differentials dt^i.

Observing that the Dirac δ-function of an odd variable (dt is odd if t is even) coincides with the variable itself (as can be seen using Berezin integration), we rewrite $\eta_{\mathcal{S}}$ as a form that will turn out to be very useful for generalization (omitting wedge symbols):

$$\eta_{\mathcal{S}} = \delta(t^1)...\delta(t^{n-s})\delta(dt^1)...\delta(dt^{n-s}) \tag{3.3}$$

which heuristically corresponds to the localisation to $t^1 = \cdots = t^{n-s} = 0$ and $dt^1 = \cdots = dt^{n-s} = 0$. Note that if a submanifold $\mathcal{S}$ is described by the vanishing of $n-s$ functions $f^1(t) = \cdots = f^{n-s}(t) = 0$ the corresponding Poincaré dual $\eta_{\mathcal{S}}$ is:

$$\eta_{\mathcal{S}} = \delta(f^1)...\delta(f^{n-s})\delta(df^1)...\delta(df^{n-s})$$

This form, when written completely in terms of the t^i coordinates, contains also the *derivatives of the* δ's because of the expansion of $\delta(f)$ and $\delta(df)$ in terms of t^i.

If we change (in the same homology class) the submanifold $\mathcal{S}$ to $\mathcal{S}'$ the corresponding Poincaré duals $\eta_{\mathcal{S}}$ and $\eta_{\mathcal{S}'}$ are known to differ by an *exact* form:

$$\eta_{\mathcal{S}} - \eta_{\mathcal{S}'} = d\gamma$$

This can be easily proved by recalling that the Poincaré duals are closed $d\eta_{\mathcal{S}} = 0$ and any variation (denoted by Δ) of $\eta_{\mathcal{S}}$ is exact:

$$\Delta\eta_{\mathcal{S}} = d\Big(\Delta f \delta(f)\Big) \tag{3.4}$$

Given the explicit expression of $\eta_{\mathcal{S}}$, it is easy to check eq. (3.4) by expanding both members (assuming that Δ follows the Leibniz rule) and using the distributional laws of δ's.

Using this property we can show that, if $dL = 0$ (in $\mathcal{M}$ since $d_{\mathcal{S}}(i^*L) = 0$ trivially in $\mathcal{S}$), then the action does not depend on the embedding of the submanifold. Indeed varying the embedding amounts to vary the Poincaré dual, so that the variation of the integral reads

$$\Delta I[L,\mathcal{S}] = I[L,\Delta\mathcal{S}] = \int_{\mathcal{M}} L \wedge \Delta\eta_{\mathcal{S}} = \int_{\mathcal{M}} L \wedge d\xi_{\mathcal{S}} = (-)^s \int_{\mathcal{M}} dL \wedge \xi_{\mathcal{S}} \tag{3.5}$$

where $\Delta\eta_{\mathcal{S}} = d\xi_{\mathcal{S}}$.

The same arguments apply in the case of supermanifolds. Consider a submanifold $\mathcal{S}$ of dimension $s|q$ of a supermanifold $\mathcal{M}$ of dimension $n|m$. We take an embedding i :

$$i : \mathcal{S} \to \mathcal{M}$$

and an integral form $L \in \Omega^{s|q}(\mathcal{M})$ (integrable in the sense of superintegration when pulled back on $\mathcal{S}$). The **Poincaré dual** of $\mathcal{S}$ is a d-*closed* form $\eta_{\mathcal{S}} \in \Omega^{n-s|m-q}(\mathcal{M})$ such that:

$$\int_{\mathcal{S}} i^*L = \int_{\mathcal{M}} L \wedge \eta_{\mathcal{S}}$$

Again we can write:

$$\eta_{\mathcal{S}} = \delta(f^1)...\delta(f^{\cdots})\delta(df^1)...\delta(df^{\cdots})$$

where the f's are the functions defining (at least locally) the submanifold $\mathcal{S}$. Here some of them are even functions and some of them are odd functions, accordingly the Poincaré dual is a closed integral form that, written in the coordinates (x, θ), contains delta forms and their derivatives.

Again it is easy to check that any variation of $\eta_{\mathcal{S}}$ is d-exact:

$$\Delta\eta_{\mathcal{S}} = d\Big((\Delta f) f \delta'(df)\Big) \tag{3.6}$$

Note that the two formulae (3.4) and (3.6) for the variation of η_S can be combined in a formula that holds true in both cases:

$$\Delta\eta_S = d\Big(\Delta f \delta(f)\delta^{'}(df)\Big) \tag{3.7}$$

Indeed, one has $\delta^{'}(df) = 1$ or $\delta(f) = f$ when f is respectively bosonic or fermionic.

Before considering some examples, we have to spend a few words on the general form of the Poincaré dual in the case of supermanifolds:

$$\eta_{\mathcal{S}}^{(n-s|m)} = \sum_{l=0}^{d} \eta_{[i_1 \cdots i_{n-s+l}]}(x,\theta) dx^{i_1} \dots dx^{i_{n-s+l}} \partial^l \delta^m(d\theta) \tag{3.8}$$

where we have added l-derivatives ∂^l on the Dirac delta functions (for the moment we have not specified how these derivatives are distributed on $\delta^m(d\theta)$, but we have to admit all possible combinations and, for each of them, we have new coefficients $\eta_{[i_1 \cdots i_{n-s+l}]}(x,\theta)$). Acting with derivatives on Dirac delta's we decrease the form number which must be compensated by adding more 1-forms dx up to the maximum n (this implies also that the maximum number of derivatives is s). In principle, we could have also added $d\theta$, but these can be removed by integration by parts. Notice that the simple Poincaré dual given in (3.3) is included in the general expression (3.8). If we consider again the integral with L, by integration-by-parts and by using the property $d\theta\delta'(d\theta) = -\delta(d\theta)$, we can finally take into account all possible directions (namely also the $d\theta$ directions). We would like also to underline that the different coefficients $\eta_{[i_1 \cdots i_{n-s+l}]}(x,\theta)$ parametrize all possible embeddings of the submanifold $\mathcal{S}$ into the supermanifold $\mathcal{M}^{(n|m)}$. In particular they parametrize how the coordinates of the submanifold are written in terms of those of the complete supermanifold and this amounts to the choice of derivatives of Dirac delta's.

Let us consider for example $\mathbb{R}^{(0|1)}$ as a submanifold of $\mathbb{R}^{(0|2)}$, which has two coordinates θ^1 and θ^2. The form $L = \theta^1\delta(d\theta^2) \in \Omega^{(0|1)}$ can be integrated over the submanifold $\mathbb{R}^{(0|1)}$ since it is a $0-$form with $1-$picture. The embedding of $\mathbb{R}^{(0|1)}$ is chosen by setting $a\,\theta^1 + b\,\theta^2 = 0$ with $a, b \in \mathbb{R}$. We can compute the integral in two ways: the first is by using $\theta^1 = -\frac{b}{a}\theta^2$ and by re-expressing L in terms of the coordinate θ^2 only. Thus

$$L = -\frac{b}{a}\theta^2\delta(d\theta^2)$$

and the integral gives:

$$I[L,(a,b)] = \int_{\mathbb{R}^{(0|1)}} L = -\frac{b}{a}$$

The second way is as follows. The Poincaré dual of $\mathbb{R}^{(0|1)}$ into $\mathbb{R}^{(0|2)}$ is

$$\eta_S = \delta(a\,\theta^1 + b\,\theta^2)\delta(a\,d\theta^1 + b\,d\theta^2)\,.$$

The first delta function can be rewritten as $a\,\theta^1 + b\,\theta^2$ because of the anticommutativity of θ's. Multiplying η_S by L we obtain:

$$L \wedge \eta_S = \theta^1 \delta(d\theta^2)\delta(a\,\theta^1 + b\,\theta^2)\delta(a\,d\theta^1 + b\,d\theta^2) =$$

$$= \theta^1(a\,\theta^1 + b\,\theta^2)\delta(d\theta^2)\delta(a\,d\theta^1 + b\,d\theta^2) =$$

$$= b\theta^1\theta^2\delta(d\theta^2)\delta(a\,d\theta^1) = -\frac{b}{a}\theta^1\theta^2\,\delta(d\theta^1)\delta(d\theta^2)$$

Thus $\int_{\mathbb{R}^{(0|2)}} L \wedge \eta_S = -b/a$ which coincides with the computation above. The integral depends upon the embedding parameters (a,b) (and is not defined for $a=0$). Repeating the same computation with a closed form (for example $\theta^1\delta(d\theta^1)$), it is easy to see that the integral equals 1 and does not depend on the embedding parameters as expected.

3.2 Variational Principle

The action $I[L,\mathcal{S}]$ is a functional of L and $\mathcal{S}$, and therefore varying it means varying both L and $\mathcal{S}$. The latter corresponds to varying η_S. The variational principle leads to

$$\Delta I[L,\mathcal{S}] = \Delta \int_{\mathcal{S}} i^* L = \int_{\mathcal{M}} (\Delta L \wedge \eta_S + L \wedge \Delta\eta_S) = 0\,. \tag{3.9}$$

The variation has two terms. The first one contains the variation of the Lagrangian L over the entire space and the second one the variation of the embedding. However, in the second term we use the exactness of the variation of η_S ($\Delta\eta_S = d\xi_S$) and by integration by parts we can rewrite the variation of the action as

$$\Delta I[L,\mathcal{S}] = \int_{\mathcal{M}} (\Delta L \wedge \eta_S + (-)^s dL \wedge \xi_S) \tag{3.10}$$

where s is the degree of the form L. The expression ξ_S is arbitrary since it corresponds to an arbitrary variation of $\mathcal{S}$, and therefore both terms of the integral must vanish separately leading to the equations of motion

$$\Delta L = 0\,, \qquad dL = 0\,. \tag{3.11}$$

Since the variation of L under Δ is an arbitrary variation, the first equation implies the second one and therefore, on the equations of motion $\Delta L = 0$ (only), the integral $I[L, \mathcal{S}]$ is independent of $\mathcal{S}$. This is somehow rather obvious, but it is interesting to notice that in many cases $dL = 0$ holds only on a subset of the equations of motion, and in some cases it holds completely off-shell.

As an example we consider $3d-$euclidean gravity on a $3d-$submanifold S (for example a $3d-$topological sphere) embedded into $\mathbb{R}^4$. The Poincaré dual is given by $\eta_S = \delta(f)df$ where $S = \{f^{-1}(0)\}$. The action is given by

$$I[\omega, V, f] = \int_S i^* \epsilon_{abc} R^{ab}(\omega) \wedge V^c = \int_{\mathbb{R}^4} \epsilon_{abc} R^{ab}(\omega) \wedge V^c \wedge \delta(f) df\,, \tag{3.12}$$

where ω is the spin connection, $R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}$, V^a is the *dreibein* and f is the embedding function. The equations of motion are given by

$$\begin{aligned} &R^{ab}\delta(f) \wedge df = 0\,, \qquad T^a \delta(f) \wedge df = 0\,, \\ &d\left(\epsilon_{abc} R^{ab} \wedge V^c\right) = \epsilon_{abc} R^{ab}(\omega) \wedge T^c = 0\,. \end{aligned} \tag{3.13}$$

where the torsion is defined as $T^a = dV^a + \omega^a{}_b \wedge V^b$. Notice that the equation on the first line are valid for any f, and this implies that $R^{ab} = 0, T^a = 0$ on the entire space $\mathbb{R}^4$. The last equation is a consequence of the first two equations together with the Bianchi identity $DR^{ab} = 0$ (where D is the Lorenz covariant derivative), but we observe that only one is sufficient to guarantee the vanishing of the last equation. Namely, for a torsionless connection ω, $dL = 0$ off-shell and ω can be expressed in terms of V^a (second order formalism).

We consider now $4d$-Einstein gravity and we would like to embed the Einstein-Hilbert Lagrangian (defined on 4-dimensional space $\mathcal{S}$) in a bigger 10-dimensional space $\mathcal{M}$ viewed as the group manifold associated to Poincaré symmetry generated by the translations and by Lorentz transformations. The coordinates of $\mathcal{M}$ are the usual x^a and the "Lorentz coordinates" y^{ab}. The exterior bundle $\Omega^\bullet(\mathcal{M})$ is parametrised by the vielbeins V^a, ω^{ab} (they are interpreted as the usual $4d$-vielbein and the spin connection). The curvatures T^a (associated to the translations) and R^{ab} (associated to the Lorentz transformations) can be decomposed along the complete basis

$$\begin{aligned} T^a &= T^a_{\ b} V^b + T^a_{\ bc} \omega^{bc} \,, \\ R^{ab} &= R^{ab}{}_{cd} V^c \wedge V^d + R^{ab}{}_{c,de} V^c \wedge \omega^{de} + R^{ab}{}_{cd,ef} \omega^{cd} \wedge \omega^{ef} \end{aligned} \tag{3.14}$$

We denote by *inner* components the coefficients along $V^a \wedge V^b$ and *outer* the remaining ones.

The EH action is written as

$$I_{EH}[\omega, V] = \int_{\mathcal{S}} i^* \left(R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} \right) = \int_{\mathcal{M}} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} \wedge \eta_{\mathcal{S}} \tag{3.15}$$

where $\eta_{\mathcal{S}}$ is the Poincaré dual of $\mathcal{S}$ in $\mathcal{M}$. Under the conditions discussed above the equations of motion on the big space $\mathcal{M}$ are

$$\epsilon_{abcd} T^c \wedge V^d \wedge \eta_{\mathcal{S}} = 0, \quad \epsilon_{abcd} R^{ab} \wedge V^d \wedge \eta_{\mathcal{S}} = 0 \tag{3.16}$$

The field equations are 3-form equations on $\mathcal{M}$. Their content can be extracted by projecting on a complete basis of 3-forms in $\mathcal{M}$. The first equation is then found to imply $T^a = 0$ (i.e. the torsion vanishes as a 2-form on $\mathcal{M}$), and the second leads to the vanishing of the outer components of the Lorentz curvature R^{ab} and to the Einstein equations for the inner components $R^{ab}{}_{cd}$:

$$R^{ac}{}_{bc} - \frac{1}{2} \delta^a{}_b R^{cd}{}_{cd} = 0 \tag{3.17}$$

It is easy to check that the field equations imply

$$d(R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd}) = 0$$

by using $dL = DL$ (the covariant exterior derivative for any Lorentz invariant quantity L), the Bianchi identity $DR^{ab} = 0$ and the field equation $T^a = 0$.

3.3 Invariances of the Action

By construction, integrals of top forms are invariant under (infinitesimal) diffeomorphisms (hereafter simply called diffeomorphisms). Indeed the action of a Lie derivative $\mathcal{L}_\epsilon$ along a tangent vector ϵ on a top form Ω is a total derivative $d(\iota_\epsilon \Omega)$. We consider the Lagrangian $L(\mu)$ as a function of the fields μ which are p-forms, their wedge products and their exterior derivative. Thus one knows a priori that the action

$$I[L(\mu), \mathcal{S}] = \int_{\mathcal{M}} L(\mu) \wedge \eta_{\mathcal{S}} \tag{3.18}$$

is invariant under diffeomorphisms in $\mathcal{M}$, $L \wedge \eta_{\mathcal{S}}$ being a top form. The variation of this action under diffeomorphisms can be written again as a sum of two pieces:

$$\delta_\epsilon I = 0 = \int_{\mathcal{M}} \mathcal{L}_\epsilon L \wedge \eta_{\mathcal{S}} + \int_{\mathcal{M}} L \wedge \mathcal{L}_\epsilon \eta_{\mathcal{S}} \tag{3.19}$$

Consider first ϵ to be a tangent vector lying along $\mathcal{S}$, so that we are dealing with diffeomorphisms in the $\mathcal{S}$ submanifold (we call them for short x-diff's). The x-diff's do not change the embedding of $\mathcal{S}$ inside $\mathcal{M}$, and therefore leave unchanged the Poincaré dual $\eta_{\mathcal{S}}$. Thus for x-diff's

$$\delta_\epsilon I = 0 = \int_{\mathcal{M}} \mathcal{L}_\epsilon L \wedge \eta_{\mathcal{S}} \tag{3.20}$$

and we find that varying *only the Lagrangian* L under x-diff's leaves the action (3.18) invariant. Since the fields μ appear only in the Lagrangian, *the action I is invariant under x-diff's variations of the fields μ.* In terms of the embedding i, the variation of I under x-diff's given in eq. (3.20) can be written as

$$\delta_\epsilon \int_{\mathcal{S}} i^* L = 0 = \int_{\mathcal{S}} i^*(\mathcal{L}_\epsilon L) \tag{3.21}$$

The l.h.s. corresponds to a variation of the fields in $i^*(L)$, the r.h.s. corresponds to a diffeomorphism variation of L.

The situation is different when the tangent vector ϵ lies outside the tangent space of $\mathcal{S}$ (we call the corresponding diffeomorphisms y-diff's). In this case, the embedding of $\mathcal{S}$ inside $\mathcal{M}$ changes under the action of the Lie derivative $\mathcal{L}_\epsilon$, and the second term of eq. (3.19) is present.

Recalling that $d\eta_{\mathcal{S}} = 0$, this term reduces, after integration by parts, to $(-)^s \int_{\mathcal{M}} dL \wedge \iota_\epsilon \eta_{\mathcal{S}}$. Thus varying only L under y-diff's leaves the action (3.18) invariant if $dL = 0$. We conclude that y-diff's applied to the fields μ are invariances of the action I if $dL = 0$.

Actually the condition for y-diff's on μ to be invariances of I is weaker: indeed it is sufficient to have $\iota_\epsilon dL = 0$. This can be checked directly by varying L in the action under y-diff.s:

$$\int_{\mathcal{M}} \mathcal{L}_\epsilon L \wedge \eta_{\mathcal{S}} = \int_{\mathcal{M}} [(\iota_\epsilon dL) \wedge \eta_{\mathcal{S}} + d(\iota_\epsilon L) \wedge \eta_{\mathcal{S}}] \tag{3.22}$$

Integrating by parts the second term and recalling that $d\eta_{\mathcal{S}} = 0$ *proves that the action I is invariant under y-diff's applied to the fields* μ when $\iota_\epsilon dL = 0$.

The last equation (3.22) can also be used to study the dependence of the action upon the embedding functions. We know that $\int_{\mathcal{M}} L \wedge \mathcal{L}_\epsilon \eta_{\mathcal{S}} = -\int_{\mathcal{M}} \mathcal{L}_\epsilon L \wedge \eta_{\mathcal{S}}$ from eq. (3.19). Thus any variation of the embedding (generated by $\mathcal{L}_\epsilon$, with an arbitrary ϵ outside $\mathcal{S}$) can be compensated by a y-diff's on L. On the other hand we have seen that y-diff's on L do not change the action when $\iota_\epsilon dL = 0$ with ϵ in the y-directions, and therefore this is also the condition for I to be independent on the particular embedding of $\mathcal{S}$.

Let us come back to our example, pure gravity in the group manifold approach, where the "big space" $\mathcal{M}$ is (a smooth deformation of) the Poincaré group manifold, and the "small space" $\mathcal{S}$ is the usual Minkowski spacetime. Usual x-diff's on the fields V^a and ω^{ab} leave the action invariant, while y-diff's, i.e. diffeomorphisms along the Lorentz directions of $\mathcal{M}$, are invariances when applied to V^a and ω^{ab} if $\iota_t dL = 0$ ($t = t^{ab}\partial_{y^{ab}}$ being the tangent vectors in the Lorentz directions, dual to the spin connection ω^{ab}). Let us check whether this condition holds. Replacing again the exterior derivative d with the covariant exterior derivative D, and using the Bianchi identity $DR^{ab} = 0$ and definition of the torsion, we find the condition:

$$\iota_t dL = \iota_t (R^{ab} \wedge T^c \wedge V^d)\ \epsilon_{abcd} = 0 \tag{3.23}$$

Using now the Leibniz rule for the contraction, and $\iota_t(V^a) = 0$, leads to the condition that all outer components of R^{ab} and T^a must vanish. These conditions are *part* of the field equations previously derived. In particular they do not involve the "inner" field equations, i.e. the

Einstein equations. On this "partial shell" the action is invariant under y-diff's ("Lorentz diffeomorphisms") applied to the fields.

The vanishing of outer components of the curvature is also called *horizontality* of the curvature.

When horizontality of R^A in the y-directions holds, the dependence of fields $\mu^A(x,y)$ on y is completely determined by their value $\mu^A(x,0)$ on the embedded hypersurface $\mathcal{S}$. Indeed in this case an infinitesimal y-diffeomorphism on $\mu^B(x,0)$ can be written as

$$\mu^B(x,\delta y) = \mu^B(x,0) + d\delta y^B + C^B_{\ CD}\mu^C(x,0)\delta y^D \tag{3.24}$$

and shows that $\mu^B(x,\delta y)$ is determined by the value of the field μ at $y=0$. This equation can be integrated to reconstruct the y-dependence of $\mu^B(x,y)$ (at least in a sufficiently small connected neighborhood of $y=0$).

A milder form of horizontality occurs when the outer components of the curvature R^A do not vanish, but are proportional to linear combinations of inner components of R^A. The curvature is then said to be *rheonomic*. In this case a diffeomorphism in the outer directions involves only the values of $\mu^A(x,0)$, and of its x-space derivatives $\frac{\partial}{\partial x^\mu}\mu^A(x,0)$, contained in the inner components of R^A. Again the value of $\mu^A(x,0)$ on the hypersurface $\mathcal{S}$ determines the y-dependence of $\mu^A(x,y)$ on the manifold $\mathcal{M}$. This situation is very common in supergravity theories, where some of the outer directions are fermionic, and diffeomorphisms in these directions are interpreted as supersymmetry transformations.

3.4 Field transformation rules

Let us have a closer look at the variation of the fields μ under (infinitesimal) diffeomorphisms. The transformation rule is given by the action of the Lie derivative on μ:

$$\delta_\epsilon\mu = \mathcal{L}_\epsilon\mu = d\iota_\epsilon\mu + \iota_\epsilon d\mu \tag{3.25}$$

When μ^A is the vielbein of a (deformed) group manifold $\mathcal{M}$ (the index A running on the Lie algebra of G), the variation formula (3.25) takes the suggestive form:

$$\delta_\epsilon\mu^B = d\epsilon^B + C^B_{\ CD}\mu^C\epsilon^D + \iota_\epsilon R^B \equiv (\nabla\epsilon)^B + \iota_\epsilon R^B \tag{3.26}$$

where $C^B_{\ CD}$ are the G-structure constants, $\epsilon = \epsilon^A t_A$ is a generic tangent vector expanded on the tangent basis t_A dual to the cotangent (vielbein) basis μ^B, and ∇ is the G-covariant exterior derivative. To prove this one just uses the definition of the group curvatures:

$$R^A = d\mu^A + \frac{1}{2} C^A_{\ BC} \mu^B \wedge \mu^C \tag{3.27}$$

that allow to re-express $d\mu^A$ in terms of R^A and bilinears of vielbeins.

When the group curvatures R^A are horizontal in the directions of some subgroup H of G, the diffeomorphisms along the H-directions become *gauge transformations*, as one sees immediately from the diffeomorphism variation formula (3.26): indeed in this case the contracted curvature term vanishes, and the variation amounts to the covariant derivative of the parameter ϵ. Thus the group-geometric approach provides a unified picture of the symmetries (gauge or diffeomorphisms): they all originate from diffeomorphism invariance in $\mathcal{M}$.

In our example of pure gravity where $\mathcal{M}$ is a deformed Poincaré manifold, the y-diff's transformation rules (3.26) are obtained by choosing the tangent vector ϵ in the Lorentz directions, $\epsilon = \epsilon^{ab} t_{ab}$, and by using the horizontality of T^a and R^{ab} in the Lorentz directions. One finds

$$\delta_\epsilon V^a = \epsilon^a_{\ b} V^b, \quad \delta_\epsilon \omega^{ab} = D\epsilon^{ab} \equiv d\epsilon^{ab} - \omega^a_{\ c} \epsilon^{cb} + \omega^b_{\ c} \epsilon^{ca} \tag{3.28}$$

reproducing the Lorentz gauge variations of the vielbein and the spin connection. The infinitesimal parameter of the diffeomorphism transformation ϵ^{ab} in the Lorentz coordinates is then re-interpreted as the local Lorentz gauge parameter. The Einstein-Hilbert action on $\mathcal{S}$ is invariant under these transformations.

3.5 Supersymmetry

In the group-geometric approach to supergravity theories, the "big" manifold $\mathcal{M}$ is a supergroup manifold, and there are fermionic vielbeins ψ (the gravitini) dual to the fermionic tangent vectors in $\mathcal{M}$. The diffeomorphisms in the fermionic directions are a particular instance of the general rule (3.26). When rheonomy holds, the fermionic diffeomorphisms are seen as (local) supersymmetry variations of the fields. To illustrate this mechanism, we consider the example of $D = 4$ simple supergravity, for which G is the superPoincaré group. The fields μ^A are in this case the vielbein V^a, the gravitino (a Majorana 1-form fermion) ψ, and the spin connection ω^{ab} corresponding respectively to the translations, supersymmetries and Lorentz rotations of the superPoincaré Lie algebra. The general curvature definition (3.27) becomes, using the structure constants of the Lie superalgebra:

$$T^a = dV^a - \omega^{ac}V^c - \frac{i}{2}\bar{\psi}\gamma^a\psi, \quad \rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi, \quad R^{ab} = d\omega^{ab} - \omega^{ac}\omega^{cb} \tag{3.29}$$

defining respectively the supertorsion, the gravitino field strength and the Lorentz curvature. All forms live on $\mathcal{M}$ = (deformed) super-Poincaré group manifold. The action is a 4-form integrated on a $\mathcal{S}$ (diffeomorphic to Minkowski spacetime) submanifold of $\mathcal{M}$:

$$I[V,\omega,\psi] = \int_{\mathcal{S}} R^{ab}V^cV^d\epsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_a\rho V^a \tag{3.30}$$

The field equations, when projected on all the $\mathcal{M}$ directions, give the following conditions on the curvatures:

$$R^{ab} = R^{ab}{}_{cd}V^c \wedge V^d - (\epsilon_{abcd}\bar{\rho}_{cd}\gamma_5\gamma_e + \delta_e^{[a}\epsilon^{b]cdf}\bar{\rho}_{df}\gamma_5\gamma_c) \wedge \psi \wedge V^e \tag{3.31}$$

$$T^a = 0 \tag{3.32}$$

$$\rho = \rho_{ab}V^a \wedge V^b \tag{3.33}$$

where the spacetime (inner) components $R^{ab}{}_{cd}$, ρ_{ab} satisfy the propagation equations

$$R^{ac}{}_{bc} - \frac{1}{2}\delta^a_b R^{cd}{}_{cd} = 0, \quad \gamma^{abc}\rho_{bc} = 0 \tag{3.34}$$

respectively the Einstein and the gravitino field equations. Eq.s (3.31)-(3.33), an output of the equations of motion, are rheonomic conditions. Indeed the only nonvanishing outer components (those of R^{ab}) are given in terms of the inner components ρ_{ab}.

The symmetries of the theory are encoded in the general diffeomorphism formula (3.26), and are given by ordinary $\mathcal{S}$ diffeomorphisms, local Lorentz rotations (diff.s in the Lorentz directions) and local supersymmetry transformations (diff.s in the fermionic directions). The latter read:

$$\delta_\epsilon V^a = i\bar{\epsilon}\gamma^a\psi, \quad \delta_\epsilon\omega^{ab} = 2\bar{\theta}^{ab}{}_c\epsilon V^c, \quad \delta_\epsilon\psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon \tag{3.35}$$

where $\bar{\theta}^{ab}{}_c$ are the ψV^c components of R^{ab} given in (3.31).

At this juncture, one may wonder whether the action is invariant under supersymmetry transformations: as discussed in a previous subsection, this will be the case if the contraction of dL along fermionic tangent vectors vanishes. Computing this contraction we find that it does vanish provided the rheonomic conditions (3.31)-(3.33) hold ([5], p.685). Thus the action is invariant only on the "partial shell" of the rheonomic conditions, and this invariance does not require the propagation equations.

However, the *closure* of the supersymmetry transformations does require also the propagation equations (3.34) to hold[3] , and therefore the supersymmetry algebra closes only on shell.

The situation is drastically different when auxiliary fields are available to close the supersymmetry algebra off-shell. Then one finds that the fermionic contractions of dL vanish identically without requiring any condition. This can be checked for example in the so-called new minimal $D = 4$, $N = 1$ supergravity (or Sohnius-West model [18]), where the super-Poincaré algebra is enlarged and auxiliary fields (a 1-form and a 2-form) enter the game. In fact in this case the natural algebraic framework is that of free differential algebras [5], a generalization of Lie algebras, whose dual formulation in terms of Cartan-Maurer equations is generalized to contain also p-form fields.

4 Ectoplasmic Integration with Integral Forms

We would like to put in relation the so-called Ectoplasmic technique (Ethereal Integration Theorem) with integral forms. The main point is to prove, by using the integral forms, the so-called "ectoplasmic integration theorem". This theorem states that, given a function L of the superspace (also known as superspace action) on a curved supermanifold $\mathcal{M}$ whose geometry is described by the supervielbein E_M^A (see eq. (2.14)), its integral

$$I_{\mathcal{M}} = \int_{\mathcal{M}} EL\,[d^n x d^m \theta] \tag{4.1}$$

[3]this can be understood by checking Bianchi identities: after enforcing rheonomic constraints on the curvatures, Bianchi identities are not identities anymore, and other conditions may arise for them to hold. These other conditions are the propagation equations.

where E is the superdeterminant of E_M^A, is equal to the following integral

$$I_{\mathcal{S}} = \int_{\mathcal{S}} e\mathcal{D}^m L \mid_{\theta=0} d^n x \tag{4.2}$$

where e is the determinant of the vielbein e_n^a of the bosonic submanifold $\mathcal{S}$ of $\mathcal{M}$ (more precisely, $\mathcal{S}$ is identified with the bosonic submanifold of $\mathcal{M}$ obtained by setting to zero the fermionic coordinates). The expression $\mathcal{D}^m L \mid_{\theta=0}$ denotes the action of a differential operator $\mathcal{D}^m$ on the function L evaluated at $\theta = 0$. $\mathcal{D}^m$ is a symbol denoting a differential operator of order m in the super derivatives. The form of the differential operator is difficult to compute by usual Berezin integration since one has to evaluate the supervielbein E_M^A (at all orders of the θ-expansion), compute its superdeterminant and finally expand the product EL. That procedure leads to the form $\mathcal{D}^m L \mid_{\theta=0}$, where $\mathcal{D}^m$ is a combination of super derivatives, ordinary derivatives and non-derivative terms and the coefficients depend upon curvature, torsion and higher derivative supergravity tensors. The relation between $I_{\mathcal{M}}$ and $I_{\mathcal{S}}$ is easy in the case of flat superspace since there is no superdeterminant to be computed and all supergravity tensors drop out.

In order to circumvent this problem, Gates *et al.* proposed a new method to evaluate $\mathcal{D}^m L \mid_{\theta=0}$. First, one has to select a closed superform (that we will denote by $L^{(n|0)}$) with degree equal to the dimension of the bosonic submanifold. The form must be closed on the complete supermanifold, namely $dL^{(n|0)} = 0$, where d is the differential on the full supermanifold. The closure of the superform (and also its non-exactness) and the existence of a constant tensor imply that a given component of $L^{(n|0)}$ can be written in terms of this tensor times an arbitrary function $\Omega(x, \theta)$ on the supermanifold. All other components of $L^{(n|0)}$ are either vanishing or written as combination of derivatives of the arbitrary function $\Omega(x, \theta)$. The coefficients of those combinations are related again to supergravity tensors. The total result $L^{(n|0)}$ is a superform whose coefficients are given in terms of $\Omega(x, \theta)$, a combination of derivatives and supergravity fields. The **Ethereal conjecture** is that the unknown function $\Omega(x, \theta)$ coincides with the superspace action L evaluated at $\theta = 0$.

The first step is to translate the definitions given by Gates *et al.* in term of integral forms. Then, we show that the integrals of eq. (4.1) and eq. (4.2) can be viewed as integrals of integral forms that can be related via the Poincaré dual. Finally, by changing the Poincaré dual by a different embedding of the bosonic submanifold into the supermanifold, we are able to show that indeed the function $\Omega(x,\theta)$ does coincide with the superspace action L.

4.1 From Ectoplasm to Integral Forms

The integral of $L^{(n|0)}$ (which we will denote in the following with $\omega^{(n|0)}$) on the bosonic submanifold $I_{\mathcal{S}}$ is defined as follows

$$I_{\mathcal{S}} = \int_{\mathcal{S}} i^* \omega \equiv \int_{\mathcal{M}^m} \hat{\omega}^{(n|0)}\Big|_{\theta=0} \tag{4.3}$$

where $\mathcal{S} \equiv \mathcal{M}^n \subset \mathcal{M}^{(n|m)} \equiv \mathcal{M}$ is the bosonic submanifold (obtained by setting to zero the anticommuting variables in the transition functions) and $\hat{\omega}^{(n|0)}\Big|_{\theta=0}$ is obtained from $\omega^{(n|0)}$ by setting to zero both the dependence on θ and on 1-forms $d\theta$

$$i^*\omega = \hat{\omega}^{(n|0)}\Big|_{\theta=0} = \omega_{[i_1 \dots i_n]}(x,0) dx^{i_1} \wedge \dots \wedge dx^{i_n} \tag{4.4}$$

Notice that this superform can be integrated on the bosonic submanifold being a genuine n-form, and if the manifold $\mathcal{S}$ is curved we get

$$I_{\mathcal{S}} = \int_{\mathcal{S}} e \epsilon^{a_1 \dots a_n} \omega_{[a_1 \dots a_n]}(x,0) \tag{4.5}$$

where we have denoted by Latin letters $a_1, \dots, a_n$ the flat indices and e is the determinant of the vielbein e_i^a.

The first crucial observation is that $I_{\mathcal{S}}$ can be also rewritten, following the prescription described in sec. 2, as follows

$$I_{\mathcal{S}} = \int_{\mathcal{M}^{(n|m)}} \omega \wedge \eta_S = \int_{\mathcal{M}^{(n|m)}} \omega^{(n|0)} \wedge \theta^m \delta^m(d\theta) \tag{4.6}$$

where, as usual, we denote by θ^m the product of all fermionic coordinates θ^α and by $\delta^m(d\theta)$ the wedge product of all Dirac delta functions $\delta(d\theta^\alpha)$. Then, the Poincaré dual in this case is $\eta_S = \theta^m \delta^m(d\theta)$ which is the product of "picture changing operators" embedding the bosonic submanifold $\mathcal{S}$ into the supermanifold $\mathcal{M}$ in the simplest way $\theta^1 = \theta^2 = \cdots = 0$.

The integration is performed over the entire supermanifold. A simple computation leads to the original result (4.3). This is clear since integrating over the $d\theta$ has the effect that all components of $\omega^{(n|0)}$ in the $d\theta$ directions are set to zero, leading to $\hat{\omega}^{(n|0)}$. The Berezin integral over the coordinates θ is simplified since the presence of the product θ^m forces us to pick up the first component of $\hat{\omega}^{(n|0)}$, namely $\hat{\omega}^{(n|0)}|_{\theta=0}$ leading to the integral.

4.2 Closure and Susy

The important point about (4.3) is the invariance under supersymmetry. The variation under supersymmetry of $\hat{\omega}^{(n|0)}$ is given by a local translation in superspace

$$\Delta_\epsilon\Big(\hat{\omega}^{(n|0)}\big|_{\theta=0}\Big) = (\Delta_\epsilon \omega^{(n|0)})\big|_{\theta=0} = \epsilon^\alpha\Big(\frac{\partial}{\partial\theta^\alpha} + (\gamma^i\theta)_\alpha\partial_i\Big)\hat{\omega}^{(n|0)}\big|_{\theta=0} = \epsilon^\alpha\frac{\partial}{\partial\theta^\alpha}\hat{\omega}^{(n|0)}\big|_{\theta=0} \tag{4.7}$$

where the first equality is due to the variation of the field components in the expression of $\hat{\omega}^{(n|0)}$ (and therefore it does not matter whether it is computed at $\theta = 0$), the second equality is just the expression of a susy transformation as a supertranslation in superspace. The last term can be rewritten as follows:

$$(\partial_\alpha \hat{\omega}_{i_1\dots i_n})dx^{i_1}\wedge\cdots\wedge dx^{i_n} = -n\,\partial_{[i_1}\omega_{i_2\dots i_n]\alpha}dx^{i_1}\wedge\cdots\wedge dx^{i_n} \tag{4.8}$$

where we have used the closure of the superform $\omega^{(n|0)} = \omega^{(n|0)}_{M_1\dots M_{n+1}}\,dZ^{M_1}\wedge\cdots\wedge dZ^{M_{n+1}}$, (recall (4.4)) which implies

$$\partial_{[M_1}\omega_{M_2\dots M_{n+1})} = 0 \tag{4.9}$$

where the superindices $M_1,\dots,M_{n+1}$ are graded-symmetrized. In this way, the r.h.s. of (4.7) is a derivative w.r.t. to bosonic coordinates x^i and therefore, by integrating over $\mathcal{M}^n$, the integral $I_{\mathcal{S}}$ in (4.3) vanishes. So, the key requirement to guarantee the supersymmetric invariance of $I_{\mathcal{S}}$ is the **closure** of $\omega^{(n|0)}$ as a superform in the full superspace.

Using (4.6), we observe that the integral form $\omega^{(n|0)}\wedge\theta^m\delta^m(d\theta)$ belongs to $\Omega^{(n|m)}$, namely the space of top forms. The closure of $\omega^{(n|0)}$ implies the closure of this integral form, since

$$d\Big(\theta^m\delta^m(d\theta)\Big) = 0.$$

We also notice that if $\omega^{(n|0)}$ belongs to the d-cohomology $H^*(\Omega^{(n|0)})$, so does the integral form, since $\theta^m\delta^m(d\theta)$ is in the d-cohomology $H^*(\Omega^{(0|m)})$ (which are the class of forms with zero form degree and highest picture number, see [19]). However, the converse is not true:

$$d\Big(\omega^{(n|0)}\theta^m\delta^m(d\theta)\Big) = 0 \;\;\Rightarrow\;\; d\omega^{(n|0)} = f_\alpha\theta^\alpha + g_\alpha d\theta^\alpha\,, \tag{4.10}$$

$d\omega^{(n|0)}$ cannot be proportional to $\delta(d\theta)$ since it must be a picture-zero form and f_α must belong to $\Omega^{(n|0)}$ while g_α to $\Omega^{(n-1|0)}$. However, by consistency we have $d\Big(f_\alpha\theta^\alpha + g_\alpha d\theta^\alpha\Big) = 0$, which implies that $df_\alpha = 0$ and $f_\alpha = -dg_\alpha$. This yields $f_\alpha\theta^\alpha + g_\alpha d\theta^\alpha = -d(g_\alpha\theta^\alpha)$ which can be reabsorbed into a redefinition of $\omega^{(n|0)}$, leading to a closed form.

Again we can check the susy invariance of I_S in the form (4.6). Performing the susy transformations leads to

$$\Delta_\epsilon\Big(\omega^{(n|0)}\theta^m\delta^m(d\theta)\Big) = \Big(\Delta_\epsilon\omega^{(n|0)}\Big)\theta^m\delta^m(d\theta) + \omega^{(n|0)}m(\epsilon\theta^{m-1})\delta^m(d\theta) \tag{4.11}$$

where $\Delta_\epsilon\theta^\alpha = \epsilon^\alpha$ and $(\epsilon\theta^{m-1}) \equiv \epsilon_{\alpha_1\dots\alpha_m}\epsilon^{\alpha_1}\theta^{\alpha_2}\dots\theta^{\alpha_m}$. Due to the closure of $\omega^{(n|0)}$ we are in the same situation as above: the partial derivative w.r.t. to θ can be re-expressed as an x-derivative and its integral is then zero. The second piece is zero because we integrate over θ *à la* Berezin and, since $\omega^{(n|0)}$ is computed at $\theta = 0$ (being multiplied by θ^m), the integral vanishes.

4.3 Density Projection Operator

Now, we need to understand the integral obtained in (4.3) in terms of the superform $\omega^{(n|0)}$ by using the closure of it. We adopt the description given by Gates and we follow the same derivation.

Let us now compute the expression in (4.3), namely we compute $\hat\omega^{(n|0)}$ by passing to non-holonomic coordinates as follows

$$\omega_{M_1\dots M_n}dZ^{M_1}\wedge\dots\wedge dZ^{M_n} = \omega_{\Sigma_1\dots\Sigma_n}E^{\Sigma_1}\wedge\dots\wedge E^{\Sigma_n} \longrightarrow \tag{4.12}$$

$$\hat\omega^{(n|0)}\big|_{\theta=0} = \Big(\omega_{\Sigma_1\dots\Sigma_n}E^{\Sigma_1}_{i_1}\dots E^{\Sigma_n}_{i_n}\Big)\big|_{\theta=0}\epsilon^{i_1\dots i_n}d^nx$$

where we denote by Σ the non-holonomic super indices. So, we have:

$$\hat{\omega}^{(n|0)}\big|_{\theta=0} = \Big(\omega_{I_1\dots I_n} E^{I_1}_{i_1}\dots E^{I_n}_{i_n} + \dots + \omega_{A_1\dots A_n} E^{A_1}_{i_1}\dots E^{A_n}_{i_n}\Big)\big|_{\theta=0} \epsilon^{i_1\dots i_n} d^n x \tag{4.13}$$

and $E^I_i\big|_{\theta=0} = e^I_i$ is the bosonic vielbein of the bosonic manifold $\mathcal{M}^n$ while $E^A_i\big|_{\theta=0} = \psi^A_i$ where ψ^A_i is the gravitino field of the supergravity model underlying it. Then,

$$\hat{\omega}^{(n|0)}\big|_{\theta=0} = \Big(\omega_{I_1\dots I_n} e^{I_1}_{i_1}\dots e^{I_n}_{i_n} + \dots + \omega_{A_1\dots A_n} \psi^{A_1}_{i_1}\dots \psi^{A_n}_{i_n}\Big)\big|_{\theta=0} \epsilon^{i_1\dots i_n} d^n x \tag{4.14}$$

Using $\psi^A_I e^I_i = \psi^A_i$, it yields

$$\hat{\omega}^{(n|0)}\big|_{\theta=0} = e\Big(\omega_{I_1\dots I_n} + \dots + \omega_{A_1\dots A_n} \psi^{A_1}_{I_1}\dots \psi^{A_n}_{I_n}\Big)\big|_{\theta=0} \epsilon^{I_1\dots I_n} d^n x \tag{4.15}$$

The requirement that the superform must be closed, $d\omega^{(n|0)} = 0$, expressed in terms of the non-holonomic basis, implies that

$$D_{[\Sigma_1} \omega^{(n|0)}_{\Sigma_2\dots\Sigma_{n+1})} + T^{\Gamma}{}_{[\Sigma_1\Sigma_2} \omega^{(n|0)}_{|\Gamma|\Sigma_3\dots\Sigma_{n+1})} = 0\,, \tag{4.16}$$

where $T^{\Gamma}_{[\Sigma_1\Sigma_2)}$ are the components of the torsion computed in the non-holonomic basis. The form $\omega^{(n|0)}$ is defined up to gauge transformations

$$\delta\omega^{(n|0)}_{[\Sigma_1\dots\Sigma_{n+1})} = D_{[\Sigma_1}\Lambda_{\Sigma_2\dots\Sigma_{n+1})} + T^{\Gamma}{}_{[\Sigma_1\Sigma_2}\Lambda_{|\Gamma|\Sigma_3\dots\Sigma_{n+1})}\,, \tag{4.17}$$

the notation $|\Gamma|$ excludes the index Γ from the graded symmetrization.

The coefficients of the torsion satisfy the Bianchi identities

$$D_{[\Sigma_1} T^{\Gamma}{}_{\Sigma_2\Sigma_3)} + T^{\Lambda}_{(\Sigma_1\Sigma_2} T^{\Gamma}{}_{|\Lambda|\Sigma_3]} = R^{\Gamma}{}_{[\Sigma_1\Sigma_2\Sigma_3)} \tag{4.18}$$

where $R^{\Gamma}_{[\Sigma_1\Sigma_2\Sigma_3)}$ are the components of the curvature. We also recall that $[D_\Sigma, D_\Gamma] = T^{\Lambda}{}_{\Sigma\Gamma} D_\Lambda + R^I{}_{J,\Sigma\Gamma} M_I{}^J$ where $M_I{}^J$ are the Lorentz generators.

The Bianchi identities become non-trivial equations when some of the components $\omega_{\Sigma_1\dots\Sigma_n}$ are set to a given value. For example by choosing some of $\omega_{I_1\dots I_p A_{p+1}\dots A_n}$ with spinorial indices equal to zero, such that the next one

$$\omega_{I_1\dots I_{p-1}A_p\dots A_n} = \Omega(x,\theta)\, t_{I_1\dots I_{p-1}A_p\dots A_n}$$

can be set equal a constant tensor $t_{I_1\ldots I_{p-1}A_p\ldots A_n}$ (combination of Dirac gamma matrices and invariant tensors) where $\Omega(x,\theta)$ is a superfield. The other components can be fixed by solving the Bianchi identities and it is easy to show that

$$\omega_{I_1\ldots I_n} = f^{A_1\ldots A_m}_{I_1\ldots I_n} D_{A_1}\ldots D_{A_m}\Omega + \ldots$$

where the dots are other tensors constructed out of curvature and derivative of it. The coefficients $f^{A_1\ldots A_m}_{I_1\ldots I_n}$ are combinations of constant tensors. Inserting the solution of the Bianchi identities into the cumulative expression (4.15) one gets an expression of the density projector $\mathcal{D}^m$

$$\begin{aligned}\hat{\omega}^{(n|0)}\big|_{\theta=0} &= e\Big(f^{A_1\ldots A_m}_{I_1\ldots I_n} D_{A_1}\ldots D_{A_m}\Omega + \cdots + f_{A_1\ldots A_n}\psi^{A_1}_{I_1}\ldots\psi^{A_n}_{I_n}\Omega\Big)\big|_{\theta=0}\epsilon^{I_1\ldots I_n}d^nx \\ &\equiv e\mathcal{D}^m\Omega\big|_{\theta=0}d^nx \qquad (4.19)\end{aligned}$$

The exponent m denotes the maximal number of spinorial derivative. This conclude this review part on the density projection operator and we are finally in position to present a proof of the theorem.

4.4 Proof of the Ectoplasmic Integration Theorem

At this point we need to study the other side of the Ectoplasmic Integration Theorem, namely we have to describe the integral $I_{\mathcal{M}}$ in terms of a superform. For that we recall eq. (2.14): the integral of a top integral form $\omega^{(n|m)}$ reads

$$\begin{aligned} I_{\mathcal{M}} &\equiv \int_{\mathcal{M}^{(n|m)}} \omega^{(n|m)} = \\ &= \int_{\mathcal{M}^{(n|m)}} \omega^{(n|m)}_{[I_1\ldots I_n][A_1\ldots A_m]}(x,\theta)E^{I_1}\wedge\ldots\wedge E^{I_n} \wedge\delta(E^{A_1})\wedge\cdots\wedge\delta(E^{A_m}) = \\ &= \epsilon^{I_1\ldots I_n}\epsilon^{A_1\ldots A_m}\int_{\mathcal{M}^{(n|m)}} E\omega^{(n|m)}_{[I_1\ldots I_n][A_1\ldots A_m]}(x,\theta)d^nx\delta^m(d\theta) = \\ &= \int E\,\epsilon^{I_1\ldots I_n}\epsilon^{A_1\ldots A_m}\omega^{(n|m)}_{[I_1\ldots I_n][A_1\ldots A_m]}(x,\theta) \qquad (4.20)\end{aligned}$$

where in the last step we have stripped out the integration over the 1-forms d^nx and over the Dirac delta's $\delta^m(d\theta)$. We are then left with the integral over the bosonic coordinates and the Berezin integral over the fermionic coordinates. The latter can be performed by taking the derivatives with respect to the fermionic coordinates of the product $E\omega$ where $\omega = \epsilon^{I_1 \dots I_n} \epsilon^{A_1 \dots A_m} \omega^{(n|m)}_{[I_1 \dots I_n][A_1 \dots A_m]}(x, \theta)$.

The main point here is the following: the superfield $\Omega(x, \theta)$ appearing in the expression of $\hat{\omega}^{(n|0)}$ – obtained by "integrating" the Bianchi identities with some constraints – has apparently nothing to do with the superfield $\epsilon^{A_1 \dots A_m} \omega^{(n|m)}_{[I_1 \dots I_n][A_1 \dots A_m]}(x, \theta)$ appearing in (4.20). Thus, to prove the ectoplasmic integration formula one has to verify that they indeed coincide. In order to do that we observe that the superform $\hat{\omega}^{(n|0)}$ belongs to the space $\Omega^{(n|0)}$ which has vanishing picture number. Thus, in order to integrate it we need to change its picture by inserting Picture Changing Operators of the form

$$Y = M^A(x, \theta) \delta(\psi^A) + N^A(x, \theta, dx) \delta'(\psi^A) + \dots \tag{4.21}$$

where ψ^A are the gravitino superfields (also denoted by E^A in the present work) and the dots stand for terms with higher derivative of Dirac delta functions. The functions $M^A(x, \theta)$, $N^A(x, \theta, dx), \dots$ are needed to impose $dY = 0$.

Therefore, we can construct a top integral form from $\hat{\omega}^{(n|0)}$ as

$$\omega^{(n|m)} = \hat{\omega}^{(n|0)} \epsilon_{A_1 \dots A_m} \Phi^{A_1} \dots \Phi^{A_m} \delta^m(\psi) \,, \tag{4.22}$$

which has the correct picture number and the correct form number to be integrated on $\mathcal{M}^{(n|m)}$. The symbol Φ^{A_i} denotes a function of θ's such that $d\Phi^A \delta(\psi^A) = 0$, giving a new arbitrary expression for the picture changing operator. Then it is easy to show that, by integrating by parts (using the superderivative appearing in the coefficients of $\hat{\omega}^{(n|0)}$), the integral form obtained is proportional to the superfield $\Omega(x, \theta)$, and being the integral top-form sections of a one-dimensional line bundle (the Berezinian), we conclude that the superfields appearing in the expansion of $\hat{\omega}^{(n|0)}$ and of $\epsilon^{A_1 \dots A_m} \omega^{(n|m)}_{[I_1 \dots I_n][A_1 \dots A_m]}(x, \theta)$ are simply proportional and they can be chosen to be the same.

In other words, this corresponds to modifying the picture changing operators, but remaining in the same cohomology class. That implies that the two integrals are indeed equal since the delta functions appearing in Y soak up the gravitinos appearing in the density projection operator $\mathcal{D}^m$.

Let summarize the main steps of the proof. We start by showing that both the integral $I_{\mathcal{M}}$ and $I_{\mathcal{S}}$ can be written in terms of integral forms. The former is viewed as an integral of a density which is the coefficient of a top form of $\Omega^{(n|m)}$. The second integral $I_{\mathcal{S}}$ is converted into an integral of an integral form by introducing a suitable picture changing operator $\theta^m \delta^m(d\theta)$. However, the choice of the picture changing operator is arbitrary and therefore it can be changed into the new form (4.21), such that the gravitons ψ^A appear as arguments of the delta functions. Finally, the computation of the integral $I_{\mathcal{S}}$ projects out all components of the combination except a superfield $\Omega(x,\theta)$ which can be chosen to be equal to the density of $I_{\mathcal{M}}$.

Acknowledgments

We thank L. Andrianopoli, R. D'Auria and M. Trigiante for useful discussions concerning integral forms and their use in supergravity theories.

5 Appendix A.

We do not wish here to give an exhaustive and rigorous treatment of integral forms. A systematic exposition of the matter can be found in the references quoted in Section 2 .

As we said in section 2, the problem is that we can build the space Ω^k of k-superforms out of basic 1-superforms $d\theta^i$ and dx^i and their wedge products, however the products between the $d\theta^i$ are necessarily commutative, since the θ^i's are odd variables. Therefore, together with a differential operator d, the spaces Ω^k form a differential complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \ldots \xrightarrow{d} \Omega^n \xrightarrow{d} \ldots \tag{5.1}$$

which is bounded from below, but not from above. In particular there is no notion of a top form to be integrated on the superspace $\mathbb{R}^{p|q}$.

The space of integral forms is obtained by adding to the usual space of superforms a new set of basic "forms" $\delta(d\theta)$, together with the derivatives $\delta^{(p)}(d\theta)$, (derivatives of $\delta(d\theta)$ must be introduced for studying the behaviour of the symbol $\delta(d\theta)$ under sheaf morphisms i.e. coordinate changes, see below) that satisfies certain formal properties.

These properties are motivated and can be deduced from the following heuristic approach.

In analogy with usual distributions acting on the space of smooth functions, we think of $\delta(d\theta)$ as an operator acting on the space of superforms as the usual Dirac's delta "measure" (more appropriately one should refer to the theory of de Rham's currents [17]), but this matter will not be pursued further). We can write this as

$$\langle f(d\theta), \delta(d\theta) \rangle = f(0),$$

where f is a superform. This means that $\delta(d\theta)$ kills all monomials in the superform f which contain the term $d\theta$. The derivatives $\delta^{(n)}(d\theta)$ satisfy

$$\left\langle f(d\theta), \delta^{(n)}(d\theta) \right\rangle = - \left\langle f'(d\theta), \delta^{(n-1)}(d\theta) \right\rangle = (-1)^n f^{(n)}(0),$$

like the derivatives of the usual Dirac δ measure.

Moreover we can consider objects such as $g(d\theta)\delta(d\theta)$, which act by first multiplying by g then applying $\delta(d\theta)$ (in analogy with a measure of type $g(x)\delta(x)$), and so on. The wedge products (when defined, note that we cannot in general multiply distributions of the same coordinates) among these objects satisfy some simple relations such as (we will omit the symbol $\wedge$ of the wedge product):

$$dx^I dx^J = -dx^J dx^I \,, \quad dx^I d\theta^j = d\theta^j dx^I \,,$$

$$d\theta^i d\theta^j = d\theta^j d\theta^i \,, \quad \delta(d\theta)\delta(d\theta') = -\delta(d\theta')\delta(d\theta),$$

$$d\theta\delta(d\theta) = 0 \,, \quad d\theta\delta'(d\theta) = -\delta(d\theta).$$

The most noticeable relation is the unfamiliar minus sign appearing in $\delta(d\theta)\delta(d\theta') = -\delta(d\theta')\delta(d\theta)$ (indeed this is natural if we interpret the delta "forms" as de Rham's currents) but can be also easily deduced from the above heuristic approach. To prove this formula we recall the usual transformation property of the usual Dirac's delta function

$$\delta(ax+by)\delta(cx+dy) = \frac{1}{\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|}\delta(x)\delta(y)$$

for $x, y \in \mathbb{R}$. We note now that in the case under consideration the absolute value must be dropped in the formula above, because the scaling properties of $\delta(d\theta)$ are driven by $\int \delta\,(d\theta)\,[d\,(d\theta)] = 1$. Under a rescaling $\theta \to \lambda\theta$ ($\lambda \in \mathbb{R}$) we must have $\int \delta\,(\lambda d\theta)\,[d\,(\lambda d\theta)] = 1$, but now $d\theta$ is bosonic and hence the fermionic $d\,(d\theta)$ scales as $d\,(\lambda d\theta) = \lambda d\,(d\theta)$. Hence setting $a = 0, b = 1, c = 1$ and $d = 1$, in the correct formula:

$$\delta(ad\theta + bd\theta')\delta(cd\theta + dd\theta') = \frac{1}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \delta(d\theta)\delta(d\theta')$$

the anticommutation property of Dirac's delta function of $d\theta$'s follows.

An interesting and important consequence of this procedure is the existence of *negative degree forms*, which are those that by multiplication reduce the degree of a forms (e.g. $\delta'(d\theta)$ has degree -1).

We introduce also the *picture number* by counting the number of delta functions (and their derivatives) and we denote by $\Omega^{r|s}$ the space of r-forms with picture s. For example, in the case of $\mathbf{R}^{p|q}$, the integral form

$$dx^{[K_1} \dots dx^{K_l]} d\theta^{(i_{l+1}} \dots d\theta^{i_r)} \delta(d\theta^{[i_{r+1}}) \dots \delta(d\theta^{i_{r+s}]})$$

is an r-from with picture s. All indices K_i are antisymmetrized, while the first $r - l$ indices are symmetrized and the last s are antisymmetrized. By adding derivatives of delta forms $\delta^{(p)}(d\theta)$, even negative form-degree can be considered, e.g. a form of the type:

$$\delta^{(n_1)}(d\theta^{i_1}) \dots \delta^{(n_s)}(d\theta^{i_s})$$

is a $-(n_1 + \dots + n_s)$-form with picture s. Clearly $\Omega^{k|0}$ is just the space Ω^k of superforms, for $k \geq 0$.

Integral forms form a new complex as follows

$$\dots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \dots \xrightarrow{d} \Omega^{(p|q)} \xrightarrow{d} 0 \tag{5.2}$$

We now briefly discuss how these forms behave under change of coordinates, i.e. under sheaf morphisms. For generic morphisms it is necessary to work with infinite formal sums in $\Omega^{r|s}$ as the following example clearly shows.

Suppose $(\tilde{\theta}^1) = \theta^1 + \theta^2$, $(\tilde{\theta}^2) = \theta^2$ be the odd part of a morphism. We want to compute

$$(\delta\left(d\tilde{\theta}^1\right)) = \delta\left(d\theta^1 + d\theta^2\right)$$

in terms of the above forms. We can formally expand in series about, for example, $d\theta^1$:

$$\delta\left(d\theta^1 + d\theta^2\right) = \sum_j \frac{\left(d\theta^2\right)^j}{j!}\delta^{(j)}(d\theta^1)$$

Recall that any usual superform is a polynomial in the $d\theta$, therefore only a finite number of terms really matter in the above sum, when we apply it to a superform. Indeed, applying the formulae above, we have for example,

$$\left\langle (d\theta^1)^k, \sum_j \frac{\left(d\theta^2\right)^j}{j!}\delta^{(j)}(d\theta^1)\right\rangle = (-1)^k (d\theta^2)^k$$

Notice that this is equivalent to the effect of replacing $d\theta^1$ with $-d\theta^2$. We could have also interchanged the role of θ^1 and θ^2 and the result would be to replace $d\theta^2$ with $-d\theta^1$. Both procedures correspond precisely to the action we expect when we apply the $\delta\left(d\theta^1 + d\theta^2\right)$ Dirac measure. We will not enter into more detailed treatment of other types of morphisms.

References

[1] S. J. Gates, Jr., "Ectoplasm has no topology: The Prelude," In *Dubna 1997, Supersymmetries and quantum symmetries* 46-57 [hep-th/9709104].

[2] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt-Wehlau and W. Siegel, "Component actions from curved superspace: Normal coordinates and ectoplasm," Phys. Lett. B **421**, 203 (1998) [hep-th/9711151].

[3] S. J. Gates, Jr., "Ectoplasm has no topology," Nucl. Phys. B **541**, 615 (1999) [hep-th/9809056].

[4] S. J. Gates, Jr. and G. Tartaglino-Mazzucchelli, "Ectoplasm and Superspace Integration Measure for 2D Supergravity with Four Spinorial Supercurrents," J. Phys. A **43**, 095401 (2010) [arXiv:0907.5264 [hep-th]].

[5] L. Castellani, R. D'Auria and P. Fré, "Supergravity and superstrings: A Geometric perspective," in 3 vol.s, Singapore: World Scientific (1991) 1375-2162;

[6] L. Castellani, P. Fré and P. van Nieuwenhuizen, "A Review of the Group Manifold Approach and Its Application to Conformal Supergravity," Annals Phys. **136**, 398 (1981).

[7] R. D'Auria, P. Fré and T. Regge, "Consistent Supergravity in Six-dimensions Without Action Invariance," Phys. Lett. B **128**, 44 (1983).

[8] L. Castellani, "Chiral $D = 10$, $N = 2$ Supergravity on the Group Manifold. 1. Free Differential Algebra and Solution of Bianchi Identities," Nucl. Phys. B **294**, 877 (1987). L. Castellani and I. Pesando, "The Complete superspace action of chiral D = 10, N=2 supergravity," Int. J. Mod. Phys. A **8**, 1125 (1993).

[9] E. Witten, "Notes On Supermanifolds and Integration," arXiv:1209.2199 [hep-th].

[10] A. Belopolsky, New geometrical approach to superstrings, [arXiv:hep-th/9703183]. A. Belopolsky, Picture changing operators in supergeometry and superstring theory, [arXiv:hep-th/9706033].

[11] T. Voronov and A. Zorich: Integral transformations of pseudodifferential forms, Usp. Mat. Nauk, 41 (1986) 167-168; T. Voronov and A. Zorich: Complex of forms on a supermanifold, Funktsional. Anal. i Prilozhen., 20 (1986) 58-65; T. Voronov and A. Zorich: Theory of bordisms and homotopy properties of supermanifolds, Funktsional. Anal. i Prilozhen., 21 (1987) 77-78; T. Voronov and A. Zorich: Cohomology of supermanifolds, and integral geometry, Soviet Math. Dokl., 37 (1988) 96-101.

[12] P. A. Grassi and G. Policastro, "Super-Chern-Simons theory as superstring theory," hep-th/0412272.

[13] P.A. Grassi and M. Marescotti, Integration of Superforms and Super-Thom Class [hep-th/0712.2600v2].

[14] D. Friedan, E. J. Martinec and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. B **271** (1986) 93.

[15] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **0409** (2004) 047 [hep-th/0406055].

[16] R. Bott and L. W. Tu "Differential Form in Algebraic Topology", Springer-Verlag, 1990.

[17] P. Griffiths and J. Harris, "Principles of Algebraic Geometry", Wiley Classics Library, 1994.

[18] M. F. Sohnius and P. C. West, "An Alternative Minimal Off-Shell Version of N=1 Supergravity," Phys. Lett. B **105**, 353 (1981).

[19] R. Catenacci, M. Debernardi, P. A. Grassi and D. Matessi, "Cěch and de Rham Cohomology of Integral Forms," J. Geom. Phys. **62** (2012) 890 [arXiv:1003.2506 [math-ph]].