AN INTRODUCTION TO ALGEBRAIC TOPOLOGY

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INTRODUCTION

Algebraic topology is now considered as a usual mathematical tool for theoretical physics; its applications range from various aspects of gauge theories to string theories and field theories.
The reader will find concepts of algebraic topology in almost all lectures in this school.
Having in mind mainly the applications, we tried, in these lectures, to give a short account of the most elementary aspects of algebraic topology.
This notes should be considered only as a summary for a study of algebraic topology using some of the available text books to which we refer the reader. Among them we recommend:

1) R. Bott and L.W. Tu "Differential forms in algebraic topology" GTM 82 Springer-Verlag 1982.
2) M.J. Greenberg "Lectures on algebraic topology" W.A.Benjamin INC. 1967.
3) A.T. Fomenko, D.B. Fuchs, V.L. Gutenmacher "Homotopic Topology" Akademiai Kiado Budapest 1986.
4) E.Spanier "Algebraic topology" McGraw-Hill 1966

We start with a review of homotopy theory and the De Rham theory. After a brief summary of the concepts of algebra commonly used, we present some aspects of singular homology and cohomology.
We conclude with an application to the classification of bundles and to the problem of anomalies in gauge theories.

We thank all the friends that organized this School on Mathematical Physics at Ferrara University,and the participants, for the discussions we had on various arguments connected with the applications of "geometry" in physics.

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ELEMENTS OF HOMOTOPY THEORY .

One of the most important point of algebraic topology is to assign an algebraic object to a topological space. Hopefully in such a way that homeomorphic spaces have isomorphic algebraic objects and viceversa. This is not completely possible and one should be less ambitious.
To a topological space X is assigned a group (or a module) F(X) and to any continuos map $f : X \to Y$ a homomorphism $F(f) : F(X) \to F(Y)$ or $F(f) : F(Y) \to F(X)$ such that

a) $f = id \to F(f) = id$ id = identity

b) $F(gf) = F(g) F(f)$.

(*application of this*: if $f : X \to Y$ is a homeomorphism then $F(f^{-1}) = F(f)^{-1}$ so that $F(X) = F(Y)$.We have a necessary (but not sufficient) condition of homeomorphicity).
The first fundamental example of such a theory is homotopy of paths
Let σ and τ paths in a space X ($\sigma,\tau : [0,1] \to X$) with $\sigma(0) = \tau(0) = x_0$, $\sigma(1) = \tau(1) = x_1$ we say σ is homotopic to τ ($\sigma \approx \tau$) rel [0,1] with end points fixed if there is a map $F : I \times I \to X$ such that

$$F(s,0) = \sigma(s)$$
$$F(s,1) = \tau(s)$$
$$F(0,t) = x_0$$
$$F(1,t) = x_1$$

F is called a homotopy from σ to τ relative to [0,1].
If σ is a *loop* at x_0 ($x_1 = x_0$) and τ is the constant $\tau(s) = x_0$, if σ is homotopic to τ ($\sigma \approx \tau$) rel [0,1] then σ is homotopically trivial.
We can consider the set of equivalence classes of paths from x_0 to x_1.
If σ is a path from x_0 to x_1 and τ from x_1 to x_2, $\sigma \circ \tau$ is a path from x_0 to x_2.

$$\sigma\ \tau(t) = \begin{cases} \sigma(2t) & 0 \leqslant t \leqslant 1/2 \\ \tau(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

It is verified that $\sigma \simeq \sigma'$, $\tau \simeq \tau' \rightarrow \sigma\tau \approx \sigma'\tau'$. Thus we can multiply classes.

Let $\pi_1(X, x_0)$ the set of homotopy classes of *loops* at x_0; then $\pi_1(X, x_0)$ is a group with 1 = the constant loop and $\sigma^{-1}(t) = \sigma(1-t)$ $0 \leqslant t \leqslant 1$.

Is there a relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$?

We have:

if α *is a path from* x_0 *to* x_1 the map:

$\alpha_* : [\sigma] \rightarrow [\alpha^{-1} \sigma\alpha]$ is an isomorphisms of $\pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ (the inverse is (α_*^{-1})) so:

If X is pathwise connected, $\pi_1(X)$ is defined and is called the *fundamental group of* X.

Now let X and Y be topological spaces and f_0 and f_1 continuos maps : $X \rightarrow Y$; $f_0 \approx f_1$ if there exists $F : X \times I \rightarrow Y$ such that

$$\begin{cases} F(x, 0) = f_0(x) \\ F(x, 1) = f_1(x) \end{cases}$$ (homotopy between maps is an equivalence relation)

(example $Y, X = \mathbb{R}^n$ f = identity, $g = 0$

$F(x,t) = tx$)

When X is such that the identity map is homotopic to a *constant map*, we say that X *is contractible. Proof that* $\approx$ *is an equivalence relation*:

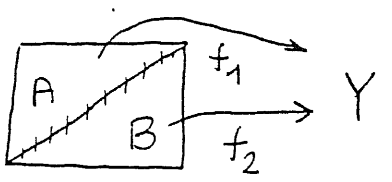
1) $f \approx f$: $F:(X \times I \rightarrow Y$ $F(x,t) = f(x)$

2) $f \approx g \rightarrow g \approx f$ $(G(x,t) = F(x, 1-t)$ is a homotopy)

3) $f \approx g$ and $g \approx h \rightarrow f \approx h$ $$H(x,t) = \begin{cases} F(x,2t) & 0 \leqslant t \leqslant 1/2 \\ G(x,2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

The only point is to show that H is continuos. This is a consequence of the important "collating lemma". If X and Y are topological spaces and $X = A \cup B$ where A and B are closed (open). Let $f_1 : A \rightarrow Y$, $f_2 : B \rightarrow Y$ such that $f_1(x) = f_2(x)$ $\forall\, x \in A \cap B$, then $g : X \rightarrow Y$

$$g(x) = \begin{cases} f_1(x) & x \in A \\ f_2(x) & x \in B \end{cases}$$ is continuos.



If X, Y, Z are topological spaces,

f_0, f_1 homotopic maps $X \to Y$

and

g_0, g_1 homotopic maps $Y \to Z$

then $g_0 f_0$ and $g_1 f_1$ are homotopic maps $X \to Z$.
Definition: Two spaces X, Y have the same homotopy type (or are homotopically equivalent spaces) ($X \approx Y$) if there exist continuos maps $f : X \to Y$ and $g : Y \to X$ such that $gf \approx i_X$ and $fg \approx i_Y$ (homeomorphic spaces have the same homotopy type but not viceversa)
(example 1) S^1 and $S^1 \times R$ (the cylinder)

$$f : S^1 \to S^1 \times R \qquad f(P) = (P,0) \qquad gf = i_X$$

$$g : S^1 \times R \to S^1 \qquad g(q,r) = q \qquad fg : (q,r) \to (q,0)$$

$$F : S^1 \times R \times I \to S^1 \times R$$
$$F(q,r,t) = (q,tr)$$
$$F(q,r,0) = (q,0) = fg(q,r)$$
$$F(q,r,1) = (q,r) = i_Y(q,r)$$

Theorem: X is contractible iff X have the same homotopy type of a point.
Corollary: if X is contractible, $\pi_1(X) = 1$. note that this is not so obvious, because every loop σ at x_0 is homotopic as a map with the constant loop but we must show they are homotopic *relative* to 0,1, i.e. homotopic as loops.
We study now the behaviour of $\pi_1(X, x_0)$ under maps.
Definition:
Let X,Y path connected spaces and $f : X \to Y$ continuos for $x_0 \in X$ let $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$:

$$f_*\langle\alpha\rangle = \langle f\alpha\rangle$$

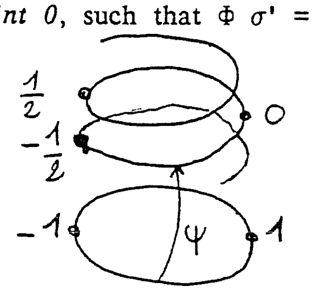
This make sense because if α and β are loops at x_0 and $\alpha \simeq \beta$ is also $f\alpha \simeq f\beta$ (the homotopy is fF).f_* is a homomorphism.
Observations: if f is a homeomorphism or a homotopic equivalence,

f_* is an isomorphism. (This also prove that if $X \approx [P] \to \pi_1(X) = 1$)

The fundamental group of S^1

We will study $\pi^1(S^1)$ as an illustration of some important concepts (covering spaces and the covering homotopy theorem) S^1 is the group of $z \in \mathbb{C}$ $|z|^2 = 1$; we have a continuos homomorphism $\Phi : \mathbb{R} \to S^1$ $\Phi(x) = e^{2\pi i x}$. Φ is an *open* mapping and maps $(-\frac{1}{2}, +\frac{1}{2}) \to S^1 - \{-1\}$ homeomorphically; let ψ be its inverse on that set: *then*

1) If σ is a path in S^1 with initial point 1, there is a *unique* path σ' in $\mathbb{R}$ *with initial point 0*, such that $\Phi\ \sigma' = \sigma$.



2) it τ is a path in S^1 with initial point 1 such that

$$F : \sigma \approx \tau \quad \text{rel}(0,1)$$

Then $\exists$ unique $F' : I \times I \to \mathbb{R}$ such that

$$F' : \sigma' \approx \tau' \text{ and } \Phi F' = F$$
$$\text{rel } (0,1)$$

Let Y be I or I x I, $f : Y \to S^1$ either σ or F, $0 \in Y$ will be either 0 or (0,0).

Since Y is compact, f is uniformly continuos, and $\exists\ \varepsilon > 0\ /\ |y-y'| < \varepsilon \to |f(y) - f(y')| < 1$ in particular

$$f(y) \neq -f(y') \quad \text{so} \quad \psi\left[\frac{f(y)}{f(y')}\right] \quad \text{is defined.}$$

Now take N such that $|y| < N\ \varepsilon\ \forall\ y \in Y$. Set

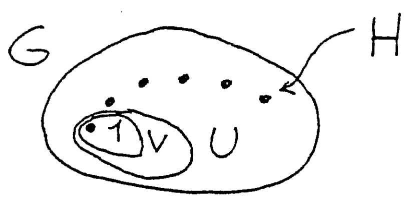
$$f'(y) = \psi\left[f(y)/f\left(\frac{N-1}{N}y\right)\right] + \psi\left[f\left(\frac{N-1}{N}y\right)/f\left(\frac{N-2}{N}y\right)\right] + \cdots \psi\left[f\left(\frac{1}{N}y\right)/f(0)\right].$$

Then $f'(0) = 0$, $\Phi f' = f$, f' is continuos. If f'' has the same properties, $f'-f''$ would be a continuos map into the kernel of Φ that is Z, since y is connected $f'-f''$ is constant hence $f' = f''$.
In the case $y = I \times I$ $f = F$, $f' = F'$ and $F' : \sigma' \approx \tau'$ in fact on $0 \times I$ $\Phi F' = F = 1$ hence $F'(0 \times I) \subset Z$ so by connectedness again $F'(0 \times I) = 0$ and similarly $F'(1 \times I)$ is constant.
Corollary: *the end point of* σ' *depends only on* $\langle\sigma\rangle$.
Define $\chi : \pi_1(S',1) \to Z$ by $\chi\langle\sigma\rangle = \sigma'(1) \in Z$ (*recall* $\Phi\sigma' = \sigma$). χ is well defined and is a *homomorphism*:
given $[\sigma]$, $[\tau] \in \pi_1(S',1)$, let $m = \sigma'(1)$, $n = \tau'(1)$ and $\tau''(s) = \tau'(s) + m$ is a path from m to n+m in R.
Then $\Phi^{*}\tau'' = \tau$ *and* $\sigma'\tau''$ *is the lifting of* $\sigma\ \tau$ *with initial point 0 and end point* n+m. Hence $x([\sigma][\tau]) = \chi[\sigma] + \chi[\tau]$ χ *is onto*: given n define $\sigma'(s) = ns$. If $\sigma = \Phi^{*}\sigma'$ $\chi[\sigma] = \sigma'(1) = n$.
χ *is one to one*:
suppose $\chi[\sigma] = 0$ if σ' is a loop in R at 0 (R is contractible) $\sigma' \simeq 0$ hence, applying Φ, $\sigma \approx 1$, $[\sigma] = 1$.
From this we have $\pi_1(S^1) = Z$

The only property of S^1 used is that $S^1 = R/Z$ were R is a topological group and Z a discrete subgroup of R.
Theorem : If G is a simply connected i.e. ($\pi_1(G) = 0$, G pathwise connected) topological group and H a discrete normal subgroup, then

$$\pi_1(G/H, 1) \simeq H$$

We have only to find a neighborhood V of 1 in G which is mapped homeomorphically onto a neighborhood of 1 in G/H by $\Phi : G \to G/H$ (so that we have ψ as before).
But since H is dicrete $\exists\ U / U \cap H = \{1\}$.
By continuity of $g_1 g_2^{-1}$ $\exists\ V\ 1 / g_1, g_2 \in V \to g_1 g_2^{-1} \in U$.



In fact in this case let

$$\Phi(g_1) = Hg_1$$
$$\Phi(g_2) = Hg_2$$

and if $\Phi(g_1) = \Phi(g_2) \to Hg_1 g_2^{-1} \in H$ but $g_1 g_2^{-1} \in U \to g_1 = g_2$

Corollary $\pi_1(S^1 \times S^1) = Z \oplus Z$

The following theorem is important and contrasts with the property of homology or cohomology for product spaces.

Theorem: given spaces X, Y, $x_0 \epsilon x$, $y_0 \epsilon y$ we have

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \oplus \pi_1(Y, Y_0)$$

sketch of proof: the isomorphism is obtained as follows:

$$\begin{array}{ccc} & X \times Y & \\ P_X \swarrow & & \searrow P_Y \\ X & & Y \end{array}$$

They induce homomorphisms

$$\begin{array}{ccc} & \pi_1(X \times Y, (x_0, y_0)) & \\ (P_X)_* \swarrow & & \searrow (P_Y)_* \\ \pi_1(X, x_0) & & \pi_1(Y, y_0) \end{array}$$

hence $((P_X)_*, (P_Y)_*)$ is the homomorphism. This homomorphism has the following inverse: given loops σ at x_0, τ t y_0 assign to the pair $[\sigma],[\tau]$ the class of the loop (σ,τ) : (σ,τ) t = $(\sigma(t), \tau(t))$.

Higher homotopy groups

If X is a topological space with a base point p, for $n \geqslant 1$ the n^{-th} homotopy *group* $\pi_n(X)$ is defined to be the homotopy classes of maps from the n–cube I^n to X that send the faces δI^n of I^n to the point p. From another point of view, if we identify δI^n to a point we get the sphere S^n with a base point. Thus $\pi_n(X)$ can be interpreted as the homotopy classes of maps (*base point preserving*) from S^n to X. The important point here is that, althought in general the set of homotopy classes of maps [X,Y] from X to Y is not a group, the sets $[S^n,X]$ (b.p.p.) *are* groups. The group operation is defined as follows: if α and $\beta : I^n \rightarrow X$, representing the classes $[\alpha]$ and $[\beta]$, $[\alpha]$ $[\beta]$ is the class of the map:

$$\gamma(t_1 \dots t_n) = \begin{cases} \alpha(2t_1, t_2 \dots t_n) & 0 \leqslant t_1 \leqslant \frac{1}{2} \\ \beta(2t_1 - 1, t_2 \dots t_n) & \frac{1}{2} \leqslant t_1 \leqslant 1 \end{cases}$$

(This generalizes the product of path).
The most important fact is that $\pi_q(X)$ is Abelian *for q>1.* It is useful to introduce $\pi_0(X)$ that is (when X is a manifold) the *set* of connected component of X (if X is a space $\pi_0(X)$ is the set of path connected component of X). It has a distinguished element *, the component containing the base point of X. This observation is crucial in defining exact sequences of homotopy groups.
The homotopy sequence of a fiber bundle (Steenrod).
If $\pi : E \to B$ is a fiber bundle with fiber F then

$$\pi_q(F) \xrightarrow{i*} \pi_q(E) \xrightarrow{\pi*} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \xrightarrow{i*} \pi_{q-1}(E) \xrightarrow{\pi*} \pi_{q-1}(B) \to$$

is exact (the kernel of any map is the image of the previous one). The kernel of a set map between pointed sets is the inverse image of the base point. We will assume that E and B have base points e_0 and b_0 and that F is identified with $\pi^{-1}(b_0) \subset E$ via a map:

$i : F \to E$ giving the diagram $\begin{array}{c} F \to E \\ \quad\ \downarrow \pi \\ \quad\ B \end{array}$ with $\pi i = b_0$ constant map

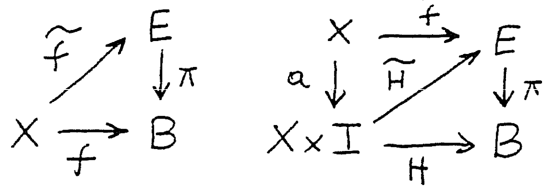
Then i_* and π_* are defined as follows: (this generalizes the case of π_1). If f is a map $(X, x_0) \to (Y, y_0)$ and (M, m_0) a space,
$f_* : [M, m_0; X, x_0) \to [M, m_0; Y, f(x_0)]$

$$(f_*\omega)\ m = f(\omega(m)) \quad \text{where} \quad \omega \in [M, m_0, X, x_0] \text{ and } m \in M$$

∂ is harder to describe. We need the *covering homotopy property* (a generalization of the lemma used in computing $\pi_1(S^1)$).
Let (E,π,B) be a fiber bundle and let $H : X \times I \to B$ be a homotopy between maps $f,g : X \to B$. Suppose that f is *covered* by a map $\tilde{f} : X \to E$ ($\pi\tilde{f} = f$). Then there is a homotopy $\tilde{H} : X \times I \to E$ such that

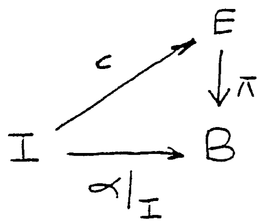
$$\tilde{H}(x,0) = \tilde{f}(x) \quad \forall\, x \in X \quad \text{and}$$

$$\pi\tilde{H} = H$$



$a(x) = (x,0) \quad \forall x$.

We can now define ∂. For simplicity consider first q=1. A loop $\alpha : I \to B$ representing $[\alpha] \in \pi_1(B)$ can be lifted to a path α in E, then $\partial[\alpha] = \alpha(1)$ in $\pi_0(F)$.
For q=2, let $I \to I^2$ the inclusion $t \to (t,0)$. A map $\alpha : I^2 \to B$ representing $[\alpha] \in \pi_2(B)$ can be considered a homotopy of $\alpha|_I : I \to B$ $\alpha|_I(t) = \alpha(t,0)$ $\alpha(t,q) : I \times I \to B$.
Then the constant map $c : I \to f_0 \in E$ covers $\alpha|_I$



then there is a homotopy $\tilde{\alpha} : I^2 \to E$ which covers α and such that $\tilde{\alpha}|_I = c_0$.
Finally, $\partial[\alpha]$ is the homotopy class of the map ψ

$$\psi : I \to F$$
$$\psi(t) = \tilde{\alpha}(t,1)$$

For q>2 δ is defined in the same way. An important property of π_n is that if X is contractible then $\pi_n(X)$ is trivial = (0) $\forall n$. In general if X and Y have the same homotopy type then $\pi_n(X)$ and $\pi_n(X)$ are isomorphic groups. (For "reasonable" spaces the following partial converse is true: if $f : X \to Y$ induces an isomorphisms $\pi_n(X) \to \pi_n(Y)$ $\forall_n$ then $X \approx Y$).
The homotopy sequence is the basic elementary tool for computing homotopy groups.

Homotopy groups of classical groups

We assume the fact that $\pi_q(S^n) = 0$ if $q<n$, Z if $q = n$.
It is known that if H is a closed Lie Subgroup of a lie group *G* then $G \to G/H$ is a fiber bundle with fiber H.
The case $R/Z = S^1$ gives

$$\to \pi_i(Z) \to \underset{\overset{\shortparallel}{0}}{\pi_i(R)} \to \pi_i(S^1) \to \underset{\overset{\shortparallel}{0}}{\pi_{i-1}(Z)} \to$$

$$\pi_i(S^1) = 0 \qquad i>1$$

The case $\frac{U(n)}{SU(n)} = S^1$ gives, for $i \geqslant 2$, $\pi_i(SU(n)) = \pi_i(U(n))$;

for $i = 1$ $\quad 0 \to \pi_1(SU(n)) \to \pi_1(U(n) \to \pi_1(S^1) \to 0$.
$$\pi_1(S^1) = Z$$

The case $\frac{0(n)}{S0(n)} = Z_2$ gives, for $i \geqslant 1$, $\pi_i(S0(n) = \pi_i(0(n))$.

The case $\frac{U(n)}{U(n-1)} = \frac{SU(n)}{SU(n-1)} = S^{2n-1}$ gives that

for $i \leqslant 2n-1$, $\pi_i(U(n) = \pi_i(U(n+1))$ and
$\pi_i(SU(n) = \pi_i(SU(n+1))$

The case $\frac{0(n)}{0(n-1)} = \frac{S0(n)}{S0(n-1)} = S^{n-1}$

gives for $i \leqslant n-2$ $\quad \pi_i(0(n)) = \pi_i(0(n-1))$
$\pi_i(S0(n)) = \pi_i(S0(n+1))$.

we can then compute the low homotopy groups of classical groups.

If $i = 0$, $0 \leqslant 2n-1$ $\quad \forall\, n \geqslant 1$, hence :

$\pi_0(U_n) = \pi_0(SU(n) = \pi_0(U_1) = 0$ and $0 \leqslant n-2$
$\pi_0(0(n)) = \pi_0(0(2)) = Z_2$ $\quad \forall\, n \geqslant 2$
$\pi_0(S0(n)) = \pi_0(S0(2)) = 0$ $\quad \forall\, n \geqslant 2$.

if $i = 1$ $\quad 1 \leqslant 2n-1$ $\quad \forall\, n \geqslant 1$, hence

$\pi_1(U(n)) = \pi_1(U(1)) = Z$
$\pi_1(SU(n)) = \pi_1(SU(1)) = 0$

$1 \leqslant n-2$ $\quad \forall\, n \geqslant 3$

so we must calculate $\pi_1(S0(3))$ but it is known that $S0(3) \approx S^3/Z_2$
hence $\pi_1(S0(3)) = Z_2 = \pi_1(S0(n)) = \pi_1(0(n))$ $\quad \forall\, n \geqslant 3$
For $S0(2) \approx U(1)$, we have $\pi_1(S0(2)) = Z = \pi_1(0(2))$
If $i = 2$,
observe that $\pi_2(U(1)) = \pi_2(SU(1)) = 0$.
Moreover $SU(2) \approx S^3$ and $\pi_2(SU(2)) = \pi_2(U(2)) = 0$
But for $n \geqslant 2$ $\quad 2 \leqslant 4-1$ hence

$$\pi_2(SU(n) = \pi_2(U(n)) = \pi_2(U(2) = 0 .$$

Moreover $\pi_2(O(1)) = 0$, and we know that $\pi_2(O(n) = \pi_2(SO(n))$ $\forall$ n .

since $2 \leqslant n-1$ for $n \geqslant 3$ $\pi_2(O(3)) = \pi_2(O(n))$.

We must compute $\pi_2(O(3)) = \pi_2(SO(3))$;

we have $SO(3) = S^3/Z_2$ and

$$\underset{0}{\underset{\shortparallel}{\pi_2(Z_2)}} \to \underset{0}{\underset{\shortparallel}{\pi_2(S^3)}} \to \pi_2 SO(3) \to \underset{0}{\underset{\shortparallel}{\pi_1(Z_2)}}$$

so $\pi_2(SO(3)) = 0$.

If $i = 3$ $\pi_3(U(2)) = \pi_3(SU(2) = \pi_3(S^3) = Z$

since $3 \leqslant 2n-1$ $\forall$ $n \geqslant 2$ we have

$\pi_3(U(n)) = \pi_3(SU(n)) = Z$ $\forall$ $n \geqslant 2$ and $\pi_3(U(1)) = 0$.

DE RHAM THEORY

The de Rham theory is the prototype of all cohomology theories. As we shall see, it is defined in terms of the differentiable structure of a manifold and one of its important features will be its topological nature.
Manifolds are locally homeomorphic to R^n and we start with R^n itself. Let $\Omega^*(R^n)$ be the Grassman algebra of differential forms on R^n i.e. $\Omega^*(R^n) = (C^\infty$ functions on $R^N) \otimes \Omega^*$ where Ω^* is the algebra generated by products $dx_1 \ldots\ldots dx_n$ with the relations $dx_i{}^2 = 0$ $dx_i\, dx_j = -dx_j\, dx_i$.
If $\omega \in \Omega^*(R^n)$ then $\omega = \Sigma\, f_{i_1..iq}\, dx_{i_1}\ldots dx_{iq}$ and $f_{i_1\cdot\cdot iq}$ are C^∞ functions. We write $\omega = \Sigma f_I dx_I$.
The algebra can be splitted: $\Omega^*(R^n) = \oplus^n{}_{p=0}\, \Omega^p(R^n)$ where Ω^p is generated by the products of p generators.
There is a differential operator (exterior differentiation)

$$d: \Omega^q(R^n) \rightarrow \Omega^{q+1}(R^n)$$

defined as follows

$$\text{if } f \in \Omega^0(R^n) : df = \Sigma \frac{\partial f}{\partial x_i} dx_i$$

$$\text{if } \omega = \Sigma\, f_I dx_I : d\omega = \Sigma\, df_I dx_I .$$

It is known that

$$d^2 = 0 \qquad \text{and} \qquad d(\tau\omega) = d\tau\omega + (-1)^{\deg \tau}\, \tau d\omega.$$

The following object

$$\Omega^0(R^n) \xrightarrow{d} \Omega^1(R^n) \xrightarrow{d} \Omega^2(R^n) \xrightarrow{d}$$

is called the de Rham complex and is an example of a *cochain complex* (see later).
We observe that $\Omega^p(R^n)$ is a module over the C^∞ function and a vector space over R.
Definition
The q–de Rham cohomology of R^n is the vector space over R:

$$H^q_{DR}(R^n) = \frac{\mathrm{Ker}\ d}{\mathrm{Im}\ d}$$

It is a measure of the deviation from exactness of the de Rham complex. Note that we may also speak of $\Omega^*(U)$ and $H^q_{DR}(U)$ for U open in R^n.

examples: $n = 1$ $H^0_{DR}(R) = R$· because

$$H^0_{DR}(R) = \mathrm{Ker} d = \text{constant functions}$$

and

$$H^1_{DR}(R) = 0 \quad \text{because} \quad \mathrm{Ker} d = \Omega^1(R) \quad \text{and}$$

if $\omega = g(x)dx$ then $g(x)dx = df$ where $f = \int^x_0 g(t)dt$

In general (Poincare' lemma) $H^*(R^n) = \begin{cases} 0 & * = p \neq 0 \\ R & * = 0 \end{cases}$

We now extend this theory to an arbitrary differentiable manifold. It is known that for a differentiable manifold the operator d can be defined independently of the coordinate system. A differential q-form is a C^∞ section of the bundle $\wedge^p(M)$ or, locally,

$$\omega = \Sigma f_I dx_I .$$

A smooth map of differential manifolds $f : M \to N$ induces a map $f^* : \Omega^*(N) \to \Omega^*(M)$ which commutes with d. We recall the definition of f^*. If $\omega \epsilon \Omega^p(N)$ and $v_1 \dots v_p$ are vector fields on M : $f^*\omega(v_1\dots v_p) = \omega(f_*v_1,\dots f_*v_p)$ where $f_* : TM \to TN$, $f_*v\ (g) = v(gf)$.
In local coordinates x^i on M and y^k on N we have

$$f^*dy^k = \frac{\partial f^k}{\partial x^i} dx^i .$$

We define again:

$$H^q_{DR}(M) = \frac{\mathrm{Ker} d}{\mathrm{Im} d} = \frac{Z^q(M)}{B^q(M)}$$

example: $H^1_{DR}(S^1) = R$ $[H^0_{DR}(S^1) = R]$

$\mathrm{Ker} d = \Omega^1(S^1)$
$\mathrm{Im} d = \{df,\ f \epsilon C^\infty(S^1)\}$

If θ is a polar coordinate, $\partial/\partial\theta$ is a non null vector field on S^1 and $d\theta$ is a non null 1-form.
Given any form $\omega = g(\theta)d\theta$ with $g(0) = g(2\pi)$ we have that $(\omega - 1/2\pi \int_0^{2\pi} g(t)dt)d\theta$ is exact.

Hence $H_{DR}{}^1(S^1) = \{cd\theta\}_{c\in R} = R$

If $\psi : X \to Y$ is smooth, $\psi^* : Z^q(Y) \to Z^q(X)$ and $\psi^* : B^q(Y) \to B^q(X)$. In fact $d(\psi^*\omega) = \psi^*(d\omega) = \psi^*(0) = 0$ and if $\omega = d\tau$, $f^*\omega = f^*d\tau = d(f^*\tau)$. We have then a homomorphisms

$$\psi^* : H^K{}_{DR}(Y) \to H^K{}_{DR}(X)$$

which is an isomorphisms if ψ is a diffeo.

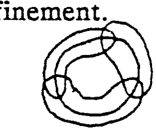
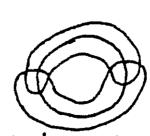
We recall now a few results from the theory of differentiable manifold. (A differentiable manifold will be assumed paracompact and Hausdorff).
A partition of unity on M is a collection of non negative C^∞ functions $\{\rho_\alpha\}_{\alpha\in I}$ such that:

$$\forall_p \ \exists \ V \ / \ \textstyle\sum_\alpha \rho_\alpha|_V \text{ is a finite sum, and } \textstyle\sum_{\alpha\in I} \rho_\alpha = 1$$

We have that:

i) Given an open cover U_α of M ($\alpha\in I$) there is $\{\rho_\alpha\}_{\alpha\in I}$ such that $\operatorname{supp}\rho_\alpha \subset U_\alpha$.
ii) There is also $\{\rho_\beta\}_{\beta\in J}$ such that $\operatorname{supp}\rho_\beta$ is *compact* and contained in some U_α.

A good cover is an open cover such that all finite intersections are diffeo to R^n.
We have that any manifold has a good cover, if the manifold is compact, the good cover is finite and moreover any open cover has a good refinement.

Examples:  good  not good

We come now to one of the most important tecnical point: The *Mayer-Vietoris Sequence.*
Suppose $M = U \cup V$ with U, V open; then we have the inclusions

$$U \cap V \xrightarrow[j_V]{j_U} \underset{\text{disjoint}}{U \cup V} \xrightarrow[i_V]{i_U} U \cup V = M$$

This gives a sequence on Ω^*:

$$0 \quad \Omega^*(M) \xrightarrow{i^*=i^*_U \oplus i^*_V} \Omega^*(U)\oplus\Omega^*(V) \xrightarrow{j^*=j^*_U - j^*_V} \Omega^*(U\cap V) \quad 0$$

where, if α is a form on M

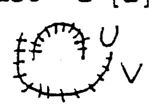
$$i^*\alpha = (\omega,\tau) \quad \text{where} \quad \omega = \alpha|_U \quad \text{and} \quad \tau = \alpha|_V$$

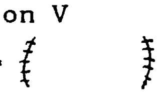
$$\text{and} \quad j^*(\omega,\tau) = j^*_U\omega - j^*_V\tau$$

The important point is that this sequence is *exact*.
Im $i^* \subseteq$ Ker j^* because $(\gamma,\delta) \in$ Im i^* if $\gamma = \omega|_U$, $\delta = \omega|_V$ and hence $\gamma-\delta|_{U\cap V} = 0$. Viceversa if $\gamma-\delta|_{U\cap V} = 0$ there exists ω on M and Ker $j \subseteq$ Im i^*. The last step is to verify that j is sujective.
We start from Ω^0.
Let f be a function on $U\cap V$ we must write f as g − h where g is on U and h on V we take a partition ρ_U, ρ_V and consider $\rho_U f$ and $\rho_V f$.
$\rho_V f$ is a function on U and $\rho_U f$ is a function on V and $\rho_U f - (-\rho_V f) = f$.
In general, if $\omega\in\Omega^p(U\cap V)$, $(-\rho_V\omega,\rho_U\omega)\in\Omega^p(U)\oplus\Omega^p(V)$ *maps* onto ω.
As we shall see later (when treating homological algebra) the Mayer−Vietoris− sequence induces an *exact sequence* in cohomology:

$$\to H^q_{DR}(M) \xrightarrow{i^*} H^q_{DR}(U)\oplus H^q_{DR}(V) \xrightarrow{j^*} H^q_{DR}(U\cap V) \xrightarrow{d^*} H^{q+1}_{DR}(M) \to$$

In this particular case $d^*[\omega] = \begin{cases} [-d\ \rho_V\ \omega] & \text{on } U \\ [d\ \rho_U\ \omega] & \text{on } V \end{cases}$

Example: $M = S^1$ 

$U\cap V =$ 

$$H^0_{DR}(S^1)\to H^0_{DR}(U)\oplus H^0_{DR}(V)\xrightarrow{j_*} H^0_{DR}(U\cap V)\to H^1_{DR}(S^1)\to H^1(U)\oplus H^1(V)$$

$$\underset{\text{Ker } j^*}{\overset{\shortparallel}{H^0_{DR}(S^1)}} \to \mathbb{R}\oplus\mathbb{R} \xrightarrow{\text{difference}} \mathbb{R}\oplus\mathbb{R} \to \underset{\text{Cokernel } j^*}{\overset{\shortparallel}{H^1_{DR}(S^1)}} \to 0$$

$$(\alpha,\beta) \longrightarrow (\alpha-\beta,\ \alpha-\beta)$$

$$\text{difference} : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}$$
$$(\alpha,\beta) \to (\alpha-\beta,\ \alpha-\beta)$$

in fact j^* sends the classes α on U and β on V into the difference of the classes on $U\cap V$.

Hence $\text{Ker } j^* \approx R$ and $\text{Coker } j^* = \frac{R \oplus R}{R} = R$

so $H^0{}_{DR}(S^1) = R$ and
$H^1{}_{DR}(S^1) = R$.

Homotopy properties of $H^*{}_{DR}(M)$

We shall describe now one of the relations between homotopy and the de Rham cohomology. The first property is that if two manifolds *have the same homotopy type then they have the same cohomology.*

Consider $M \times R \underset{s_0}{\overset{\pi}{\rightleftarrows}} M$

where π is the projection and s is the zero section $s_0(p) = (p,0)$.
We have $\pi^* : \Omega^*(M) \to \Omega^*(M \times R)$ and $s^*{}_0 : \Omega^*(M \times R) \to \Omega^*(M)$ since $\pi s_0 = 1$ we have $s^*{}_0\pi^* = 1$ but $\pi^* s^*{}_0 \neq 1$.
If $\{U_\alpha\}$ is an atlas for M, then $\{U_\alpha \times R\}$ is an atlas for $M \times R$. Every form on $M \times R$ is locally a linear combination of forms of the following type: (a local basis for the cotangent bundle of $M \times R$ is $\pi^* dx^i$, dt).

a) $(\pi^*\omega)\ \ f(p,t)$
b) $(\pi^*\omega)\ \ f(p,t)dt$ where ω is a form on M

we define an operator $K : \Omega^p(M \times R) \to \Omega^{p-1}(M \times R)$
by

$K(a) = 0$
$K(b) = (\pi^*\omega)\ \ \int^t{}_0 f$

It can be checked that $\pi^* s^* = 1 - (-1)^{q-1}(dK - Kd)$, hence $\pi^* s^* = 1$ *in cohomology* because $dK - Kd$ maps closed forms exact forms:

if ω is such that $d\omega = 0$
$(dK - Kd)\omega = d(K(\omega))$.

It follows that $H^*{}_{DR}(M \times R) = H^*{}_{DR}(M)$. (In particular, by induction we have the Poincare' lemma : $H^*(R^n) = H^*(R)$).
Now if f and g are homotopic maps $f,g : M \to N$

we have $f=Fs_1$, $g=Fs_0$ and

$f^*=s^*_1F^*$, $g^*=s^*_0F^*$ where $s_1(x)=(x,1)$, $s_0(x)=(x,0)$

and since all s^*_1 and s^*_0 both invert π^* in cohomology they are equal, and $f^* = g^*$ in cohomology.
Now if the two manifolds have the same homotopy type in the C^∞ sense, since any C^0 map between manifolds is C^0 homotopic to a C^∞ map, they also have the same homotopy type in the usual sense. Hence if $f : M \to N$ and $g : N \to M$ and both fg and gf are homotopic to 1, it follows that f^* and g^* are *isomorphisms* in cohomology.
Remark: the formula $H^*_{DR}(M \times R) = H^*_{DR}(M)$ is a particular case of the so called Künneth formula:

$$H^*_{DR}(M \times N) = H^*_{DR}(M) \otimes H^*_{DR}(N)$$

i.e. $$H^n_{DR}(M \times N) = \oplus_{p+q=n} H^p_{DR}(M) \otimes H^q_{DR}(N)$$

We only describe the isomorphism.

The two projections $M \times N \xrightarrow{\pi_1} M$, $\xrightarrow{\pi_2} N$

give rise to a map φ on forms

$$\varphi : \Omega^*(M) \otimes \Omega^*(N) \to \Omega^*(M \times N)$$
$$\varphi(\omega\otimes\mu) = \pi^*_1\omega\wedge\pi^*_2\mu \ .$$

This map descends to cohomology and give the isomorphism.
Application: $H^*_{DR}(S^1 \times S^1) = H^*_{DR}(S^1) \otimes H^*_{DR}(S^1)$ so

$$H^0(S^1 \times S^1) = H^0(S^1) = R$$
$$H^1(S^1 \times S^1) = H^0(S^1) \oplus H^1(S^1) \oplus H^1(S^1) \oplus H^0(S^1) = R\oplus R$$
$$H^2(S^1 \times S^1) = H^1(S^1) \oplus H^1(S^1) = R\oplus R = R$$

Finite dimensionality of de Rham cohomology

If the manifold M has a finite good cover (in particular if M is compact) then its cohomology spaces are finite dimension. Infact from the sequence

$$\to H^{q-1}(U\cap V) \xrightarrow{d^*} H^q(U\cup V) \xrightarrow{i^*} H^q(U) \oplus H^q(V) \to$$

we have $H^q(U \cup V) \simeq Ker\ i^* \oplus Im\ i^* \simeq im\ d^* \oplus Im\ i^*$

Infact $H^q(U \cup V) \simeq imd^* \oplus B$ where $B \subset H^q(U \cup V)$.

Now let $h = i^*|_B$, since $Ker\ i^* = Imd^*$ and $Imd^* \cap B = 0$ it follows that *Ker* $h = 0$ and $B \approx Imh$.
Now let $z \in Im\ i^*$, then there exists $y \in H^q(U \cup V)$ such that $z = i^*(y)$. Since $H^q(U \cup V) = Ker\ i^* \oplus Imh$,

$y = a + b$ where $a \in Ker\ i^*$, $b \in Imh$, hence

$$z = i^*(y) = i^*(a+b) = i^*(b) = h(b)$$

This implies $Im\ i^* = Im\ h$.
Thus if $H^q(U)$, $H^q(V)$ and $H^{q-1}(U \cap V)$ are finite dimensional, so is $H^q(U \cup V)$.
Suppose now the cohomology of any manifold with a good cover with at most p open sets is finite dimensional. Consider a manifold having a good cover with p+1 open sets $\{U_0 \dots U_p\}$. Now $(U_0 \cup U_1 \dots U_{p-1}) \cap U_p$ has a good cover with p open sets $\{U_{0,p}, U_{1,p}, U_{p-1,p}\}$ by hypothesis the q-cohomology of $(U_0 \cup U_1 \dots U_{p-1})$, U_p, and $(U_0 \cup U_1) \cap U_p$ as finite dimensional and so is the q-cohomology of $U_0 \cup U_1$ $U_{p-1} U \cup U_p$. The initial step of the induction is Poincare' lemma.

Poincare' duality

If M is compact and orientable (i.e. it admits an atlas with $\det(\partial x_\alpha{}^i/\partial x_\beta{}^j)>0$) the integral of a top form τ (a form of degree n = dim. M) is defined:

$$\int_M \tau = \sum_\alpha \int_{U_\alpha} \rho_\alpha \tau$$

where $\int_{U_\alpha} \rho_\alpha \tau$ means $\int_{R^n} (\Phi^{-1}{}_\alpha)^*(\rho_\alpha \tau)$ where

Φ_α is some trivialization and $\{\rho_\alpha\}$ a partition of unity. It can be shown that the integral is independent of the oriented atlas and the partition. If M is not compact we take τ with compact support.
The fundamental theorem is Stokes' theorem:
If ω is an (n-1) form with compact support on an oriented manifold M with boundary of dim n and if ∂M (boundary of M) is given the

induced orientation:

$$\int_M d\omega = \int_{\partial M} \omega \quad .$$

We consider now *compact orientable* manifolds. The exterior product and the integration descends to cohomology and so is defined a pairing:

$$\int : H^q{}_{DR}(M) \otimes H^{n-q}{}_{DR}(M) \to \mathrm{R}$$

if $[\alpha] \in H^q{}_{DR}(M)$, $[\beta] \in H^{n-q}{}_{DR}(M)$, $<[\alpha],[\beta]> = \int_M[\alpha\wedge B]$

Poincare' duality says that this pairing is not degenerate i.e.

$$H^q{}_{DR}(M) \approx (H^{n-q}{}_{DR})^*$$

(Poincare' duality holds also when M is not compact but *has a finite good covering*:

$$H^q{}_{DR}(M) \simeq (H_c{}^{n-q}(M))^*$$

where H_c is the cohomology of compactly supported forms).
We recall that given two finite dimensional vector spaces V,W a pairing $<,> : V\otimes W \to \mathrm{R}$ is not degenerated if $v \to <v,>$ is an isomorphism $V \to W^*$ (equivalently $<v,w> = 0 \ \forall \ w \to v = 0$).
In particular it follows from Poincare' duality that *if M is compact oriented and connected* $H^n{}_{DR}(M) = \mathrm{R}$.

ELEMENTS OF ALGEBRA

Homological algebra is, in some sense, the abstract support of algebraic topology and is now invading many chapters of mathematics just as group theory or linear algebra.
So, before continuing the study of the various cohomology theories, we recall some of the most important concepts, namely the functors ⊗, Hom, Tor and Ext.
Let R denote a commutative ring with identity $1 \neq 0$. (We have in mind $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_n$). An R–module is an additive abelian group X together with a multiplication $(a, x) \to ax$ $a \in R$, $x \in X$ that satisfies:

$$(a+b)x=ax+bx, \quad a(x+y)=ax+ay, \quad 1x=x, \quad a(bx)=(ab)x.$$

example 1: (this is the typical example we have in mind) take $R = \mathbb{Z}$ (integers) and X an abelian group and define

$$ax=\underbrace{x+x+\dots+x}_{a\text{-times}}$$

Just as for vector spaces we have $0x=0$ and $(-1)x=-x$ and the notions of submodule and quotient module; if A is a submodule of x, to the quotient group $Q=X/_A$ can be given the structure of a module: if $x+A$ denotes an element of $X/_A$ we have $x+A+y+A = x+y+A$ and $\alpha(x+A) = \alpha x+A$.
If S is a subset of X we call (S) the submodule of X given by all finite linear combinations of elements in S. (S) is also called the module generated by S. An element x is said to be a *torsion element* of X if there is $\lambda \neq 0$ such that $\lambda x = 0$ (this cannot happen if R is a field as in the case of vector spaces).
The torsion elements of X form a submodule $\tau(X)$ of X; X is *torsion–free* if $\tau(X) = 0$.When $R=\mathbb{Z}$, $X/\tau(X)$ is torsion free.
Let now X, Y modules over R and f an homomorphism $X \to Y$: as usual we define

$$\mathrm{Ker} f = \{f^{-1}(0)\} \quad \text{and} \quad \mathrm{Coker}\, f = Y/_{\mathrm{Image}(f)}.$$

An *exact sequence* is

$$\dots \to A \xrightarrow{f} B \xrightarrow{h} C \to \dots$$

where A,B,C modules and f,h homomorphisms such that Imf=Kerh.

Any exact sequence of the form

$$0 \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{o} 0 \qquad \begin{array}{l} i = \text{inclusion} \\ o = \text{0-homomorphism} \end{array}$$

is called *short-exact.*

Examples: 1) $0 \to X \xrightarrow{f} Y \xrightarrow{g} Y/_X \to 0$ is *short-exact*

where X is a submodule of Y, f is the inclusion and g the projection.

2) $0 \to \text{Ker}f \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{\pi} \text{Coker}f \to 0$

where X,Y modules, f an homomorphism, i the inclusion and π the projection, is *exact*.

As for vector spaces we have the concept of *direct sum* of modules X Y; X and Y can be considered as submodules of X⊕Y and in this case X∩Y = 0. A sudmodule A of X is a *direct summand* of X if there exists a submodule B such that X = A⊕B.

We say that an exact sequence $\to X \xrightarrow{f} Y \xrightarrow{g} Z \to$ *splits at Y* if Im(f) = Ker(g) is a direct summand of Y. It is an easy exercise to prove that in this case Y = Im(f)⊕Im(g), so, if a short exact sequence

$0 \to A \to B \to C \to 0$ splits, then
$B \cong A \oplus C$ (where $\cong$ means isomorphic)

Semi-exact sequences

A sequence $\to X \xrightarrow{f} Y \xrightarrow{g} Z \to$ is called *semi-exact* if Imf ⊆ Kerg.

Example: $A \subset X$ but $A \neq X$

$$0 \to A \xrightarrow{i} X \to 0$$

is semi-exact but not exact. The quotient $Q = X/_A$ serves as a measure of the deviation from exactness. This suggests that $\text{Kerg}/_{\text{Imf}}$, the so called *derived module at Y* should be important.

The modules of a semi-exact sequence will be indexed by integers; if we use decreasing integers the sequence

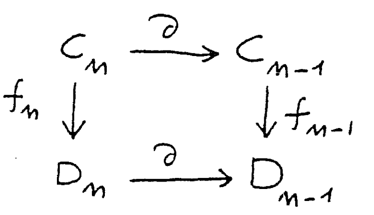
$$C : \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \dots$$

with $\partial^2 = 0$ is called a *chain complex* and the homomorphisms ∂ the *boundary operators*.
The Kernel of ∂ in C_n is denoted by $Z_n(C)$ and is called the module of the *n-cycles of C*, the image of ∂ in C_n is called the module of the *n-boundaries* of C and finally $H_n(C) = Z_n(C)/B_n(C)$ the *n-homology module of C*.
For "upper sequences" or *cochain complex*

$$C : \overset{\delta}{\to} C^{n-1} \overset{\delta}{\to} C^n \overset{\delta}{\to} C^{n+1} \overset{\delta}{\to} \qquad \text{with } \delta^2 = 0$$

we have the *n-cocycles*, the *n-coboundaries* and $H^n(C) = Z^n(C)/B^n(C)$ the *n-cohomology module of C*.
Consider now two chain complexes (all the following can be repeated for cochain complexes) C and D, a *chain transformation* is a family $f = \{f_n : C_n \to D_n\}$ of homomorphisms such that

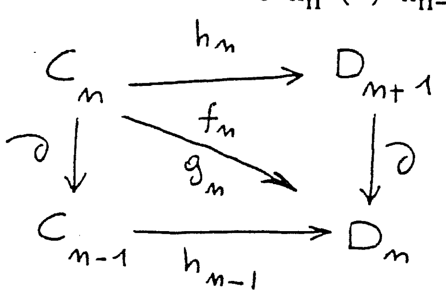
$$\partial \circ f_n = f_{n-1} \circ \partial$$



f_n carries $Z_n(C)$ into $Z_n(D)$ and $B_n(C)$ into $B_n(D)$, hence $H_n(f) : H_n(C) \to H_n(D)$ is defined.
Two homomorphisms $f,g : C \to D$ are said *homotopic* ($f \simeq g$) if there exists $h = \{h_n : C_n \to D_{n+1}\}$ such that

$$\partial\, h_n\ (\mp)\ h_{n-1}\, \partial = f_n - g_n$$

in



we have that if $f \simeq g$ then $H_n(f) = H_n(g)$. Infact if $x \in H_n(C)$, take $p(z) = x$ for $z \in Z_n(C)$ then

$$f_n(z) - g_n(z) = \partial\, h_n(z) \in B_n(D)$$

and hence $H_n(f)(x) = H_n(g)(x)\ \forall\ x$.

Let us now consider a short exact sequence of chain complexes

$$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$$

where we mean that,

$$\forall n \qquad 0 \to C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \to 0$$

is exact. We have the following lemma:

$$H_n(C) \xrightarrow{H_n(f)} H_n(D) \xrightarrow{H_n(g)} H_n(E) \quad \text{is } \textit{exact}$$

Obviously $\operatorname{Im} H_n(f) \subset \operatorname{Ker} H_n(g)$ because

$$H_n(g)\, H_n(f) = H_n(g \circ f) = 0 \qquad \text{since } g \circ f = 0$$

We prove that $\operatorname{Ker}((H_n(g)) \subset \operatorname{Im}(H_n f)$.
If $\alpha \in H_n(D)$ and $\alpha \in \operatorname{Ker} H_n(g)$. We can choose a $z \in \alpha \subset Z_n(D) \subset D_n$; we have $g_n(z) \in B_n(E)$ because $H_n(g)\ \alpha = 0$. Hence there exists $y \in E_{n+1}$ such that $\partial\ y = g_n(z)$. Since g_{n+1} is onto $y = g_{n+1}(x)$ $x \in D_{n+1}$.

Then $\quad g_n(z - \partial(x)) = g_n(z) - g_n(\partial(x)) = g_n(z) - \partial(g_{n+1}(x)) =$
$= g_n(z) - \partial(y) = 0,$

and $\quad z - \partial(x) \in \operatorname{Ker} g_n = \operatorname{Im} f_n$.

Hence $z = \partial(x) = f_n(w)$ $w \in C_n$, moreover $f_{n-1}(\partial w) = \partial\ f_n\ w = \partial(z - \partial x) = \partial z = 0$ but f_{n-1} is injective and hence $\partial w = 0$ and $w \in Z_n(C)$.
Let now $\beta = p(w) \in H_n(C)$; since $z \in \alpha$ and $\partial\ x \in B_n(D)$

$$f_n(w) = z - \partial\ x \in \alpha \qquad \text{and hence}$$
$$H_n(f)(\beta) = \alpha\ .$$

We construct now the *connecting homomorphisms* ∂: $\quad \forall n$

$$\partial : H_n(E) \to H_{n-1}(C)$$

in order to obtain an *exact* sequence called the *homology sequence* of the sequence $0 \to C \to D \to E \to 0$:
$\ldots\ldots \to H_n(C) \to H_n(D) \to H_n(E) \to H_{n-1}(C) \to H_{n-1}(D) \to$
(for cohomology we have

$\to H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to$).

We first define a function

$$\Phi : Z_n(E) \to H_{n-1}(C)$$

Let $z \in Z_n(E)$. Since $g_n : D_n \to E_n$ is onto, $z = g_n(u)$ for $u \in D_n$. Consider $\partial u \in D_{n-1}$. Since $g_{n-1}(\partial u) = \partial g_n u = \partial z = 0$ we have $\partial u \in \operatorname{Ker} g_{n-1} = \operatorname{Im} f_{n-1}$. Since f_{n-1} is injective, there exists a unique $v \in C_{n-1}$ with $f_{n-1}(v) = \partial u$ and $f_{n-2} \partial v = \partial f_{n-1} v = \partial^2 u = 0$, and this implies $\partial v = 0$. Hence $v \in Z_{n-1}(C)$ and $w = p(v) \in H_{n-1}(C)$.
It can be shown that:

1) w is independent of $u \in D_n$ and depends only on $z \in Z_n(E)$
2) Φ is a homomorphism
3) $B_n(E) \subset \operatorname{Ker} \Phi$.

This last observation says that Φ induces a homomorphism $\partial : H_n(E) \to H_{n-1}(C)$ and we obtain the sequence.

Free Modules

Let R be our ring and S any set.
We will construct a module F(S) called the free module over R on S.
Let $F(S) = \{\Phi : S \to R \ / \ \Phi(s) = 0$ except for a finite number of $s\}$.

$$(\Phi + \Psi)(s) = \Phi(s) + \Psi(s)$$
$$\alpha\Phi(s) = \alpha(\Phi(s))$$

Next define a function $f : S \to F(S)$:

$$f(s)(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \quad \forall\, t \in S$$

f is injective and we may identify S with f(S) in F(S).
Hence f(S) becomes a subset of F(S) which generates F(S)

$$\Phi = \textstyle\sum_{s \in S} \Phi(s) f(s) .$$

Now let us consider a family of modules $\{X_s \mid s \in S\}$ where $X_s = R$ $\forall$ s.

By definition of direct sum we have that the direct sum of $\{X_S\}$ is isomorphic to F(S).
A module X is said *free* if it is isomorphic to F(S) for some S.
Theorem: any X is isomorphic to a quotient of a free module.
Let X be any module and consider F(X).
Then h : F(X) $\rightarrow$ X is an onto homomorphisms. Let K = Ker h.
Then X $\approx$ F(X)/K by the isomorphism theorem.
It can be shown that a module X is free if it admit a basis. A basis is a linearly independent subset S of X which generates X.If the ring is Z and X is finitely generated,X is free iff is torsion-free.

Tensor products

Let A and B modules over R. Consider F(A x B) and G the submodule generated by the elements

$$(\alpha_1 a_1 + \alpha_2 a_2,\ b) - \alpha_1(a_1,\ b) - \alpha_2(a_2,\ b)$$
$$(a,\ \beta_1 b_1 + \beta_2 b_2) - \beta_1(a,\ b_1) - \beta_2(a,\ b_2)$$

We define $A \otimes_R B = \dfrac{F(A \times B)}{G}$ and denote

$$\pi(a,\ b) = a \otimes b$$

we have by definition

$$(\alpha_1 a_1 + \alpha_2 a_2) \otimes b = \alpha_1(a_1 \otimes b) + \alpha_2(a_2 \otimes b)$$
$$(\gamma\ a) \otimes b = \gamma(a \otimes b) = a \otimes (\gamma\ b)$$
$$a \otimes (\beta_1 b_1 + \beta_2 b_2) = \beta_1(a \otimes . b_1) + \beta_2(a \otimes b_2) \quad \text{so}$$

$$A \otimes B \ni t = \sum_{finite} a_i \otimes b_i$$

Example: $Z_2 \otimes Z_3 = 0$ infact

$$1 \otimes 1 = (3 - 2)(1 \otimes 1) =$$
$$= (-2)1 \otimes 3(1) = 0 \otimes 0 \quad \text{etc.}$$

Tensor product of homomorphisms:

$f : A \rightarrow A'$, $g : B \rightarrow B'$
$f \otimes g : A \otimes B \rightarrow A' \otimes B'$
$(f \otimes g)(a \otimes b) = f(a) \otimes f(b)$ and extended by linearity.

Properties of the tensor product

It can be shown that:

1) if $A = \oplus_\mu A_\mu$, $B = \oplus_\nu B_\nu$
$A \otimes B = \oplus_{(\mu,\nu)} A_\mu \otimes B_\nu$

2) $A \otimes B \approx B \otimes A$

3) $(A \otimes B) \otimes C \approx A \otimes (B \otimes C)$

Properties of the tensor product of homomorphisms

If $f : A \to A'$ and $g : B \to B'$ are *onto* then $f \otimes g$ is also onto and Ker h = {module generated by $a \otimes b$ with $a \in$ Ker f or $b \in$ Ker g}. (The proof is a long exercise; it should be noted that if f and g were not onto *both* conclusions would be false). It follows that if f and g are isomomorphisms, also $f \otimes g$ is.

Theorem: If M is a module and $A \to B \to C \to 0$ is exact then

$$A \otimes M \overset{f\otimes 1}{\to} B \otimes M \overset{g\otimes 1}{\to} C \otimes M \to 0 \quad \text{is exact.}$$

Proof: denote $f' = f \otimes i$ and $g' = g \otimes i$

Since g and i are onto, g' is also onto and Ker g' = {generated by $y \oplus w$ with $y \in$ Ker g}.
Since Im f = Ker g, then is $x \in A$ and $f(x) = y$.
Hence $y \otimes w = f(x) \otimes i(w) = f'(x \otimes w) \in \text{Im } f'$.
Viceversa, since

$$g' \; f' = (g \otimes i) \; (f \otimes i) = (g \; f) \otimes (i \; i) = 0 \otimes i = 0$$

we have that $\text{Im } f' \subset \text{Ker } g'$.

Modules of homomorphisms

If A and B are given modules over R, the set Hom(A, B) of R linear mappings is a module over R with sum $(\Phi+\Psi)x = \Phi(x) + \Psi(x)$ and product $(\alpha\Phi)(x) = \alpha \; \Phi(x)$. Now let $f : A' \to A$ and $g : B \to B'$ denote homomorphisms.

We can define a function

$$h : Hom(A, B) \to Hom(A', B')$$
$$h(\Phi) = g \circ \Phi \circ f .$$

h is a homomorphism of Hom(A, B) into Hom(A', B') and is denoted by the symbol Hom(f, g). As for tensor product, it can be proved the following theorem:

Theorem: If M is an arbitrary module over R and if

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{is exact, then also}$$
$$0 \to Hom(C, M) \xrightarrow{g^*} Hom(B, M) \xrightarrow{f^*} Hom(A, M) \quad \text{is exact.}$$

Where $f^* = Hom(f, i)$, $g^* = Hom(g, i)$ and $i : M \to M$ is the identity.

Moreover if $0 \to A \to B \to C$ is exact, then also

$$0 \to Hom(M, A) \xrightarrow{f_*} Hom(M, B) \xrightarrow{g_*} Hom(C, M) \quad \text{is exact}$$

where $f_* = Hom(i, f)$, $g_* = Hom(i, g)$.

Tor and Ext

Let A be a module, a free resolution of A is an exact sequence

$$\to A_K \to A_{K-1} \to \dots \to A_1 \to A_0 \to A \to 0$$

such that A_K is free, Free resolutions always exist. For any A we know that there exists an exact sequence

$$0 \to Ker\ h \to F(A) \to A \to 0.$$

(If A is an abelian group, i.e. a Z–module, also Ker h is free).

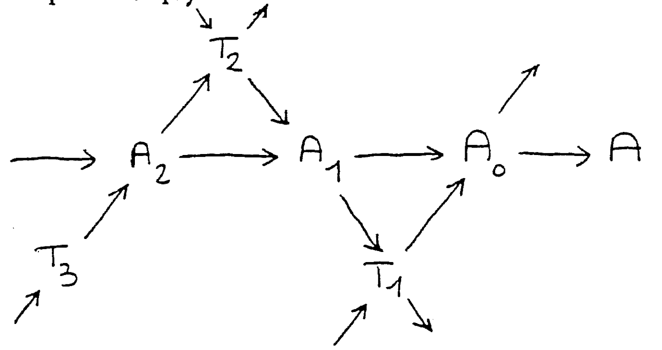
Let now $F(A) = A_0$, $Ker\ h = T_1$, $F(T_1) = A_1$ we have

$$0 \to T_1 \to A_0 \to A \to 0 \quad \text{and also}$$

$$0 \to T_2 \to A_1 \to T_1 \to 0$$

$$0 \to T_3 \to A_2 \to T_2 \to 0 \qquad \text{etc.}$$

where $A_i = F(T_i)$; we obtain



The resolution of A is not uniquely determined, nevertheless the further constructions will not depend on the choice.
Let us return to the case of Abelian groups.
Let $0 \to B \to F \to A \to 0$ a free resolution of A and let G be a Z module.
We know that the two sequences

$$0 \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(F, G) \overset{f^*}{\to} \mathrm{Hom}(B, G)$$

and $\quad B\otimes G \overset{f\otimes 1}{\to} F\otimes G \to A\otimes G \to 0 \quad$ are exacts.

Tor and Ext measure the failure of this two sequences to be short exact.

$$\mathrm{Tor}(A, G) = \mathrm{Ker}(f\otimes 1)$$
$$\mathrm{Ext}(A, G) = \mathrm{Coker}\ f^* = \mathrm{Hom}(B, G)/_{\mathrm{Im}\ f}$$

It can be shown that these definitions are independent of the free resolution used.
Note that

$$\mathrm{Tor}(A, G) = \mathrm{Tor}(G, A) \quad (\text{this follows from } X\otimes Y \approx Y\otimes X)$$

and that

$$\mathrm{Tor}(A'\oplus B', G) = \mathrm{Tor}(A', G)\oplus \mathrm{Tor}(B', G).$$

We will need the cases in which the modules involved are Z_q or Z or R. We have as a free resolution

$$0 \to Z \overset{f}{\to} Z \to Z_n \to 0$$

where f is the multiplication by n.
Tensoring with Z_m gives

$$\mathrm{Tor}(Z_n, Z_m) \to Z_m \overset{(f\otimes 1)}{\to} Z_m \to Z_{(n,m)} \to 0$$

where (n, m) is the G.C.D of n and m.
The Kernel of the multiplication by n in Z_m is $Z_{(n,m)}$.
So $\mathrm{Tor}(Z_n, Z_m) = Z_{(n,m)}$
Now, if A is free we can take as a free resolution

$$0 \to 0 \to A \overset{1}{\to} A \to 0$$

and $\mathrm{Tor}(A, G) = \mathrm{Ker}\ 1\otimes 1 = 0 = \mathrm{Tor}(G, A)$.
We have then

Tor	$\mathbb{Z}$	$\mathbb{Z}_m$
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	0	$\mathbb{Z}_{(m,n)}$

note that if $A = \oplus_i Z_{ai}$, $G = \oplus_j Z_{bj}$ then

$\mathrm{Tor}(A, G) = \oplus_{i,j} Z_{dij}$ where $dij = \mathrm{G.C.D.}(ai,bj)$.

We will need also $\mathrm{Tor}(Z, R) = 0$, $\mathrm{Tor}(Z_n, R) = 0$
$\mathrm{Tor}(R, R) = 0$. This follows from the following result:
consider two Abelian groups X and Y.

Let $S = \{(x, y, n) \in X \times Y \times Z \ / \ nx = 0, \quad ny = 0\}$

Let F the free group generated by S and G the subgroup containing elements of the form

$$(x_1 + x_2, y, n) - (x_1, y, n) - (x_2, y, n) ,$$
$$(x, y_1 + y_2, n) - (x, y_1, n) - (x, y_2, n)$$
$$(x, y, m n) - (mx, y, n)$$
$$(x, y, m n) - (x, my, n) ,$$

Then Tor(X, Y) ≈ F/G. It follows that if X or Y is torsion free then Tor(X, Y) = 0.
Moreover if T and N are finitely generated abelian group, Tor(T, N) = $\tau(T) \otimes \tau(N)$ where τ means the torsion part.
As for Ext we have that if A is free Ext(A, G) = 0. Moreover Ext(Z_m, Z) = Z_m infact

$$0 \to Z \xrightarrow{f} Z \to Z_m \to 0$$

$$f^* : \mathrm{Hom}(Z, Z) \to \mathrm{Hom}(Z, Z)$$

$$\overset{\shortparallel}{Z} \quad \xrightarrow{f} \quad \overset{\shortparallel}{Z}$$

Im $f^* = m Z$ and

$$\mathrm{Ext}(Z_m, Z) = Z/_{mZ} = Z_m$$

Also we will use Ext $(Z_m, R) = 0$

$$f^* : \mathrm{Hom}(Z, R) \to \mathrm{Hom}(Z, R)$$

$$\mathrm{Im}\ f^* : \mathrm{Hom}(Z, R)$$

$$\mathrm{Ext}(Z_m, R) = 0$$

For any abelian group Ext(Z, G) = 0 and Ext(Z_n, G) = $G/_{nG}$; for finitely generated T and N, Ext(T, N) = $\tau(T) \otimes N$.

ELEMENTS OF SINGULAR HOMOLOGY AND COHOMOLOGY

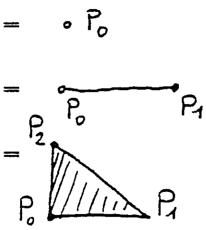
If we denote with R^∞ the space $\bigcup_0^\infty R^n$, let $P_i = (0....1....0..)$ the i-th basis vector and $P_0 = (0\ 0\ 0\ 0\ ......)$.
The q-simplex Δq is the set:

$$\Delta_q = \left\{ \sum_0^q t_i\ P_i,\ \sum_0^q t_i = 1 \right\} \quad 0 \leqslant t_i \leqslant 1$$

$\Delta_0 =$

$\Delta_1 =$

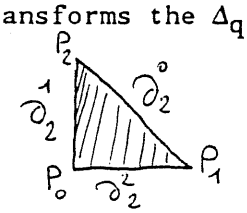
$\Delta_2 =$



The i-th face map $\partial^i{}_q : \Delta_{q-1} \to \Delta_q$

$$\partial^i{}_q \left[\sum_0^{q-1} t_j\ P_j \right] = \sum_{j=0}^{i-1} t_j\ P_j + \sum_{j=i+1}^{q} t_j\ P_j$$

transforms the Δ_{q-1} simplex into the faces of Δ_q.



$$\begin{cases} \partial^0{}_2(\Delta_1) = \partial^0{}_2(t_0P_0 + t_1P_1) = t_1P_1 + t_2P_2 \subset \Delta_2 \\ \partial^1{}_2(\Delta_1) = t_0P_0 + t_2P_2 \subset \Delta_2 \\ \partial^2{}_2(\Delta_1) = t_0P_0 + t_1P_1 \subset \Delta_2 \end{cases}$$

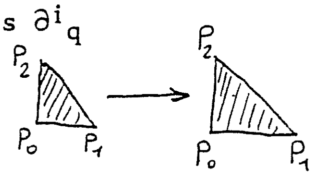
$$\begin{cases} \partial^0{}_1(\Delta_0) = P_1 \subset \Delta_1 \\ \partial^1{}_1(\Delta_0) = P_0 \subset \Delta_0 \end{cases}$$

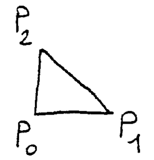
If X is a topological space, a *singular q-simplex* is a continuos map. $s : \Delta_q \to X$ and a *singular q-chain* is a *finite* linear combination with coefficient in Z of singular q-simplices. $S_q(X, Z)$ is the Abelian group (free Z-module) of singular q-chains.
There is a boundary operator ∂ such that $\partial^2 = 0$

$$\partial : S_q(X, Z) \to S_{q-1}(X, Z)$$

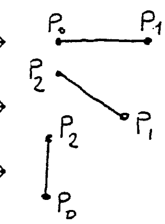
and we have a chain complex.

$$\partial s = \sum_{0}^{q} (-1)^i \, s \, \partial^i{}_q$$

Example: $X = R^2$ $s =$ identity s : 

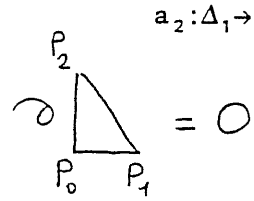
$\partial s = \partial^0{}_2 - \partial^1{}_2 + \partial^2{}_2 =$ 

$= a_0 + a_1 + a_2$

where $a_0 : \Delta_1 \rightarrow$, $a_1 : \Delta_1 \rightarrow$, $a_2 : \Delta_1 \rightarrow$ 

hence $\partial a_0 = a_0 \, \partial^0{}_1 - a_0 \, \partial^1{}_1$

$\partial a_0 \; : \; P_0 \rightarrow P_1 - P_0$
$\partial a_1 \; : \; P_0 \rightarrow P_2 - P_1$
$\partial a_2 \; : \; P_0 \rightarrow P_0 - P_2$

so 

and we have $\partial^2 = 0$.
The q-homology module of this complex is denoted by

$$H_q(X, Z) = \frac{Z_q(X, Z)}{B_q(X, Z)} \quad \text{and} \quad H_*(X, Z) = \oplus_{q \geq 0} H_q(X, Z)$$

Example: if X is a connected manifold $H_0(X, Z) = Z$.

Example: $H_q(*, Z)$ $*$ a point.
There is a unique q-simplex $\sigma_q \; \forall_q$ (the constant map); we have:

for $q = 0$ $Z_0 = S_0$ and $B_0 = 0$ hence $H_0(*, Z) = Z$
for $q > 0$

$\partial\sigma_1 = \sigma_1 \, \partial_1{}^0 - \sigma_1 \, \partial_1{}^1 = 0$ $\qquad \sigma_1 : \Delta_1 \rightarrow *$

$\partial\sigma_2 = \sigma_2 \, \partial_2{}^0 - \sigma_2 \, \partial_2{}^1 + \sigma_2 \, \partial_2{}^2 = \sigma_2 \, \partial_2{}^2$ $\qquad \sigma_2 : \Delta_2 \rightarrow *$

but

$$\sigma_2 \, \partial_2{}^2 \, \Delta_1 = \sigma_2(\Delta_1) = * = \sigma_1(\Delta_1)$$

in general: $\partial\sigma_q = \begin{cases} \sigma_{q-1} & q \text{ even} > 0 \\ 0 & q \text{ odd} \end{cases}$

so $Z_q = \{\sigma_q \ / \ \partial\sigma_q = 0\} = \begin{cases} 0 & q \text{ even} > 0 \\ S_q & q \text{ odd} \end{cases}$

$B_q = \{\sigma_q \ / \ \sigma_q = \partial\sigma_{q+1}\} = \begin{cases} S_q & q \text{ odd} \\ 0 & q \text{ even} > 0 \end{cases}$

Hence $H_q(*, Z) = 0$

Example: $H_q(R^n)$ $\quad q > 0 \quad$ $(H_0(R^n) = Z)$

$$\begin{array}{ccccc} \to S_q(R^n, Z) & \to & S_{q-1}(R^n, Z) & \to \\ \pm 1 \downarrow \ \downarrow 0 & & \pm 1 \downarrow \ \downarrow 0 & \\ \to S_q(R^n, Z) & \to & S_{q-1}(R^n, Z) & \to \end{array}$$

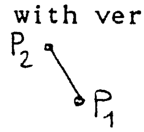
we have a map $K : S_q \to S_{q+1}$ such that

$$\partial K - K\partial = (-1)^{q+1}$$

so ± 1 and 0 are homotopic (as chain homomorphisms) and $H_p(\pm 1) = H_p(0) = 0$ but $H_p(\pm 1)$ is an isomorphisms and so $H_p(R^n, Z) = 0$. K is said the cone map.
The formula is complicated:

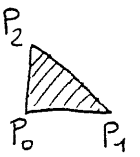
$$Ks\left[\sum_{j=0}^{q+1} t_j P_j\right] = \left[1 - t_{q+1}\right] s \left[\sum_{j=0}^{q} \frac{t_j}{1 - t_{q+1}} P_j\right]$$

this is the *cone* in R^n with vertex in 0 and base s.

Example s: $\Delta_1 \to$ 

$= t_1P_1 + t_2P_2$

$$Ks(t_0P_0 + t_1P_1 + t_2P_2) = (1 - t_2)\ s\left[\frac{t_0P_0}{1-t_2} + \frac{t_1P_1}{1-t_2}\right]$$

Ks(Δ_2) = 

when $t_2 = 0$ the cone start from s
when $t_2 = 1$ the cone ends up to 0

We "verify" that $\partial K - K\partial = 1$ in this case

∂K $= \partial$  $=$ 

$K\partial$ $= K(\cdot\ \cdot\ ) =$  and 

Topological invariance

Let $f : M \to N$ a continuos map, if $\sigma \in S_q(M, Z)$, $f\ \sigma \in S_q(N, Z)$ and we obtain a homomorphism $H_q(f) : S_q(M, Z) \to S_q(N, Z)$ by $H_q(f)(\sum \nu_\sigma \sigma) = \sum \nu_\sigma(f\ \sigma)$

Now $\partial\ H_q(f) = H_{q-1}(f)\ \partial$ infact recall that $\partial s = \sum(-1)^i\ s\ \partial_q^{\ i}$ so

$$\partial\ H_q(f)(s) = \sum(-1)^i (f\ s)\ \partial_q^{\ i}$$

$$H_q(f)\partial s = \sum(-1)^i f\ (s\ \partial_q^{\ i}).$$

Moreover, $H_q(gf) = H_q(g) \cdot H_q(f)$.

Hence $H_q(f) : H_q(M, Z) \to H_q(N, Z)$

and if $f : M \to N$ is a homomeorphism, $H_q(f)$ is an isomorphism.

Homotopical invariance. As for the de Rham modules, also $H_n(M, Z)$ are homotopy invariants:
if f and g are homotopic maps $M \to N$ then $H_q(f) = H_q(g)$. In particular if M and N have the same homotopy type then $H_q(M, Z) = H_q(N, Z)$. Infact if $f : M \to N$ and $g : N \to M$ and both $f \circ g$ and $g \circ f$ are homotopic to an identity, it follows that $H_q(f)$ and $H_q(g)$ are isomorphisms in homology.
The homotopical invariance is proved along the same way used for de Rham.
Consider the maps $s_t : M \to M \times I$ $\quad s_t(x) = s(x, t)$
$s_0 \approx s_1$. Let now $F : X \times I \to X'$ the homotopy from f to g.

$$F \circ s_0 = f, \qquad F \circ s_1 = g.$$

It is sufficient to prove the theorem for s_0 and s_1: infact

$$H_q(f) = H_q(Fs_0) = H_q(F)\,H_q(s_0) = H_q(F)H_q(s_1) = H_q(g).$$

To prove it for s_0 and σ, we construct an operator:

$$P : S_q(X, Z) \to S_{q+1}(X \times I, Z) \qquad \text{such that}$$

$$\partial P + P\partial = S_q(s_1) - S_q(s_0)$$

P gives a chain homotopy between $S_q(s_1)$ and $S_q(s_0)$.

$$P(\sigma) = S_{q+1}(\sigma \times \mathrm{id})(P(\delta_q)) \qquad \text{where } \delta_q : \Delta_q \to \Delta_q$$

(note that $S_q(\sigma)(\delta_q) = \sigma$)

P is a generalization of the *cone* operator and is said the *prism* operator. To explain the definition of P, first note that really $P(\sigma) : \Delta_{q+1} \to X \times I$ infact

$$\delta_q \in S_q(\Delta_q, Z), \qquad P(\delta_q) \in S_{q+1}(\Delta_q \times I, Z)$$

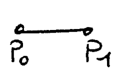
$$\sigma \mathrm{xid} : \Delta_q \times I \to X \times I$$

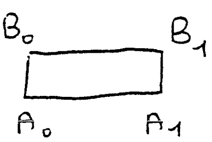
$$S_{q+1}(\sigma \mathrm{xid}) : S_{q+1}(\Delta_q \times I, Z) \to S_{q+1}(X \times I, Z)$$

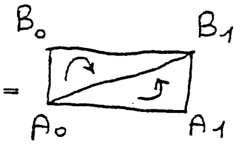
The definition of $P(\delta_q)$ is the following:
Denote $\Delta_q = (P_0, P_1 \dots P_q)$ where P_0, P_1, P_q are the vertices of Δ_q.

$\Delta_q \mathrm{x} I$ has vertices $\underset{A_0}{(P_0,0)}$, $\underset{A_1}{(P_1,0)}$, $\underset{A_q}{(P_q,0)}$, $\underset{B_0}{(P_0,1)} \underset{B_1}{\dots\dots} \underset{B_q}{\dots}$

$$P(\delta_q) = \sum_1^q (-1)^i (A_0 \dots A_i\ B_i \dots B_q)$$

$q = 1 \qquad \delta_1 = P_0\,P_1$ 

$\Delta_1 \mathrm{x} I =$ 

$$P(\delta_1) = (A_0\ B_0\ B_1) - (A_0\ A_1\ B_1) =$$ 

It is left as exercise to verify the chain homotopy property.
We obtain also as corollary that

$$H_p(M \times R) = H_p(M).$$

Mayer Vietoris for homology

We quote the exact sequence of Mayer–Vietoris.
If $X = U \cup V$, U, V open sets, the exact sequence is :

1) $$\ldots \to H_q(U \cap V) \xrightarrow{H_q(f)} H_q(U) \oplus H_q(V) \xrightarrow{H_q(g)} H_q(X) \xrightarrow{\partial_X} H_{q-1}(U \cap V) \to \ldots$$

where $H_q(f)$ is induced by $a \to (-a, a)$
and $H_q(g)$ is induced by $(a, b) \to (a + b)$.

1) is induced by

$$0 \to S_q(U \cap V) \xrightarrow{f} S_q(U) \oplus S_q(V) \xrightarrow{g} \hat{S}_q(X) \to 0$$

$\hat{S}_q(X)$ are the *small* chains i.e. chains made up of simplices each of which lies in U or in V. It can be shown that $\hat{H}_q = H_q$.

Application to S^1 = 

$$0 \to H_1(S^1, Z) \to Z \oplus Z \xrightarrow{f} Z \oplus Z \xrightarrow{g} H_0(S^1, Z)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad Z$$

$$f(a, b) = (a - b, b - a)$$

and $H_1(S^1, Z) = \text{Ker } f = Z$

and $H_p(S_1) = 0 \qquad p \geqslant 2$.

Application to S^2 and S^n $\qquad S^2 =$ 

$\qquad U \cap V \approx S^1 \times R$

We have: $H_0(S^2) = Z$

$$H_2(U)\oplus H_2(V) = 0 \to H_2(S^2) \to H_1(S^1 \times R) \to H_1(U)\oplus H_2(V)$$
$$H_1(S^1 \times R) = Z \qquad H_1(U)\oplus H_2(V) = 0$$

$H_2(S^2) = Z$

$$H_1(U)\oplus H_1(V) \to H_1(S^2) \to H_0(S^1 \times R) \xrightarrow{f} H_0(U)\oplus H_0(V) \to H_0(S^2)$$
$$H_1(U)\oplus H_1(V) = 0 \qquad H_0(S^1 \times R) = Z \qquad H_0(U)\oplus H_0(V) = Z\oplus Z \qquad H_0(S^2) = 0$$
$$a \mapsto (-a, a)$$

$H_1(S^2) = \text{Ker } f = 0$

$H_p(S^2) = 0 \qquad \forall\, p \geq 3$

by induction $\quad S^n =$ 

$\quad U \cap V \approx S^{n-1} \times R$

$H_0(S^n) = H_n(S^n) = Z \qquad H_p(S^n) = 0$.

Finite "dimensionality" of $H_p(M)$, (M compact)

In the case M compact, Mayer-Vietoris (as for de Rham) says that the "dimension" of $H_p(M)$ is finite. In this context the word dimension is wrong because $H_p(M)$ is a Z module and not a vector space. We must speak of *rank*. The *rank* of a Z module A is the minimal number of generators of $A/_{\tau(A)}$. Examples Z has rank = 1, $Z\oplus Z$ has rank = 2, $Z\oplus Z_2$ has rank = 1

Homology with coefficients

If A is any abelian group, we can define also $S_q(X, A)$ the free Z module of chains with coefficients in A.
Accordingly we have $H_q(X, A)$.
We quote the universal coefficient theorem, which comes from homological algebra:

$$H_q(X, A) = H_q(X, Z)\otimes A \oplus \text{Tor}(H_{q-1}(X), A)$$

we have also

$$H_q(X, R) = H_q(X, Z) \otimes R$$

Remark: rank of $H_q(X, Z)$ = dim of $H_q(X, R)$

if dim $H_q(X, R) = \beta_q$ β_q are called Betti's numbers and (if $< \infty$) $\chi(X) = \sum_q (-1)^q \beta_q$ is the Euler characteristic.

Relations between homotopy groups and homology groups

The relation between π_1 and H_1 is easy to describe.
a) There is a homomorphism $\chi : \pi_1(X, x_0) \to H_1(X, Z)$ which sends $[\gamma] \in \pi_1$ into the homology class of the singular simplex γ.

b) If X is path connected, χ *is surjective* and Ker $\chi = [\pi_1, \pi_1]$ the subgroup of commutators.
No statement can be made in general when $q > 1$. The following theorem of Hurewicz gives some informations. We give a weak version of it.
If X is a simply connected compact manifold then $\forall$ $r \geqslant 2$ if $\pi_q(x) = 0$ for $1 \leqslant q < r$ then $H_q(X)=0$ for $1 \leqslant q < r$ and $H_r(X)=\pi_r(X)$ and viceversa. The first non trivial π_r or H_r occur in the same dimension and are equal. (This theorem holds also when X is path connected, simply connected *and* is homotopy equivalent to a space with a good cover).

Application: $\pi_q(S^n) = 0$ for $0<q<n$ and $\pi_n(S^n) = Z$ because we know that $H_q(S^n) = 0$ for $0<q<n$ and $H_n(S^n) = Z$.

Singular cohomology

A singular q–cochain on X is a linear functional on the Z module $S_q(X, Z)$;

$$S^q(X, Z) = \mathrm{Hom}(S_q(X), Z)$$

$$S^q(X, Z) \stackrel{\delta}{\to} S^{q+1}(X, Z) \stackrel{\delta}{\to} \ldots$$

is a cochain complex with $\delta\omega(c) = \omega(\partial c)$, $\delta^2 = 0$ and

we have $$H^q(X, Z) = \frac{\mathrm{Ker}\delta}{\mathrm{Im}\delta} = \frac{Z^q(X, Z)}{B^q(X, Z)} = \frac{\text{cocycles}}{\text{coboundaries}}$$

Example: $H^0(X, Z) = Z$ if X is path connected; infact $\delta\omega = 0$ (ω on X) if $\omega(\partial c) = 0$ $\forall$ c and ω is constant on each path–component.

We now compute

$$H^q(R^n, Z) \qquad q \geqslant 1$$

Define L: $S^q(R^n, Z) \to S^{q-1}(R^n, Z)$ to be the adjoint of the cone map K : $S_q(R^n, Z) \to S_{q+1}(R^n, Z)$ defined before

$$L\sigma(c) = \sigma(K(c))$$

Then we have:

$$(\delta L - L\delta)\sigma(c) = \sigma((K\partial - \partial K)c) = ((-1)^{q+1}\sigma)c$$

hence

$\delta L - L\delta$ is a homotopy operator between ±1 and 0

and hence $\qquad H^p(R^n, Z) = 0 \qquad p \geqslant 1.$

Cohomology with coefficients

If A is any abelian group we can also define $H^p(X, A)$ by considering $S^q(X, A)$ i.e. cochains with coefficients in A (Z–linear functionals on $S_q(X, A)$).
The universal coefficient theorem gives:

$$H^q(X, A) = \mathrm{Hom}(H_q(X, Z), A) \oplus \mathrm{Ext}(H_{q-1}(X, Z), A).$$

In particular if $H_q(X, Z)$ and $H_{q-1}(X, Z)$ are finitely generated Z modules, putting A = Z:

$$H^q(X, Z) = \mathrm{Hom}(H_q(X, Z), Z) \oplus \text{torsion part of } H_{q-1}(X, Z)$$

We have also:

$H^q(X, R) = \mathrm{Hom}(H_q(X, Z), R) = H_q(X, R)$ as dual vector spaces.

(Note that if $H_q(X, Z) = Z \oplus Z \oplus \ldots \oplus \oplus_i Z_{ai}$

$$\mathrm{Hom}(Z \oplus Z \oplus \ldots \oplus_i Z_{ai}, R) = \mathrm{Hom}(Z \oplus Z \oplus \ldots Z, R) = R \oplus R \ldots \oplus R$$
$$H_q(X, R) = (Z \oplus \ldots \oplus_i Z_{ai}) \otimes R = R \oplus R \ldots \oplus R).$$

We have also:

$$H^q(X, \mathbb{R}) = H^q(X, \mathbb{Z}) \otimes \mathbb{R}$$

De Rham Theorem

We quote the result: if X is a compact orientable manifold without boundary

$$H^q(X, \mathbb{R}) = H_{DR}{}^q(X)$$

The isomorphism is given by:
given a q-form ω representing a class in $H^q{}_{DR}(X)$ we obtain a q-cochain ω representing a class in $H^q(X, \mathbb{R})$ by

$$\omega(c) = \int_c \omega \qquad \text{where c is a q-chain.}$$

Note $\delta\omega(c) = \omega(\partial c) = \int_{\partial c} \omega = \int_c d\omega = 0$.
This result,via the Chech cohomology (see the lectures of C.Reina) can be proved also for general manifolds.

Remark on integer classes

An integer class in $H_{DR}{}^n(X)$ is a class such that gives an integer when integrated on an integer c.
Note that {integer classes in $H_{DR}{}^n(X)\} \neq H^n(X, \mathbb{Z})$ because $H_{DR}{}^n(X) = H^n(X, \mathbb{R}) = H^n(X, \mathbb{Z}) \otimes \mathbb{R}$ and torsion elements are lost during the embedding $H^n(X, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \otimes \mathbb{R}$ $(a \to a \otimes 1)$.
We give now an example of this fact.

$$SO(3) = \frac{SU(2)}{\mathbb{Z}_2} \qquad \text{we know that:}$$

$\pi_0(SO(3)) = 0$ and $\pi_1(SO(3)) = \mathbb{Z}_2$ so $H_0(SO(3), \mathbb{Z}) = \mathbb{Z}$

$H_1(SO(3), \mathbb{Z}) = \mathbb{Z}_2$; we have also $H_2(SO(3), \mathbb{Z}) = 0$.

So $H^2(SO(3), \mathbb{Z}) = \mathbb{Z}_2$ while $H_{DR}{}^2(SO(3)) = 0$.

This is a general consequence of the fact that π_1 is cyclic. If, for example, $\pi_1(M) = \mathbb{Z}_p$ where M is a manifold, $H_1(M, \mathbb{Z})$ has torsion and also $H^2(M, \mathbb{Z})$ has torsion.

TWO APPLICATIONS

Classification of Bundles

It is known that in physics fields are to be regarded as sections of suitable bundles associated to the principal bundle on which the gauge field is a connection. So the problem of the topological classification of bundles is crucial, because it shows the nature of the topological charges in the theory.

Recall that a fiber bundle (E, π, M) with fiber F, base space M and group G acting on F on the left is required to be locally trivial i.e. $\forall x \in M$, $\exists U \subset M$ and $h_U : U \times F \to \pi^{-1}(U)$ omeomorphically such that $\pi(h_U(x, v) = x \quad \forall x \in U, \forall V \in F$.

If V is another open containing x such that $h_V : V \times F \to \pi^{-1}(V)$, then there exists $g_{VU} : x \to G$ such that $h_U(x, v) = h_V(x, g_{VU}(x)v)$.

For *vector bundles* the fiber F is a vector space and G is required to act linearly. For *principal bundles* the fiber F is = G and G acts on itself by the group law.

We will consider all objects as topological manifold or topological groups.

$\{g_{VU}\}$ satisfying $g_{VU} = g_{UV}^{-1}$ and $g_{VU}\, g_{UZ}\, g_{ZV} = 1$ is called *a system of transition function* associated to the covering U, V, Z...

Two systems are called equivalent if there exist a set of functions $r_U : U \to G$ such that

$$g'_{VU} = r_V^{-1} g_{VU} r_U \,.$$

It is known that a bundle is given (modulo isomorphisms) by a system of transition functions and, viceversa, a system of transition functions (modulo equivalence) gives a bundle.

In simple cases, this observation allows a classification of bundles.

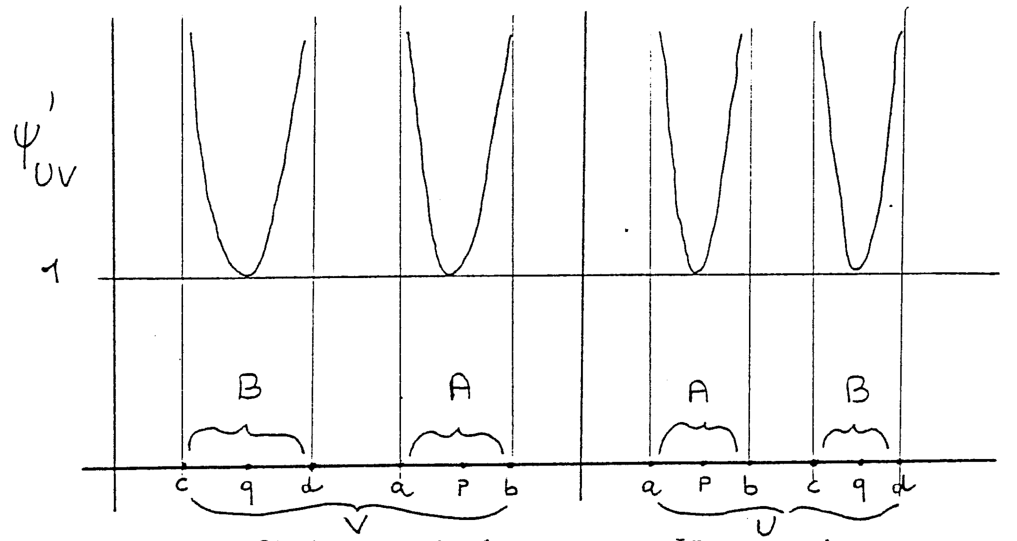
Example: line bundles over S^1 (Fiber $\mathbb{R}$, group $\mathbb{R}^*$) cover S^1 with two open sets $\quad A \cup B = U \cap V$

we have two bundles:

The trivial one is given by $g_{UV}(x) = 1$ and a non trivial one is given by $g'_{UV}(x) = \begin{cases} 1 & x \in A \\ -1 & x \in B \end{cases}$

We want to show that any other is isomorphic to one of this two.

If we have any bundle given by $\Psi_{UV}(x) \in \mathbb{R}^*$ if $\Psi_{UV|A}$ and $\Psi_{UV|B}$ have the same sign (e.g.+), we can find an equivalent system Ψ'_{UV} such that $\Psi_{UV}(p) = \Psi'_{UV}(b) = 1$ where $p \in A$ and $q \in B$:



Now we can find an equivalent system Ψ''_{UV} putting:

$$r'_V = \begin{cases} \Psi'_{UV} & \text{in } (c, q] \\ 1 & \text{in } [q, p] \\ \Psi'_{UV} & \text{in } [p, b) \end{cases} \qquad r'_U = \begin{cases} 1/\Psi'_{UV} & \text{in } (a, p] \\ 1 & \text{in } [p, q] \\ 1/\Psi'_{UV} & \text{in } [q, d) \end{cases}$$

so $\Psi''_{UV} = r'^{-1}_V \Psi'_{UV} r'_U = 1 = g_{UV}$ and the bundle is trivial.
If the sign is different, an analogous procedure will give that $\Psi''_{UV} \approx g'_{UV}$.
The general case, for arbitrary M (base) and G (group) is a formidable task and we will limit to a few cases. We call $B_G(M)$ the set of isomorphism classes of G–bundles over M.
Before proceeding it is useful to note that with the tensor product the line bundles form a group (identity is the trivial bundle and the inverse is the bundle given by the inverse of the transition function) (recall that the transition function of the tensor product of line bundle is the product of the transition functions of the two line bundles). We have just shown that (line bundles over S^1) = Z_2.
Note that also $\pi_0(R^*) = Z_2$.
This is a general fact: $B_G(S^n) = \pi_{n-1}(G)$ (Do not worry here about the group structure of $B_G(S^n)$).
Recall that if f is a map $M \to N$ and E a bundle over N, $f^*E = \{(x, p) \in M \times E \;/\; f(x) = \pi(p)\}$ is the pull–back bundle on M. An important result is that if $f \approx g$ then $f^*E \approx g^*E$, thus we have a "map" from [M, N] into the set of G–bundles over M. ($[f] \to f^*E$).
Definition: A n–universal G bundle (EG, π, BG) is a principal G bundle such that the map

$$[M, BG] \to B_G\, M$$

is one to one and onto for any n-dimensional manifold.
n–Universal bundles *exist* for any n and any G and it can be shown that a bundle (E, π, M) is n–universal if $\pi_0(E) = \pi_1(E) = \dots \pi_n(E) = 0$.
There exist also ∞–universal bundles called simply universal bundles. We do not enter into the construction of EG and BG but only give examples.
1–Universal bundle for $G = Z_2$. We have $RP^2 = S^2/Z_2$ so S^2 is the total space of a principal Z_2 bundle over RP^2. Since $\pi_0(S^2) = \pi_1(S^2) = 0$ this bundle is 1–universal so $[S^1, RP^2] = B_Z(S^1)$ i.e. $B_Z(S^1) = \pi_1(RP^2) = Z_2$ as we already knew. (The group R^* can be always reduced to Z_2).

2–Universal bundle for G = U(1): the Hopf bundle is 2–universal

$$S^1 \to S^3 \downarrow S^2 \qquad \text{so} \qquad B_{U(1)}(S^2) = [S^2, S^2] = Z$$

The long exact homotopy sequence for the universal bundle (EG, π ,BG) gives (recall $\pi_n(EG) = 0 \quad \forall n$)

$$0 \to \pi_n(BG) \to \pi_{n-1}(G) \to 0 \qquad n \geq 1$$

and so $\pi_n(BG) = \pi_{n-1}(G)$

Hence, since for spheres $[S^n, BG] = \pi_n(BG)$, we have

$$B_G(S^n) = \pi_{n-1}(G)$$

To know that universal bundles exist in general is interesting but the structure of BG is usually so complicated that [M, BG] cannot be computed. An important exception is the case in which BG is an Eilemberg–Maclane space.
An Eilemberg–Maclane space is K(A, n) which satisfy:

1) K is a path–connected topological space, n an integer ≥ 1, A an abelian group.
2) $\pi_1(K(A, n)) = A$ if $i = n$, $= 0$ if $i \neq n$.

It can be shown that

$$[M, K(A, n)] = H^n(M, A)$$

Examples: Consider the fiber bundle $Z_2 \to S^n \downarrow RP_n$

The exact homotopy sequence gives

$$\pi_1(RP^n) \approx Z_2$$
$$\pi_i(RP^n) \approx 0 \quad \text{if} \quad 1<i<n$$

Now to the sets $S^\infty = U^\infty{}_{n=1}S^n$, $RP^\infty = U^\infty{}_{n=1}\ RP^n$, can be given a topology and it can be shown that $\pi_i(S^\infty) = 0 \quad \forall i$. From the homotopy sequence we have $\pi_1(RP^\infty) = Z_2$, $\pi_i(RP^\infty) = 0 \quad i\neq 1$.
So S^∞ is a universal Z_2–bundle, and $RP^\infty = K(Z_2, 1)$ and $B_Z(M) = [M, BZ_2] = [M,(Z_2, 1)] = H^1(M, Z_2)$.
On the other hand

$$\begin{array}{ccc} U(1) & \to & S^\infty \\ & & \downarrow \\ & & CP^\infty \end{array}$$

is a universal U(1)-bundle and moreover $CP^\infty = K(Z, 2)$.
(This follows from the bundle $S^{2n-1} \to CP^{n-1}$

$$(z_1 \ \ldots\ldots\ z_n) \to [z_1 \ \ldots\ldots\ z_n])$$

So $\quad B_{U(1)}(M) = [M, BU(1)] \doteq [M, K(Z, 2)] = H^2(M, Z)$

The element in $H^2(M, Z)$ corresponding to a particular bundle is said the *first (topological) Chern class*. Note that the Chern class usually obtained by means of the curvature form of a connection is an integer class in $H_{DR}{}^2(M)$ and in general does not classifies U(1) bundles (see the remark on integer classes).

Cohomology and Anomalies in Gauge Theories

In this section we briefly describe an application of algebraic topology to the problem of anomalies in gauge theories (for details see R. Catenacci, G.P Pirola, C. Reina and M. Martellini Phys. Lett. 172B 1986).
We recall that the main object of study is the vacuum functional W(A) for chiral fermions in an external gauge potential A.
W(A) is a smooth complex valued functional $W : \mathcal{A} \to C$ defined on the space $\mathcal{A}$ of all gauge potentials. $\mathcal{A}$ carries a principal action of the group G of pointed gauge transformation; the map $A \to A\ \ g = g^{-1}Ag + g^{-1}dg$ is free and the projection $\pi : \mathcal{A} \to \mathcal{A}/g = \Theta$ over the space of orbits is a principal bundle. We say that an anomaly is present if W is not invariant under the action $A \to Ag$. Algebraic topology allows a general study of the possible transformation properties of W. We can consider W as a section of a trivial line bundle over $\mathcal{A}$ and when G acts on $\mathcal{A}$, the

corresponding actions on the bundle are of the form

$$(A, W) \rightarrow (Ag, f(A, g) W)$$

where $f(A, g) \in \mathbb{C}^*$.

Associativity requires $f(A, g_1 g_2) = f(Ag_1, g_2) f(A, g_1)$. Moreover two actions $f(A, g)$, $f'(A, g)$ can be considered as equivalent if there exists a vertical automorphism of the bundle $H : \mathcal{A} \rightarrow \mathbb{C}^*$ such that

$$f'(A, g) = H(Ag)\ f(A, g)\ H(A)^{-1} .$$

We see that the problem of the classification of the functions f admits a cohomological interpretation: we define a 0–cochain a map $f : A \rightarrow \mathbb{C}^*$ and a 1–cochain a map $f : A \times G \rightarrow \mathbb{C}^*$ (and a n–cochain $\Psi(A, g_1, .. g_n)$. The coboundary operator δ is

$$(\delta h)(A, g) = h(Ag)\ h(A)^{-1}$$

$$\delta f(A, g_1, g_2) = f(Ag_1, g_2)\ f(A, g_1)\ f(A, g_1 g_2)^{-1}$$

It is obvious that $\delta^2 = 1$ (multiplicative notation!) and the trivial actions are the 1–coboundary; while the actions are the 1–cocycles. We have a cohomology group denoted by $H^1(G, \mathbb{C}^*)$.
It is not difficult to show that this group is isomorphic to $H^2(\Theta, Z)$. Infact to any equivalent class of f one can associate a line bundle over Θ and viceversa.
We recall now that line bundles over a space X are classified by $H^2(X, Z)$ (the topological chern classes).
It is known that for the interesting case of a SU(n) gauge theory on S^4, $H^2(\Theta, Z) \approx Z$ and so the anomalies are integer multiplies of a fundamental class. As a further example we will consider the fermionic vacuum functional in gauge theories with parity preserving fermionic content.
The vacuum functional W is now real and can be considered as a section of a real line bundle. Anomalies are now of the form $f : A \times G \rightarrow R^*$ where f is a class in $H^1(G, R^*)$. This group classifies real line bundles and is isomorphic to $H^1(\Theta, Z_2)$.
It is known that on S^4 with gauge group SU(2) $\pi_0(G) = Z_2$ and we get $\pi_1(\Theta) = \pi_0(G) = Z_2$ (recall that $\mathcal{A}$ is contractible). Thus:

$$H^1(\Theta, Z_2) = \mathrm{Hom}(H_1(\Theta, Z), Z_2) \oplus \mathrm{Ext}(H_0(\Theta, Z), Z_2) = \mathrm{Hom}(Z_2, Z_2) = Z_2 .$$