RIVISTA DEL NUOVO CIMENTO Vol. 29, N. 1 2006
DOI 10.1393/ncr/i2006-10008-2

The legacy of pseudospheres: From geometry to physics(*)

B. Bertotti([1]), R. Catenacci([2]) and C. Dappiaggi([1])

([1]) *Dipartimento di Fisica Nucleare e Teorica, Università di Pavia & INFN, Sezione di Pavia via A. Bassi 6, I-27100 Pavia, Italy*

([2]) *Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale & INFN, Sezione di Alessandria - Via Bellini 25/G, I-15100 Alessandria, Italy*

(ricevuto il 3 Luglio 2006)

Summary. — The definition of a sphere depends on the definition of distance in the embedding three-dimensional space; the classification is straightforward, but should be clearly understood to appreciate the richness and the variety of this concept in geometry and mathematical physics. There are essentially three *loci* of points with the same "distance" from a centre O: the ordinary sphere S_2, the single-sheet hyperboloid AdS_2 and the two-sheet hyperboloid $\mathbb{H}$. The extraordinary fertility of AdS_2 and $\mathbb{H}$ began in the XIX century, when the Italian mathematician E. Beltrami discovered an intrinsic metric on a disk on the plane which is a canonical realization of two-dimensional hyperbolic geometry with constant curvature, but did not recognize that it is just a stereographic projection of $\mathbb{H}$. Subsequently, AdS_2 became an important geometrical building block in special relativity, in cosmology and, in 1959, in a solution of Einstein-Maxwell's field equations corresponding to a uniform electromagnetic field. This space-time, here called BR after Bertotti and Robinson's papers of 1959, consists in the combination of two (generalized) spheres and can be obtained in a purely geometric way. The BR geometry has played a relevant role in the search for new unifying fundamental laws, in particular in very high-energy physics, and has provided examples to test and exemplify new physical principles. In sect. **4** we briefly outline three general areas. The first area is the extension of a classical field theory to the complex domain (in particular, in relation to quantum gravity). Ideally, the most interesting complexification of a Riemannian manifold consists in endowing it with a Kählerian structure; it turns out that, while this is possible for a definite signature in many cases, in space-time a Kählerian manifold is obtained only if it is just the BR solution (or, trivially, if it is flat). The other two areas are: the exploration of the quantum properties of the horizon of a black hole and the holographic principle; string theory, and the fundamental role of a scalar field, the *dilaton*. We give, and clarify, examples in which the BR solution has been applied.

PACS 04.20.-q – Classical general relativity.
PACS 02.40.-k – Geometry, differential geometry, and topology.
PACS 04.60.-m – Quantum gravity.

(*) Much material of this paper, with more emphasis on the history, will appear in the Proceedings of the meeting *Un grande matematico dell '800: omaggio a Eugenio Beltrami*, held in Milan on 14-15 October 2004. They will be published by the Istituto Lombardo di Scienze e Lettere, via Borgonuovo 25, I-20121 Milan (Italy).

1. – Pseudospheres

The requirement of a positive metric must be abandoned. In a (real) flat embedding space $(\mathbb{R}^3, \mathrm{d}\sigma^2)$ with metric

$$\mathrm{d}\sigma^2 = \epsilon_\xi \mathrm{d}\xi^2 + \epsilon_\eta \mathrm{d}\eta^2 + \epsilon_\zeta \mathrm{d}\zeta^2, \tag{1}$$

consider the *fundamental quadric* [1]

$$\epsilon_\xi \xi^2 + \epsilon_\eta \eta^2 + \epsilon_\zeta \zeta^2 = \epsilon R^2. \tag{2}$$

$\epsilon_\xi, \epsilon_\eta, \epsilon_\zeta$ and ϵ take the values $+1$ or -1. Each case will be denoted with the symbol $(sgn\,\epsilon_\xi\ sgn\,\epsilon_\eta\ sgn\,\epsilon_\zeta,\ sgn\,\epsilon)$; since the two cases $(+++,-)$ and $(---,+)$ are forbidden, with a permutation of the embedding coordinates, we can choose $\epsilon_\zeta = \epsilon$.

The manifolds (2) with the induced metric (1), are complete surfaces in $\left(\mathbb{R}^3, \mathrm{d}\sigma^2\right)$ with constant scalar curvature $\mathcal{R} = 2\epsilon/R^2 \neq 0$; when simply connected, they are the only ones with these properties. Their geodesics are the intersections with planes through the origin O. Note that ϵ is the sign of the square of the normal vector $\mathbf{n}$, and can be directly found taking a particular point, *e. g.*, P. When viewed as subsets of the Euclidean $\mathbb{R}^3$, there are three topologically distinct kinds (see fig. 1). With a definite metric ($\epsilon_\zeta = \epsilon_\eta = \epsilon_\zeta = 1$), ϵ must also be 1; this is the usual sphere S^2 (Note that it is the only one that can be embedded with the induced metric in the Euclidean $\mathbb{R}^3$.) We have also the "negative sphere" $(- - -, -)$. When the metric (1) is indefinite, the fundamental quadric is a hyperboloid. A double-sheet hyperboloid can be realized either as $(- - +, +)$ or $(+ + -, -)$; in both cases the induced metric ds^2 is also definite, $(--)$ in the first case and $(++)$ in the second. $(- + -, -)$ and $(+ - +, +)$ are single-sheet hyperboloids; at every point there are two null lines, the intersections with the tangent plane. The induced metric is indefinite, $(-+)$ in the first case and $(+-)$ in the second. If the coordinates ξ and η are interchanged the same is true for $(-++,+)$ and $(+--,-)$.

All these metrics appear in pairs, obviously equivalent from the point of view of the geometry of the geodesics; they differ only in the sign of scalar curvature $\mathcal{R}$. In all cases they are the locus of the points at the same σ-distance from the origin O and, therefore, they possess the three-dimensional isometry group which leaves the embedding metric invariant. Their Riemann and Ricci tensors are

$$R_{ijkh} = \epsilon/R^2(g_{ik}g_{jh} - g_{ih}g_{jk}),\ \ R_{ij} = \epsilon/R^2 g_{ij},\ \ \mathcal{R} = g^{ik}g^{jl}R_{ijkl} = 2\epsilon/R^2\ \ , \tag{3}$$

where Latin indexes denote two intrinsic coordinates.

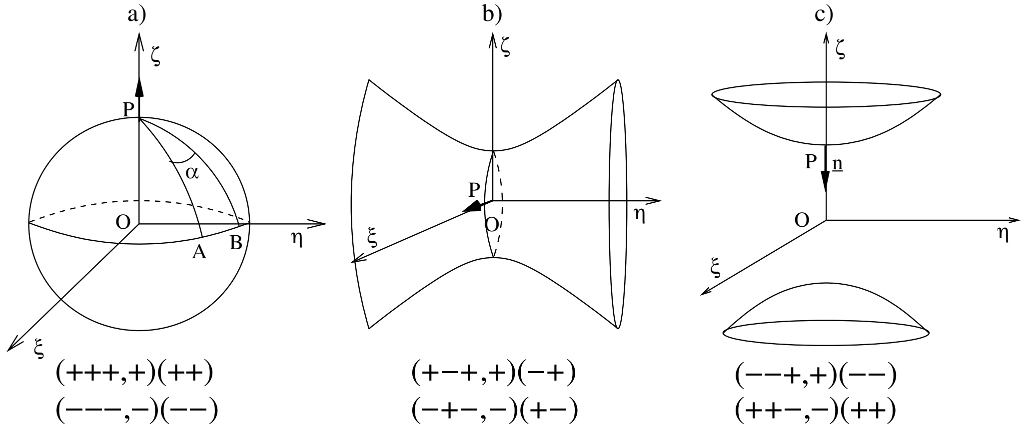

Fig. 1. – Topological classification of the fundamental quadrics (2) in $\mathbb{R}^3$, with constant scalar curvature $\mathcal{R} = 2\epsilon/R^2$. a) The sphere S^2. The triangle ABP illustrates Gauss' *Theorema egregium* (9). b) The single-sheet hyperboloid. If viewed as a space-time, it is called dS_2 (for "de Sitter" in two dimensions) or AdS_2 (for "Anti de Sitter") according to whether its axis is time-like or space-like (see table below). c) The two-sheet hyperboloid $\mathbb{H}$. As explained in the text, each of these topologically different cases can be realized metrically in two ways. The direction of the vector $\mathbf{n}$ corresponds to one of them. Each realization is denoted by the signs of the four ϵ's; the second bracket gives the intrinsic signature.

These two-dimensional metrics can be written in a "conformally flat" form:

$$\mathrm{d}s^2 = \frac{\epsilon_\xi \mathrm{d}u^2 + \epsilon_\eta \mathrm{d}v^2}{\left(1 + \frac{\epsilon}{4R^2}\left(\epsilon_\xi u^2 + \epsilon_\eta v^2\right)\right)^2}, \tag{4}$$

where

$$u = \frac{2\xi}{1 + \left(1 - \frac{\epsilon}{R^2}\left(\epsilon_\xi \xi^2 + \epsilon_\eta \eta^2\right)\right)^{1/2}}, \qquad v = \frac{2\eta}{1 + \left(1 - \frac{\epsilon}{R^2}\left(\epsilon_\xi \xi^2 + \epsilon_\eta \eta^2\right)\right)^{1/2}},$$

are coordinates in a "plane" with metric $\mathrm{d}l^2 = \epsilon_\xi \mathrm{d}u^2 + \epsilon_\eta \mathrm{d}v^2$. The pseudospheres can also be mapped with hyperbolic variables ($-\infty < \chi < \infty, 0 \leq \phi < 2\pi$). For the single-sheet case the intrinsic metrics are

$$\mathrm{d}s^2 = \epsilon R^2 \left(-\mathrm{d}\chi^2 + \cosh^2\chi \mathrm{d}\phi^2\right); \tag{5}$$

for the two-sheet case

$$\mathrm{d}s^2 = \epsilon R^2 \left(-\mathrm{d}\chi^2 - \sinh^2\chi \mathrm{d}\phi^2\right). \tag{6}$$

The two-dimensional surfaces generated by fundamental quadrics (fig. 1) can be combined to construct Riemannian manifolds with an even number of dimensions, endowed with interesting symmetries. For space-time, however, the correct number of time-like (one) and space-like (three) intrinsic coordinates must be ensured, narrowing down the choice. Consider a four-dimensional manifold $\Sigma_4 = \Sigma_+ \times \Sigma_-$, topological product of two two-dimensional manifolds, Σ_+ with coordinates (x_0, x_1), and Σ_-, with coordinates (x_2, x_3); a tensor is termed *decomposable* if a) its components with mixed indexes vanish and b) its components relative to Σ_+ (Σ_-) depend only on the respective coordinates. A Riemannian manifold Σ_4 with this property is decomposable if its metric tensor is decomposable([1]):

$$g_{\mu\nu} = g_{(+)\mu\nu} + g_{(-)\mu\nu}. \tag{7}$$

Its Ricci tensor has the same property [2]:

$$R_{\mu\nu} = (\mathcal{R}_+/2) g_{(+)\mu\nu} + (\mathcal{R}_-/2) g_{(-)\mu\nu}. \tag{8}$$

As discussed in sect. **4**, they directly lead to the Bertotti-Robinson solution of Einstein-Maxwell field equations [3, 4].

Obviously the fundamental quadrics can be generalized to n-dimensional manifolds of constant scalar curvature $\mathcal{R}$ embedded in a flat space $\mathbb{R}^{n+1}$.

2. – Geometry and physics of pseudospheres

In this section of historical interest we briefly recall some relevant applications of fundamental quadrics in geometry and physics. They have played a very important role in the development of non-Euclidean geometry (see, *e.g.*, [5-7]) and provided a striking and direct realization of trigonometry and Gauss' *Theorema egregium*

$$\alpha + \beta + \gamma - \pi = \varepsilon = KA, \tag{9}$$

which connects the sum of the internal angles of a geodetic triangle with its area A. ε is called the *excess angle* (not to be confused with ϵ). (A word of caution about the concept of the Gaussian curvature K is in order here. In a positive definite metric $K = 1/(R_M R_m)$ is defined in terms of the radii of curvature of the sections of the surface with the family of planes through its normal at a given point; R_M and R_m are, respectively, the upper and lower bounds of such radii. But in the indefinite case, a careful definition of the radius of curvature and its sign is required; more generally, in non-Euclidean geometry trigonometry should be appropriately generalized (see [8]). We skirt this complication and confine ourselves to concepts of intrinsic Riemannian geometry, without using angles at all.)

In 1868 the Italian mathematician Beltrami investigated a realization of hyperbolic geometry (with $K < 0$) in an open disk in a plane [9]. It is intrinsically defined by the metric

$$\mathrm{d}s^2_{\mathrm{B}} = R^2 \frac{(R^2 - v^2)\mathrm{d}u^2 + 2uv\,\mathrm{d}u\,\mathrm{d}v + (R^2 - u^2)\mathrm{d}v^2}{(R^2 - u^2 - v^2)^2}. \tag{10}$$

The variables u and v are confined to the open disk $u^2 + v^2 < R^2$, the points corresponding to its boundary being at infinity. In this representation geodesics are just straight segments in the (u, v)-plane, and the model is geodesically complete. Consider now the

([1]) In this paper we adopt Einstein's summation rule. Greek indices run from 0 to 3.

two-sheet quadric labelled c) in fig. 1, whose metric can be written as (6); it can easily be shown that the stereographic projection of one sheet from the origin O onto the tangent plane through P reproduces Beltrami's metric $\mathrm{d}s_{\mathrm{B}}^2$ (10) or $-\mathrm{d}s_{\mathrm{B}}^2$. Explicitly, the required mapping $(\chi, \phi) \rightarrow (u, v)$ is

$$u = R \tanh \chi \cos \phi, \quad v = R \tanh \chi \sin \phi.$$

The points on the circumference $u^2 + v^2 = R^2$ correspond to the points at infinity on the upper null cone $\xi^2 + \eta^2 = \zeta^2$ $(\zeta > 0)$. A complete representation can be obtained by formally identifying the antipodal points (ξ, η, ζ) and $(-\xi, -\eta, -\zeta)$. Beltrami solution has been extensively investigated from the projective point of view, allowing recovery of its trigonometric properties; we did not find in the literature this simple and clarifying interpretation, which throws full light also on its topological properties.

The single-sheet hyperboloid b) $(- + -, -)$ (fig. 1) also provides a realization of hyperbolic geometry. Denoting with (u, v) the ξ and η coordinates in the projection plane $\zeta = R$, the projection P_0 of a point $P = R(\cosh \chi \cos \phi, \sinh \chi, \cosh \chi \sin \phi)$ on the hyperboloid has $u = R \cot \phi, v = R \tanh \chi \csc \phi$, leading to the metric

$$\mathrm{d}s_{\mathrm{B2}}^2 = R^2 \frac{(v^2 - R^2)\mathrm{d}u^2 - 2uv\mathrm{d}u\mathrm{d}v + (u^2 + R^2)\mathrm{d}v^2}{(R^2 + u^2 - v^2)^2},$$

to be compared with the original Beltrami metric (10). The straight lines in the (u, v)-plane are still geodesics, but Beltrami's disk is replaced by $v^2 - u^2 < R^2$.

The four-dimensional generalization of the single-sheet hyperboloid b) has played important roles in physics. In special relativity the energy E and the momentum $\mathbf{p} = (p_x, p_y, p_z)$ of a free particle with rest-mass m fulfill([2])

$$E^2 - p_x^2 - p_y^2 - p_z^2 = m^2. \tag{11}$$

This is the extension of the two-sheet hyperboloid $(- - +, +)$ to four dimensions; the two sheets correspond to particles and antiparticles, respectively. The fact that this, *de facto*, introduces non-Euclidean geometry has been fully recognized first by Varićak (1865-1942) in 1912 [10] (see also [11]), in particular in relation to the relativistic law of addition of velocities.

De Sitter's four-dimensional manifold dS_4 generalizes the single-sheet hyperboloid $(- + -, -)$; in a flat five-dimensional space $\mathbb{R}^5$ with metric

$$\mathrm{d}\sigma^2 = -\mathrm{d}\zeta^2 + \mathrm{d}\eta^2 - \mathrm{d}\xi^2 - \mathrm{d}v^2 - \mathrm{d}\omega^2, \tag{12}$$

it is defined by the fundamental quadric

$$-\zeta^2 + \eta^2 - \xi^2 - v^2 - \omega^2 = -R^2. \tag{13}$$

The induced metric has the correct space-time signature $(+ - --)$. This manifold is invariant under the 10-parameter isometry group consisting of the linear transformations

([2]) The signature $(+---)$ is assumed for space-time, at variance with the usual choice $(-+++)$ in quantum field theory; the velocity of light is unity.

in $\mathbb{R}^5$ which leave (12) invariant; when restricted to the fundamental quadric, in the limit $R \to \infty$ it corresponds to the Lorentz group combined with the four-parameter translational group (the Poincaré group $SO(3,1) \ltimes T^4$)([3]). All points of dS_4 are equivalent.

Just like the sphere is the "perfect" surface in ordinary space (and for this reason it played such an important role in Hellenistic and Tolomean cosmologies), we have here the *perfect space-time*: an obvious cosmological model in which there is no origin, nor history. This is the *Steady State Theory* (see the pioneer papers [12, 13] and [14]), now abandoned in face of compelling evidence for evolution (see [15] and references therein). Its symmetry properties guarantee that De Sitter's world dS_4 is a solution of

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad (\Lambda = 1/R^2). \tag{14}$$

If in the fundamental quadric (13) η and ζ are exchanged, we obtain the four-dimensional anti-de Sitter's Universe, in which the time t is cyclic: every physical field is a periodic function of t. With obvious generalizations, dS_d is the de Sitter Universe in d dimensions (embedded in $\mathbb{R}^{d+1}$), one of which is time-like and open; similarly, in the AdS_d universe time is cyclic.

3. – Bertotti-Robinson solution and the *Already Unified Theory*

With the geometrical tools discussed in sect. **1** and in the framework of Maxwell-Einstein theory, it is easy to get the physical interpretation of the topological product of two fundamental quadrics. A (covariantly) constant and not null([4]) electromagnetic field determines the traceless energy momentum tensor $\tau_{\mu\nu}$ in terms of its energy density ρ (with dimensions L^{-2}); if, in addition, we have a cosmological constant Λ, the field equations read

$$R_{\mu\nu} = \tau_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{15}$$

The field laws of general relativity can be expressed as second-order partial differential equations for the metric components and are usually solved in a specific coordinate system. This has two serious drawbacks: the question of geodesic completeness and of topology remains beyond reach; the geometric identity of the solution remains hidden. In fact, two different metric fields, expressed as functions of some coordinates, may correspond to the same global geometry. On the contrary, the construction of the BR solution, both in this paper and in the original article [16,3], is geometric and global, hence free of these drawbacks. Robinson has investigated the case with vanishing cosmological constant [4], while earlier Kasner [17] found the solution without electromagnetic field.

In the wake of Einstein and Schrödinger's attempts to build a unified theory of gravity and electromagnetism, the "Already Unified Theory" [18, 19] was developed, based upon the principle that *geometry is all*; Einstein-Maxwell equations can be formulated in a purely geometric form, which allows to extract the electromagnetic field from its imprint on the metric. There are three questions:

1. Does a solution of Maxwell-Einstein (15) fulfill purely geometric conditions?

([3]) The symbol $\ltimes$ stands for semidirect group product.
([4]) The null case is degenerate and it will not be discussed here.

2. Are they sufficient to determine it?
3. Can the electromagnetic field be recovered?

Spacetime is a connected manifold endowed with a local metric g. At a point P the operators $\partial_\mu = \partial/\partial x^\mu$ provide a basis for the tangent space $T_P M$. The dual space $T^*_P M$ is the set of 1-forms, linear mappings from $T_P M$ to $\mathbb{R}$. The natural basis in $T^*_P M$, called dx^μ (not infinitesimal!), is defined for each ∂_ν by saying that this number is δ^μ_ν, to wit, 1 when $\mu = \nu$ and 0 otherwise. In [3] the BR solution was obtained with ordinary tensor calculus; here, also in view of the developments of sect. **5**, it is convenient to use the language of forms (see, *e. g.*, [20, 21]):

$$\begin{aligned} \theta^0 &= n_\mu \mathrm{d}x^\mu, \quad \theta^3 = l_\mu \mathrm{d}x^\mu, \\ \theta^1 &= -\bar{m}_\mu \mathrm{d}x^\mu, \quad \theta^2 = -m_\mu \mathrm{d}x^\mu. \end{aligned} \tag{16}$$

They are four independent, local and null complex one-forms. The null fields n_μ and l_μ (real) and m_μ (complex) fulfill

$$g(l,n) = g_{\mu\nu} l^\mu n^\nu = 1, \qquad g(m,\bar{m}) = g_{\mu\nu} m^\mu \bar{m}^\nu = -1.$$

In this null tetrad the metric tensor reads

$$g = \theta^0 \otimes \theta^3 + \theta^3 \otimes \theta^0 - \theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1, \tag{17}$$

where $\otimes$ denotes the tensor product. The dual of a (possibly complex) two-form $F = F_{ij}\theta^i \wedge \theta^j$ is defined as(5)

$$*F_{ij} = \frac{i}{2}\epsilon_{ijpq}F^{pq}.$$

This is an anti-idempotent operator, with $**F = -F$. A two-form F is called self-dual if $*F = iF = \sqrt{-1}F$; hence

$$\tilde{F} = \frac{1}{2}(F - i * F)$$

is the self-dual part of F.

The complex-valued two-forms:

$$\begin{aligned} Z^1 &= 2\sqrt{2}\theta^0 \wedge \theta^1, \\ Z^2 &= 2\sqrt{2}\theta^2 \wedge \theta^3, \\ Z^3 &= 2\left(\theta^1 \wedge \theta^2 - \theta^0 \wedge \theta^3\right) \end{aligned} \tag{18}$$

provide a basis for the space Λ^2_{SD} of the self-dual complex two-forms $\tilde{F}$. In the space of complex two-forms, two bilinear forms are defined:

$$(F, G) = \frac{1}{4}F_{ij}G^{ij}, \quad \{F, G\} = -\frac{1}{2}\left(F_{ij}G^j_k + *F_{ij} * G^j_k\right)\theta^i \otimes \theta^k.$$

(5) In this section Latin indices from i on range from 0 to 3 and label the components in the null tetrad. In an arbitrary frame, $*F_{\mu\nu} = \sqrt{-\det g}\,\epsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}/2$.

(F, G) is a scalar which, when restricted to Λ^2_{SD}, defines a metric and a covariant basis:

$$Z_1 = Z^2, Z_2 = Z^1, Z_3 = -Z^3.$$

$\{F, G\}$ is a complex, traceless and symmetric tensor.

The two forms Z^a are parallel transferred by Cartan's structure equations:

$$\mathrm{d}Z^a = \epsilon^{abc}\sigma_b \wedge Z_c. \qquad (a, b, c = 1, 2, 3). \tag{19}$$

The three one-forms σ_b are defined by the parallel transfer ∇_X of θ^i along a vector field X:

$$\nabla_X \theta_i \otimes \theta^i = \frac{1}{2}\left(\sigma_a(X) Z^a + \overline{\sigma_a(X) Z^a}\right). \tag{20}$$

The bar denotes the complex conjugate. The quantities σ_a are constructed with the Christoffel symbols.

The source-free Maxwell equations and the electromagnetic energy-momentum tensor τ are

$$\mathrm{d}\tilde{F} = 0, \qquad \tau = \{\tilde{F}, \overline{\tilde{F}}\}.$$

The electromagnetic case corresponds to a real $F_{\mu\nu}$; in this case, the mathematical and physical content of Maxwell theory is fully encoded in its (complex) self-dual part. In particular, $(\tilde{F}, \tilde{F})$ is the only invariant(6) one can build with $\tilde{F}$; when $\left(\tilde{F}, \tilde{F}\right) \neq 0$ the field is called non-null, and there is a frame in which

$$\tilde{F} = \frac{\sqrt{2}}{2} A Z^3, \tau = A\bar{A}\{Z^3, \bar{Z}^3\}.$$

A is a differentiable complex function over spacetime. The two-planes $\theta^0 \wedge \theta^3$ and $\theta^1 \wedge \theta^2$ are the *blades* of the electromagnetic field.

The first Cartan equation (19) gives

$$\mathrm{d}Z^3 = \sigma_1 \wedge Z^1 - \sigma_2 \wedge Z^2 \equiv \psi \wedge Z^3. \tag{21}$$

The one-form ψ is geometrically important, because its components are related to the well known, and widely used, Newmann-Penrose spinor coefficients [22]. The most important result of the Already Unified Theory [3,23] is that the necessary and sufficient conditions for a metric to be a solution of Einstein-Maxwell equations for a non-null electromagnetic field are

1. There is a frame in which the Ricci tensor is $Ricci = A\bar{A}\{Z^3, \bar{Z}^3\}$. This condition is equivalent to the "algebraic condition" of [3].

2. In this frame, $\mathrm{d}\psi = 0$ (the "differential condition" of [3]).

(6) This is a complex invariant; its real and imaginary parts correspond to the usual real invariants $E^2 - B^2$ and $\mathbf{E} \cdot \mathbf{B}$.

TABLE I. – *The two possible realizations of the BR universe.*

Symbol	Quadrics	$\Sigma_+ \otimes \Sigma_-$	Time
BR_1	$(- - -, -) \otimes (+ - +, +)$	$S^2 \otimes AdS_2$	cyclic with period $2\pi R_-$
BR_2	$(- + -, -) \otimes (- - +, +)$	$dS_2 \otimes \mathbb{H}_2$	$(-\infty, \infty)$

The electromagnetic self-dual two-form is then given by

$$\tilde{F} = \frac{\sqrt{2}}{2} e^{i\alpha} A Z^3,$$

where $e^{i\alpha}$ is a constant duality rotation.

The metric of the topological product of two surfaces of constant curvature trivially fulfills these two conditions. For the first one, note that when $\mathcal{R}_+ + \mathcal{R}_- = 0$ the Ricci tensor is traceless; moreover (see eq. (3))

$$Ricci = \frac{\mathcal{R}_+}{2} g_+ + \frac{\mathcal{R}_-}{2} g_- = \frac{\mathcal{R}_+}{2} g_+ - \frac{\mathcal{R}_+}{2} g_-,$$

and hence

$$\mathcal{R}_+ \left(\theta^0 \otimes \theta^3 + \theta^3 \otimes \theta^0\right) - \mathcal{R}_+ \left(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1\right) = -\mathcal{R}_+ \{Z^3, \bar{Z}^3\}.$$

For the second condition, since the metric has constant curvature, we obtain $\psi = 0$.

The correct signature of spacetime and, as explained below, the positive sign of the electromagnetic energy restricts the choice among the six quadrics of fig. 1 to just two possibilities, as shown in table I (see also fig. 1 of [16]). BR_1, with a cyclic time, has been extensively used for quantum field-theoretical applications. In these papers, sometimes, neither the sign subtleties involved in the Gaussian curvature, nor global aspects have been taken into account.

The electromagnetic field self-dual form is covariantly constant and given by

$$\tilde{F} = \frac{\sqrt{2}}{2} \sqrt{\rho} Z^3, \quad \tau = \rho \{Z^3, \bar{Z}^3\},$$

where

$$\rho = -\mathcal{R}_+ = \mathcal{R}_- > 0.$$

In BR_2 we have

$$g_+ = \theta^0 \otimes \theta^3 + \theta^3 \otimes \theta^0, \quad g_- = -\left(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1\right).$$

The null tetrads are

$$
\left\{
\begin{aligned}
\sqrt{2}\theta^0 &= \left(1-\tfrac{x^2}{R_+^2}\right)^{1/2}\mathrm{d}t+\left(1-\tfrac{x^2}{R_+^2}\right)^{-1/2}\mathrm{d}x,\\
\sqrt{2}\theta^1 &= i\left(1+\tfrac{z^2}{R_-^2}\right)^{1/2}\mathrm{d}y+\left(1+\tfrac{z^2}{R_-^2}\right)^{-1/2}\mathrm{d}z,\\
\sqrt{2}\theta^2 &= -i\left(1+\tfrac{z^2}{R_-^2}\right)^{1/2}\mathrm{d}y+\left(1+\tfrac{z^2}{R_-^2}\right)^{-1/2}\mathrm{d}z,\\
\sqrt{2}\theta^3 &= \left(1-\tfrac{x^2}{R_+^2}\right)^{1/2}\mathrm{d}t-\left(1-\tfrac{x^2}{R_+^2}\right)^{-1/2}\mathrm{d}x.
\end{aligned}
\right. \tag{22}
$$

Here x, y, z, t take their usual Cartesian meaning when $R_+, R_- \to \infty$ (see eq. (5) and (6) expressed in terms of an appropriate pseudospherical coordinates).

For BR_1 we have

$$
g_+ = -\left(\theta^1\otimes\theta^2+\theta^2\otimes\theta^1\right), \quad g_- = \theta^0\otimes\theta^3+\theta^3\otimes\theta^0,
$$

with null tetrads

$$
\left\{
\begin{aligned}
\sqrt{2}\theta^0 &= \left(1+\tfrac{x^2}{R_-^2}\right)^{1/2}\mathrm{d}t+\left(1+\tfrac{x^2}{R_-^2}\right)^{-1/2}\mathrm{d}x,\\
\sqrt{2}\theta^1 &= i\left(1-\tfrac{z^2}{R_+^2}\right)^{1/2}\mathrm{d}y+\left(1-\tfrac{z^2}{R_+^2}\right)^{-1/2}\mathrm{d}z,\\
\sqrt{2}\theta^2 &= -i\left(1-\tfrac{z^2}{R_+^2}\right)^{1/2}\mathrm{d}y+\left(1-\tfrac{z^2}{R_+^2}\right)^{-1/2}\mathrm{d}z,\\
\sqrt{2}\theta^3 &= \left(1+\tfrac{x^2}{R_-^2}\right)^{1/2}\mathrm{d}t-\left(1+\tfrac{x^2}{R_-^2}\right)^{-1/2}\mathrm{d}x;
\end{aligned}
\right. \tag{23}
$$

i.e.

$$
\mathrm{d}s^2 = \left(1+\frac{x^2}{R_-^2}\right)\mathrm{d}t^2-\left(1+\frac{x^2}{R_-^2}\right)^{-1}\mathrm{d}x^2-\left(1-\frac{z^2}{R_+^2}\right)\mathrm{d}y^2-\left(1-\frac{z^2}{R_+^2}\right)^{-1}\mathrm{d}z^2. \tag{24}
$$

In terms of pseudo-spherical coordinates $x = R_- \sinh\chi, t = R_-\phi$. In both cases, the scalar curvatures are $\mathcal{R}_+ = -2/R_+^2$, $\mathcal{R}_- = 2/R_-^2$, with $R_+^2 = R_-^2$.

We can also introduce the cosmological constant Λ by setting, instead,

$$
\mathcal{R}_+ = 2(\Lambda-\rho), \quad \mathcal{R}_- = 2(\Lambda+\rho),
$$

so that

$$
Ricci = \tau + \Lambda g.
$$

Then we can have $R_+^2 \neq R_-^2$; the positive energy condition requires $\mathcal{R}_- > 2\Lambda > \mathcal{R}_+$, $\mathcal{R}_+ < 2\Lambda$. An analogous discussion for the choice of the fundamental quadrics easily follows.

The geometric structure of the BR spacetime is best reflected in its *holonomy group*. In a Riemannian manifold M (with a generic signature), parallel transfer of vectors around a closed curve through a point P generates a mapping of the tangent space T_PM onto itself, which leaves the scalar product unchanged. The set of the mappings for all closed curves through P is the *holonomy group* at P. Under reasonable conditions the holonomy groups at all points are isomorphic and generate the holonomy group of M. For a spacetime it is a subgroup of the six-dimensional Lorentz group $\mathcal{L}$. The classification of the holonomy groups is based upon the connected Lie subgroups of the Lorentz group and leads to 15 elements ($R_1, \ldots, R_{15} = \mathcal{L}$); each of them is characterized by the set of its generators, which are just simple bivectors fields. The BR spacetimes are the only solutions of the Einstein-Maxwell's equations belonging to R_7 [24]; the Lorentz subalgebra is two-dimensional, with generators $\theta^0 \wedge \theta^3$ and $\theta^1 \wedge \theta^2$.

4. – The BR metric as a playground for high-energy physics

The Already Unified Theory previously quoted is just one of the many attempts to extend to more general field theories the extraordinary success that general relativity had achieved for gravitation. In all these attempts one investigates mathematical structures which, hopefully, embody new fundamental physical principles—like the Equivalence Principle, the main driver for General Relativity—and lead to self-consistent and verifiable unification schemes. As discussed by Bousso [25], these principles must be explored and tested. We have seen already that the pseudosphere concept ranges far beyond pure mathematics; indeed, it has been often revisited and applied in the current search for a satisfactory quantum theory of high-energy particle physics. In the following subsections we discuss three applications of the BR spacetime along this line; they belong to three general areas briefly reviewed below.

4·1. *Ascension to Heaven by complexification.* – Extending Riemannian metrics to complex coordinates is a very powerful tool to generate new solutions of Einstein's equations from a given real solution $g_{\mu\nu}(x^\rho)$. If the arguments x^ρ are allowed to take complex values and, after a (complex) coordinate transformation, are restricted again to real values, with a different *time slice*, a new solution may result. Such a procedure, for instance, generates Kerr's rotating black hole from Schwarzschild's spherical solution (see, *e. g.*, [26]). These complex solutions have been called *heavenly*; one can say, Einstein's equations in complex coordinates is a *Heaven*, from which beautiful earthly goodies gratuitously descend.

This kind of complexification produce manifolds with twice as many real dimensions; we will later address the more difficult and interesting case in which a complex structure is introduced in a real $2n$-dimensional real manifold to produce an n-dimensional complex manifold.

In quantum gravity, ascending from real to complex variables not only has proved very effective, but has also opened up completely new perspectives (see [27] for an introduction and a collection of papers). Recall that in the path-integral formulation of quantum field theory in a Minkowsky spacetime, probabilities amplitudes are defined by functional integrals of the type

$$Z = \int \mathrm{d}[\phi] \exp[iS[\phi]]; \tag{25}$$

here ϕ represents the set of fields (for instance, a real scalar), $S[\phi]$ is the classical action and $d[\phi]$ is the appropriate infinitesimal measure in ϕ-space. The integral is carried out over all the field histories which take up at t_1 and t_2 the initial and final values ϕ_1 and ϕ_2. A powerful tool to define and carry out the integration is to take an imaginary time variable $t' = it$ (the *Wick rotation*), which produces a Euclidean (negative definite) metric; it also replaces hyperbolic equations with elliptic equations, making the use of powerful mathematical tools possible. In the functional integral the exponent $iS[\phi] = S'[\phi]$ becomes the (real) Euclidean action. Having calculated the path integral in the Euclidean section (x, y, z, t') real, one then tries to continue analytically Z to the Minkowski section (x, y, z, t) real. In the application of this idea to quantum gravity (see, *e. g.*, [28]), one considers the Einstein action and a time slicing of spacetime; transition amplitudes between t_1 and t_2 are then defined with a functional integration over the class of admissible metric fields g (with the Minkowsky signature) between t_1 and t_2. The mathematical status of the required functional measure is largely unknown. When the imaginary time is introduced the signature becomes definite, and the action is stationary at those metrics which are solutions of Einstein's equations in a locally Euclidean, four-dimensional space. They are called *instantons*, and play an essential role in quantum gravity. The functional integral is carried out in the class of definite metrics and, hopefully, analytical continuation will provide the physical value of the transition amplitudes. While the programme of *Euclidean quantum gravity* is still far from physical completeness and mathematical clarity (in particular in relation to the analytical continuation), it has opened up extraordinary perspectives. As stressed by Hawking [29], one can escape the slavery of perturbation expansions around a flat spacetime, leading to an understanding of black holes in the quantum domain. The integration over the metric fields can, in principle, be performed including different topologies; indeed, very near the Big Bang the radius of curvature is of the order of Planck's length, so that quantum processes must needs include topological transitions (the *spacetime foam*). In the Euclidean domain the problem of the initial conditions for the quantum state of the Universe can be formulated and, in principle, solved; this provides quantum cosmology with a completely new foundation (see, *e. g.*, [30]). The thermal properties of quantum gravity are based upon a partition function obtained from (25) with an imaginary time.

A complex manifold is a manifold in which complex coordinates can be consistently introduced, in the sense that the manifold locally looks like $\mathbb{C}^n$ and the only allowed coordinate transformations between overlappings charts are holomorphic functions. The simple case of a *complex curve*, *i. e.* a complex manifold of complex dimension one, can introduce the constructions we will need. If $z = x + iy$ is a complex coordinate in a chart, the basis at z_0 of the complex tangent space is

$$\left.\frac{\partial}{\partial z}\right|_{z_0}.$$

Under a holomorphic transformation $z \to w(z)$ we have

$$\frac{\partial}{\partial z} \to \frac{\mathrm{d}z}{\mathrm{d}w}\frac{\partial}{\partial z} = \frac{\partial}{\partial w}.$$

At the point z_0 the transformation is a complex linear operator on the tangent space, given by the multiplication by

$$\lambda = \left.\frac{\mathrm{d}z}{\mathrm{d}w}\right|_{z_0}.$$

The tangent space can also considered as a two-dimensional real space, generated by $\partial/\partial x$ and $i\partial/\partial y$, but the holomorphic change of coordinates restricts the possible linear transformations in $\mathbb{R}^2$ to those of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where $\lambda = a+ib$. This means that in order to introduce consistently complex coordinates in a real even-dimensional manifold, some additional structure must be imposed on its tangent space.

We want to stress again that is *always* possible to trivially complexify a real manifold just by allowing the coordinates to take complex values, but in this case the number of complex dimensions will be equal to the number of real dimensions. We are discussing now a different problem: how and when can we introduce complex coordinates without "doubling" the dimension of the real manifold?

Again, the case of a real orientable surface M and one of its tangent spaces $T_P M$ can introduce the concept. We need a linear anti-idempotent operator J—called *almost complex structure*—(with $J^2 = -I$, I being the identity) acting on $T_P M$, where it plays the role that the imaginary unit $i = \sqrt{-1}$ has for complex numbers. In order to keep track of the signature of the metric, let us denote by

$$g(X', Y') = X'_1 Y'_1 + X'_2 Y'_2$$

the positive definite scalar product in its (local) Euclidean form. Similarly, for the hyperbolic case

$$g(X'', Y'') = X''_1 Y''_1 - X''_2 Y''_2.$$

The meaning of the symbol g (and of J) results from its arguments: primed or double primed in each case. The almost complex structure J must be *Hermitian*, to wit, the scalar product between two vectors X and Y in the tangent space must be the same as the scalar product between $J(X)$ and $J(Y)$; this allows a unique definition of parallel transport. Thus the choice of J depends on the signature. For the canonical forms above, respectively,

$$\begin{aligned} J(X') &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X' = \begin{pmatrix} X'_2 \\ -X'_1 \end{pmatrix}, \\ J(X'') &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} X'' = \begin{pmatrix} iX''_2 \\ iX''_1 \end{pmatrix}. \end{aligned} \tag{26}$$

A hyperbolic metric requires the matrix J to have *complex* elements.

This construction must be extended to the whole manifold. In a given coordinate system covering a patch, J is defined as a tensor field; it is said to be *integrable* if this holds in all charts. Then the manifold becomes a complex manifold.

The Euclidean case applies to the quadrics a) and c) of fig. 1 (the sphere S_2 and the two-sheet hyperboloid $\mathbb{H}$). In the Euclidean projection plane (ξ, η) $J(X')$ provides the complex coordinate $z = \xi + i\eta$ and its complex conjugate $\overline{z}$:

$$(I - iJ) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi + i\eta \\ \xi - i\eta \end{pmatrix} = \begin{pmatrix} z \\ \overline{z} \end{pmatrix}.$$

The squared distance from the origin is $z\overline{z}$. The corresponding metrics read

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{\left(1 \pm \frac{z\bar{z}}{4R^2}\right)^2}, \tag{27}$$

where the + and − signs refer, respectively, to $\mathbb{H}$ and S_2 (in the latter case, of course, they are bounded by $z\overline{z} < R^2$).

For AdS_2, the hyperbolic case b) of fig. 1, gives

$$(I - iJ)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi + \eta \\ \xi - \eta \end{pmatrix} = \begin{pmatrix} w \\ \overline{w} \end{pmatrix}.$$

In the projection of AdS_2 on the (Minkowskian!) plane (ξ, η) the appropriate "complex" variable is now $w = \xi + \eta$, with "conjugate" $\overline{w} = \xi - \eta$. The metric of AdS_2, in its two realizations corresponding to the character (time-like or space-like) of the ζ-axis, reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}w\,\mathrm{d}\overline{w}}{\left(1 \pm \frac{w\overline{w}}{4R^2}\right)^2}. \tag{28}$$

An orientable surface is always conformally flat, which is evident in both metrics.

As briefly discussed in the next section, endowing a differential manifold with an even number of dimensions with an integrable complex structure leads to a *Kählerian manifold* with n complex dimensions (with coordinates, say, z^α, with $\alpha = 1, ..., n$). Its metric is fully determined by a complex scalar function U (eq. (29)). This beautiful and simple geometry has been discovered in 1933 by Kähler in a concise paper written when he was 23 years old [31] (see, for instance, [20]) and has a large number of applications in mathematics and physics [32]. Solutions of the field equations of general relativity in such a manifold, with $n = 2$ would be very interesting in the complexification programme, in particular for quantum gravity. Several solutions have been found in the case of a definite signature; they represent *gravitational instantons*. The best example is $S^2 \times S^2$, found by Kähler [31]; see also [33]). However, the imposition of a Kählerian structure on spacetime, with indefinite signature, is highly restrictive; indeed, much more restrictive than for a definite signature. The following theorem has been proved [34]: the only Kählerian solutions of Einstein-Maxwell's equations (with cosmological constant) are the flat spacetime, the BR solution and Nariai metric [35, 36]. Thus Kählerian geometry can be used in quantum gravity only for a complex metric, for which the signature is meaningless, or for definite metrics. One way to go to Heaven seems precluded.

4·2. *The structure of the horizon of a black hole.* – Hawking's discovery that, due to quantum effects, a black hole is not really "black", but emits thermal radiation at a rate proportional to the area A of its horizon, has opened up deep and unexpected perspectives for the foundations of thermodynamics and quantum theory. The changes undergone by the black hole due to this emission obey the second law of thermodynamics, where the black-hole entropy S_{BH} is proportional to the area A of its horizon. The proportionality between S_{BH}—in the statistical meaning a dimensionless quantity—and the area requires specifying the elementary area taken up by a single bit of information (see, *e. g.*, [37]). Note that, due to dimensional reasons, the only candidate is the square of Planck's length

$$\ell_{\mathrm{P}} = \sqrt{\frac{G\hbar}{c^3}} = 1.7 \times 10^{-33}\mathrm{cm}.$$

A black hole of one solar mass, with a spherical horizon of area 6×10^{11} cm^2 has a really huge entropy!

When two black holes coalesce the entropy changes from $(A_1+A_2)/\ell_\mathrm{P}^2$ to A/ℓ_P^2, a most striking result if compared with ordinary thermodynamics, where the entropy is additive with respect to the volume of the system. Surely, if a physical system contains black holes, the second principle, which describes how information is lost, must be reformulated. At a fundamental level, perhaps, the information content of a general physical system is not the sum of the information contents of its three-dimensional parts, but resides on a two-dimensional boundary (see, *e. g.* [38] and [39]). Such boundary must include not only the horizons of the black holes it may harbour, but also a large surface in which it is contained. This is, in essence, the *holographic principle*, first discovered by 't Hooft (see, *e. g.*, [38] and [39]). Its realization, in principle, requires the following: any dynamical change occurring in the system must reflect itself in changes in the relevant fields at the boundary: these changes must be sufficient to describe what is going on in the bulk, both classically and at the quantum level. Extensive and heuristic explorations and testing of this principle have been made, and no inconsistencies or no-go results have emerged. This principle may usher a revolution and a unification in our conceptions of gravity, quantum theory and particle physics at very high energies.

To explore the holographic principle in an assigned Riemannian manifold it is convenient to use a simple geometry with a high symmetry; its spatial and null infinity should be easily defined and compactified. The best candidates which have been studied include as a component our fundamental quadric AdS_2; an important role is played by the metric $AdS_d \otimes S^d$, which reduces to BR_1 when $d = 2$ (see [40] and references therein). The BR universes, in particular BR_1, based upon AdS_2, have played an important role in this exploration and have given occasion to a large number of papers. In the search for realizations of holography, Since AdS_d is conformally flat, it may be possible to establish a correspondence between the content of a conformally-invariant dynamics in the bulk of the manifold with its content at infinity. As shown in fig. 1, however, the time coordinate in AdS is cyclic and causality cannot be implemented. To circumvent this problem, spacetime must be "unwrapped", by mapping the periodic time variable into $\mathbb{R}$.

In a generic, asymptotically flat spacetime, a horizon $\mathcal{H}$ separates an interior region (topologically an infinite cylinder around the time axis) from its exterior, extending to spatial infinity, with the following property: no future-pointing time-like line can cross $\mathcal{H}$ from inside to outside. Near a horizon quantum field theory is deeply affected; Hawking's radiation is produced, which carries away energy and information. The investigation of the geometric and the quantum structure near a horizon, therefore, is important. It turns out that the neighbourhood of the horizon of a particular solution of Einstein-Maxwell's equations corresponding to a point with mass and charge has a geometric structure identical to the BR_1 solution.

4·3. *String theory and the dilaton field.* – String theory, in which the fundamental entities subject to quantization are geometrical objects with a time-like and a space-like dimension, just like an ordinary string in spacetime, has emerged as a possible unified scheme of all physical interactions. Strings are embedded in an abstract, d-dimensional Lorentzian manifold; the $d-4$ dimensions which lie beyond four-dimensional spacetime are usually assumed to be a compact submanifold with a small volume, which can be probed only at exceedingly high energies. In bosonic string theories, in which the fundamental entities are two-dimensional surfaces in spacetime (or, in general, in pseudo-Riemannian manifolds), a complexifying rotation produces a Riemann surface and makes exceptional analytic tools available.

A common feature of string theory is the universal appearance of a scalar field ϕ—the *dilaton*—inextricably coupled with the metric. At low energies a model of classical fields interacting with gravity emerges, in which ϕ is produced by the mass-energy distribution; when its mass vanishes the scalar field has a long range effect which simulates gravity and produces discrepancies in the classical tests of general relativity and violates the Weak Equivalence Principle. A unique, complete and self-consistent low-energy approximation to string theory with its scalar field is not available; many examples have been studied in detail. It is interesting to note that, as will be briefly discussed in sect. **7**, the BR spacetime appears in some solutions, including its AdS_2 part. This is an interesting feature from the holographic point of view.

5. – Kähler geometry and the BR universe

The technical content of this section is abbreviated, stressing instead definitions and results; see also [20, 34, 41].

Let M be a (real) Riemannian manifold with an even number $2n$ of dimensions and arbitrary signature. An almost complex structure on M is a tensor field J which maps each tangent spaces $T_P M$ onto itself, with the property $J^2(X) = -X$. The metric $g(X, Y)$ of M is *Hermitian* if

$$g(J(X), J(Y)) = g(X, Y);$$

then X and $J(X)$ are orthogonal. The two-form (an antisymmetric tensor field)

$$\Omega(X, Y) = g(X, J(Y)) = -\Omega(Y, X) = \Omega(J(X), J(Y))$$

is invariant under the action of J; it is called the Kähler form of the metric g. When $\mathrm{d}\Omega = 0$ the form is closed and can be expressed locally in terms of a vector field A_μ:

$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

In spacetime, this suggests the identification of Ω with the electromagnetic field; half of Maxwell's equations are thus fulfilled. Then the manifold is called *Kählerian* and the structure $J(X)$ is (covariantly) constant. As shown by Kähler [31], the metric of such a manifold is determined by a single complex scalar U:

$$g = \frac{\partial^2 U}{\partial z^\alpha \partial \overline{z}^\beta} \mathrm{d}z^\alpha \mathrm{d}\overline{z}^\beta. \tag{29}$$

Let us see what this entails for spacetime. The metric of its tangent space $T_P M$, with vectors $X = (X'', X')$, can always be locally decomposed as in (17), and written

$$g(X, Y) = g(X'', Y'') - g(X', Y').$$

Using the expressions (26)

$$\Omega(X, Y) = g(X'', J(Y'')) - g(X', J(Y'))$$

and recalling the standard form of the null tetrad (16)

$$n_\mu = (1,1,0,0), \quad l_\mu = (1,-1,0,0), \quad m_\mu = (0,0,i,-i),$$

it is easily shown that $\Omega = iZ_3$.

A real Riemannian metric with arbitrary signature is said to have a *product structure* if there is an idempotent (with $P^2(X) = X$) and traceless operator field P which leaves the scalar product invariant [41].

$$g(P(X),P(Y)) = g(X,Y) \quad P^2(X) = X, \quad \mathrm{Tr}P = 0.$$

One can show that it can always be written in the form

$$P(X) = \{Z^3, \bar{Z}^3\}.$$

In spacetime, at a tangent space $T_P M$, the matrix P is an element of the Lorentz group consisting in the inversion of two (out of four) basis vectors. For the BR solution there is a P which changes the sign of X'', but leaves X' unchanged; the previous relation can be directly verified. For a Lorentz signature one has also

$$P(X) = J\bar{J}(X);$$

Hermitian and product structures are equivalent. For a definite signature $J = \bar{J}$, so that $J\bar{J}(X)$ is equal to $-X$ and does not give a product structure.

The requirement of integrability of a product structure to the whole manifold is crucial. It can be shown that in that case spacetime is decomposable, in the sense of eq. (7).

The transport equation (21) shows that Ω is integrable if, and only if, $\sigma_1 = \sigma_2 = 0$. Then it can be shown that the BR metric is the unique, non-flat Kählerian solution of Einstein-Maxwell equations [34]. The electromagnetic self-dual form is proportional to the Kähler structure Ω, and, therefore, is (covariantly) constant.

6. – Near-horizon limit of the Reissner-Nordstrøm black hole

The only asymptotically flat and spherically symmetric solution of Einstein-Maxwell equations corresponding to a point with mass M and charge q was found by Reissner (1916) and Nordstrøm (1918) [37]; it possesses two horizons at

$$r = M \pm \sqrt{M^2 - q^2}.$$

In the extremal case $M = |q|$ [42](7) and, in a global coordinate frame (t, r', θ, φ), it reads

$$\mathrm{d}s^2 = \left(1 - \frac{M}{r'}\right)^2 \mathrm{d}t^2 - \left(1 - \frac{M}{r'}\right)^{-2} \mathrm{d}r'^2 - r'^2 \mathrm{d}^2\Omega. \tag{30}$$

(7) In ordinary physical units, and in terms of lengths,

$$\frac{GM}{c^2} = \frac{\sqrt{G}|q|}{c^2}.$$

Here $\mathrm{d}^2\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. The two horizons coalesce and the variable t never changes its time-like character.

The structure of a quantum field theory in a given spacetime has been extensively studied with this metric as a background. The main issue concerns the properties at the quantum level of a physical system near the horizon where, eventually, semiclassical analysis fails. This corresponds to the limit $r = r' - M \to 0$; in this limit, and with a constant conformal rescaling, (30) becomes

$$\mathrm{d}s^2 = M^2\left(\frac{r^2}{M^4}\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{r^2} - \mathrm{d}^2\Omega\right). \tag{31}$$

We now let r vary over $(0,\infty)$. It can be shown that, as long as $r^2t^2 > M^4$, a coordinate transformation can be found (see [43] and [44]) that carries it into the metric

$$\mathrm{d}s^2 = \left(1+\frac{x^2}{M^2}\right)\mathrm{d}t^2 - \left(1+\frac{x^2}{M^2}\right)^{-2}\mathrm{d}x^2 - M^2\mathrm{d}^2\Omega. \tag{32}$$

The metric (31) is "half" the BR_1 solution of Einstein-Maxwell equations $(---,-)\otimes(+-+,+)$, product of a sphere S^2 and a single-sheet hyperboloid with cyclic time (AdS_2), with $R_- = R_+ = M$ (see (24)); hence $r = 0$ is just a coordinate singularity. One can say, BR_1 *embodies the near-horizon properties of* (30). Rather than displaying the transformation, we start from the fundamental definition of the (t,x) part of the BR_1 metric: it is the fundamental quadric $\xi^2 - \eta^2 + \zeta^2 = M^2$ embedded in $\mathbb{R}^3$ with metric $\mathrm{d}\sigma^2 = \mathrm{d}\xi^2 - \mathrm{d}\eta^2 + \mathrm{d}\zeta^2$. The required embedding reads

$$\begin{aligned}
\xi(r,t) &= \sqrt{-M^2+\left(\frac{rt}{M}\right)^2}\,\sinh\left[\frac{1}{2}\ln\left(\frac{t^2}{M^2}-\frac{M^2}{r^2}\right)\right],\\
\eta(r,t) &= \sqrt{-M^2+\left(\frac{rt}{M}\right)^2}\,\cosh\left[\frac{1}{2}\ln\left(\frac{t^2}{M^2}-\frac{M^2}{r^2}\right)\right],\\
\zeta(r,t) &= -\frac{rt}{M}.
\end{aligned} \tag{33}$$

The restriction $r^2t^2 > M^4$ is apparent. This curtailed spacetime (31) is labeled BR^0.

Along this line, the embedding

$$\begin{aligned}
\xi(r,t) &= \sqrt{M^2+r^2}\,\sin\left(\frac{t}{M}\right),\\
\eta(r,t) &= r,\\
\zeta(r,t) &= \sqrt{M^2+r^2}\,\cos\left(\frac{t}{M}\right)
\end{aligned} \tag{34}$$

has no restriction, and produces the full BR_1 metric (32), now called BR^+. Finally, the embedding

$$\begin{aligned} \xi(r,t) &= \sqrt{-M^2+r^2}\sinh\left(\frac{t}{M}\right), \\ \eta(r,t) &= \sqrt{-M^2+r^2}\cosh\left(\frac{t}{M}\right), \\ \zeta(r,t) &= r \end{aligned} \tag{35}$$

also reproduces BR_1, but with the restriction $r^2 > M^2$; this is called BR^-. Its metric reads

$$\mathrm{d}s^2 = \left(-1+\frac{r^2}{M^2}\right)\mathrm{d}t^2 - \left(-1+\frac{r^2}{M^2}\right)^{-1}\mathrm{d}r^2 - M^2\mathrm{d}^2\Omega. \tag{36}$$

It should be noted that, for simplicity, in the three cases we have used the same symbols (t, r) for the time and the radial coordinate, although they label different points on the quadric and do not have the same range. The entire BR_1 spacetime can be interpreted as the geodesic completion of BR^0 (31) or of BR^- (36). The three labels ^{0}, $^+$ and $^-$ refer to the sign occurring in the expressions of g_{00} and g_{11}. Due to the presence of a Killing horizon at $r = M$, BR^- is physically quite appealing.

The global structure of these spacetimes is best understood with Penrose diagrams (see [43] for an excellent introduction). Since a two-dimensional spacetime is always conformally flat, its metric can be put in the form

$$\mathrm{d}s^2 = C(u,v)\mathrm{d}u\,\mathrm{d}v,$$

where u and v are two null coordinates, chosen in such a way as to be finite at spatial infinity. It is possible, therefore, to represent an infinite spacetime in a finite sheet of paper; the u and v lines are conventionally drawn as straight lines at 45°. For BR^+ (see fig. 2)

$$C(u,v) = -\left(1+\tan^2\frac{u-v}{2}\right), \tag{37}$$

with the mapping

$$x = M\tan\frac{u-v}{2}, \quad t = M\frac{u+v}{2} \quad (0 < v < 2\pi,\ -\pi < u < \pi).$$

All points whose time coordinate t differs by $2\pi M$ are identified.

7. – Dilatonic models and the BR universe

The concept of point-particle is replaced with an extended one-dimensional object, which spans in time evolution a two-dimensional surface Σ (with indefinite signature); such a surface is embedded in a higher-dimensional Lorentzian d-dimensional manifold M with metric $g_{\mu\nu}$. Mathematically, Σ is a mapping $X_\mu : \Sigma \to M$. The string action is

$$S = \frac{1}{2\pi\alpha'}\int_\Sigma g^{\mu\nu}\partial X_\mu \partial X_\nu\,\mathrm{d}^2\sigma\,,$$

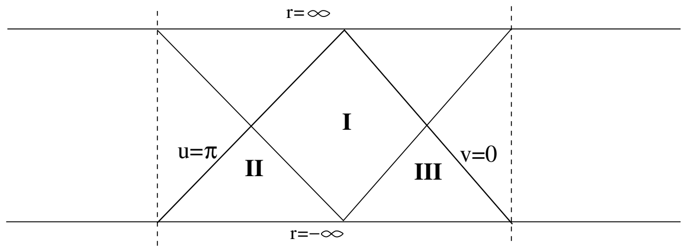

Fig. 2. – The Penrose diagram for the AdS component of the BR_1 metric; the time t runs from left to right, with period $2\pi M$, repeating indefinitely the block between the dashed lines. The r-coordinate is the one used for the complete BR_1 metric. BR^- is the restriction of BR^+ to region **I**, whereas BR^0 is the union of **I**, **II** and **III** with the exclusion of the bold lines, which represent the locus $r = 0$ in (31).

where $X^\mu(\sigma)$ are the embedding maps and α' is the so-called string coupling constant. It is known that, in order to preserve at a quantum level all the symmetries of the classical system (most notably Lorentz covariance), the dimension of the target manifold must be 26, or 10 if supersymmetry is taken into account. Since a complete analysis of the above model is at present rather difficult, it is often customary to simplify the system either considering M^d as the Cartesian product of a four-dimensional manifold $\tilde{M}^4$ times a compact manifold X^{d-4}, or performing a perturbative expansion with respect to α'. This procedure is not unique, but leads to a four-dimensional low-energy effective action; within this latter class of theories an interesting model of a coupling between electromagnetism, the dilaton and gravity is described by the following action [44]:

$$S = \int_{M^4} \mathrm{d}^4x \sqrt{|\, g\,|} e^{-2\phi}(\mathcal{R} - F_{\mu\nu}F^{\mu\nu}). \tag{38}$$

$\mathcal{R}$ is the scalar curvature. Both the electromagnetic field $F_{\mu\nu}$ and the scalar ϕ act as sources for gravity. The electromagnetic Lagrangian appears with the factor $\exp[-2\phi]$, and so is the electromagnetic energy; as a consequence, the electric and magnetic binding energies of a neutral body change when it moves in a gradient of ϕ, thereby violating the weak-equivalence principle. This is an example of the fact that in string theory the equivalence principle is generically violated.

The spherically symmetric solution depends upon two lengths R_+, R_-, and reads:

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{R_+}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{R_+}{r}\right)^{-1}\left(1 - \frac{R_-}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}^2\Omega, \\ F_{\mu\nu} &= \frac{2q}{\sqrt{3}\,r^2}\epsilon_{\mu\nu\rho\sigma}u^\rho v^\sigma, \\ \phi - \phi_0 &= -\frac{1}{4}\ln\left(1 - \frac{R_+}{r}\right), \end{aligned} \tag{39}$$

where $r > R_+$, ϕ_0 is an integration constant and

$$2M = R_+ + \frac{3}{2}R_-, \quad q^2 = R_+R_-.$$

q is the charge; u^μ, v^ν are two unitary and orthogonal space-like vectors tangent to the surface $r = \text{const}$. The solution has two horizons at $r = R_+$ and $r = R_-$. The extremal limit corresponds to $R_+ - R_- = O(r - R_+) \ll 1$ and, with the variable

$$\eta = \operatorname{arcsinh}\sqrt{\frac{r - R_+}{R_+ - R_-}},$$

yields the BR^- component of the BR_1 metric in the form

$$\mathrm{d}s^2 = q^2(4\sinh^2\eta\,\mathrm{d}t^2 - 4\,\mathrm{d}\eta^2 - \mathrm{d}^2\Omega). \tag{40}$$

An important geometrical difference between the extremal Reissner-Nordstrøm solution (31) and the metric (39) should be emphasized: the radii of curvature of the two components are equal in modulus in the former, but arbitrary in the latter.

As a second model, we quote the Jackiw-Teitelboim model (JT) of two-dimensional gravity, with the "cosmological constant" Λ. Two-dimensional dilatonic models have important and widely ranging applications, from toy models of quantum gravity to string theory (see [45]). In the JT model gravity is coupled to the dilaton scalar ϕ with the action

$$S = \frac{1}{2\pi}\int\sqrt{|\,g\,|}\,\mathrm{d}^2x e^{-2\phi}(R + 2\Lambda). \tag{41}$$

Its Euler-Lagrange equations admit the solution

$$\mathrm{d}s^2 = (\Lambda r^2 - a^2)\mathrm{d}t^2 - (\Lambda r^2 - a^2)^{-1}\mathrm{d}r^2, \quad \phi - \phi_0 = -\frac{1}{2}\ln\left(\sqrt{\Lambda}\,r\right), \tag{42}$$

where a^2 is a dimensionless integration constant. In (42) the metric coincides with the (t, r) part of (36): a BR universe arising in a Einstein-Maxwell-dilaton system has the same structure as the two-dimensional metric produced by a dilaton.

8. – Conclusion

After our long roaming, it is clear that the word "pseudospheres" in the title is inadequate: the real protagonist is a more general and abstract object, the *fundamental quadric* discussed in sect. **1**, with its different realizations and its symmetries. As shown in fig. 1, it combines all the possible signatures of the embedding space and, topologically, gives rise to three possibilities. Its unexpected prolificacy and its role as a building block for more complex structures is a consequence, in our view, of its fundamental and deep mathematical simplicity. Only if this simplicity is fully grasped, the gist of its manifold applications becomes manifest, in particular: the nature of Beltrami's realization of hyperbolic non Euclidean geometry, the underlying Kählerian structure of Bertotti-Robinson solution, the relation of this solution to the structure of the horizon of the extreme Reissner-Nordstrøm black hole, and the holographic application of the Anti de Sitter Universe.

This prolificacy, however has a drawback for the present review: it has led us to touching a bewildering variety of topics, most of which, in particular those under the wide umbrella of sect. **4**, have been discussed superficially and inadequately. Our aim was not to provide for each of them exhaustive mathematical and physical detail, but to open a window into, and give examples of, the role of the fundamental quadric and its derivatives in testing and modeling new fundamental theories. We have no final candidate to encompass gravity and quantum fields and clearly this testing and modeling is far from over; we have shown, we believe, that it can be carried out properly only at the appropriate level of mathematical rigour and beauty.

REFERENCES

[1] Eisenhart L. P., *Riemannian Geometry* (Princeton University Press) 1997.

[2] Ficken F. A., "The Riemannian and affine geometry of product-spaces", *Ann. Math.*, **40** (1939) 892.

[3] Bertotti B., "Uniform electromagnetic field in the theory of general relativity", *Phys. Rev.*, **116** (1959) 1331.

[4] Robinson I., "A solution of the Einstein-Maxwell equations", *Bull. Acad. Pol. Sci.*, **7** (1959) 351.

[5] Bonola R., *La geometria non-euclidea* (Zanichelli, Bologna) 1906.

[6] Fano G., *Geometrie non euclidee e non archimedee*, in Berzolari L., Vivanti G. and Gigli D. (Editors), *Enciclopedia delle matematiche elementari*, Vol. II, Part II (Hoepli, Milano) 1958, p. 435.

[7] Rosenfeld B. A., *A History of Non-Euclidean Geometry* (Springer-Verlag) 1988.

[8] Birman G. S. and Nomitzu K., "Trigonometry in Lorentzian geometry", *Am. Math. Mon.*, **91** (1984) 543.

[9] Beltrami E., "Saggio di interpretazione della geometria non euclidea", *G. Mat.*, **6** (1868) 284.

[10] Varićak V., "Über die nichteuklidische Interpretation der Relativitätstheorie", *Jahresber. Deut. Math. Ver.*, **21** (1912) 103.

[11] Sommerfeld A., "Über die Zusammensetzung der Geschwindigkeites in der Relativtätstheorie", *Phys. Z.*, **10** (1909) 826.

[12] Bondi H. and Gold T., "The steady-state theory of the expanding Universe", *Mon. Not. R. Astron. Soc.*, **108** (1948) 252.

[13] Hoyle F., "A new model for the expanding Universe", *Mon. Not. R. Astron. Soc.*, **108** (1948) 372.

[14] Schrödinger E., *Expanding Universes* (Cambridge University Press) 1956.

[15] Sciama D., *Modern Cosmology and the Dark Matter Problem* (Cambridge University Press) 1993.

[16] Bertotti B., "Structure of the electromagnetic field", *Phys. Rev.*, **115** (1959) 742.

[17] Kasner E., "An algebraic solution of the Einstein equations", *Trans. Am. Math. Soc.*, **27** (1925) 101.

[18] Misner C. W. and Wheeler J. A., "Classical physics as geometry"', *Ann. Phys. (N.Y.)*, **2** (1957) 525.

[19] Rainich G. Y., "Electrodynamics in the general relativity theory", *Trans. Am. Math. Soc.*, **27** (1925) 106.

[20] Nakahara M., *Geometry, Topology and Physics* (Institute of Physics Publishing) 2003.

[21] Choquet-Bruhat Y., De Witt-Morette C. and Dillard-Beck M., *Analysis, Manifolds and Physics* (North-Holland) 1977.

[22] Penrose R. and Rindler W., *Spinors and Space-Time*, Vol. **1** (Cambridge University Press) 1986.

[23] Catenacci R. and Salmistraro F., "Classical geometrodynamics revisited", *Nuovo Cimento B*, **46** (1978) 167.

[24] Hall G. S., *Symmetries and Curvature Structure in General Relativity* (World Scientific) 2004.

[25] Bousso R. "The holographic principle", *Rev. Mod. Phys.*, **74** (2002) 825.

[26] Flaherty E. J., *Hermitian and Kählerian Geometry in Relativity*, *Lecture Notes in Physics*, Vol. **46** (Springer-Verlag) 1976.
[27] Gibbons G. W. and Hawking S. H. (Editors), *Euclidean Quantum Gravity* (World Scientific) 1993.
[28] Hawking S. W., *The path-integral approach to quantum gravity*, in *General Relativity*, edited by Hawking S. W. and Israel W. (Cambridge University Press) 1979, p. 746.
[29] Hawking S. W., *Euclidean quantum gravity*, in *Recent Developments in Gravitation*, edited by Lévy M. and Deser S. (Plenum Press) 1979, p. 145.
[30] Hawking S. W., *The quantum mechanics of the Universe*, in *General Relativity and Gravitation*, edited by Bertotti B. *et al.* (Reidel Publishing Company) 1984, p. 491.
[31] Kähler E., "Über eine bemerkenswerte Hermitische Metrik", *Abh. Math. Seminar Hamburg Universität*, **9** (1933) 173.
[32] Bourguignon J.-P., *The unabated vitality of Kählerian geometry*, in *E. Kähler Mathematische Werke*, edited by Berndt R. and Riemenschneider O. (W. de Gruyter) 2003, p. 737.
[33] Catenacci R. and Reina C., "Einstein-Kähler surfaces and gravitational instantons", *Gen. Rel. Grav.*, **14** (1982) 255.
[34] Catenacci R. and Salmistraro F., "On Kähler metrics In Lorentzian manifolds", *Annales Poincaré Phys. Theor.*, **31** (1979) 355.
[35] Nariai H., "On some static solutions of Einstein's gravitational field equations in a spherically symmetric case", *Sci. Rep. Tohoku Univ.*, **34** (1950) 160.
[36] Nariai H., "On a new cosmological solution of Einstein's field equations of gravitation", *Sci. Rep. Tohoku Univ.*, **35** (1951) 62.
[37] Wald R., *General Relativity* (Chicago University Press) 1984.
[38] Susskind L., "The World as a hologram", *J. Math. Phys.*, **36** (1995) 6377.
[39] Aharony O., Gubser S. S., Maldacena J. M., Ooguri H. and Oz Y., "Large N field theories, gravity and string theory", *Phys. Rep.*, **323** (2000) 183.
[40] Cadoni M. and Carta P., "The AdS/CFT correspondence in two dimensions", *Mod. Phys. Lett. A*, **16** (2001) 171.
[41] Catenacci R. and Salmistraro F., "Almost-structures and structures in Lorentzian manifolds. I. Almost-Hermite- and almost-product-(2×2)-structures", *J. Math. Phys.*, **19** (1977) 2047.
[42] Carter B., "The complete analytical extension of the symmetry axis of the Reissner-Nordstrøm metric in the special case $e^2 = m^2$", *Phys. Lett.*, **21** (1966) 423.
[43] Carter B., *Black hole equilibrium states*, in *Les Astres Occlus*, edited by De Witt C. (Gordon and Breach) 1973, p. 57.
[44] Cadoni M. and Mignemi S., "Nonsingular four-dimensional black holes and the Jackiw-Teitelboim theory", *Phys. Rev. D*, **51** (1995) 4319.
[45] Grumiller D., Kummer W. and Vassilevich D. V., "Dilaton gravity in two dimensions", *Phys. Rep.*, **369** (2002) 327.