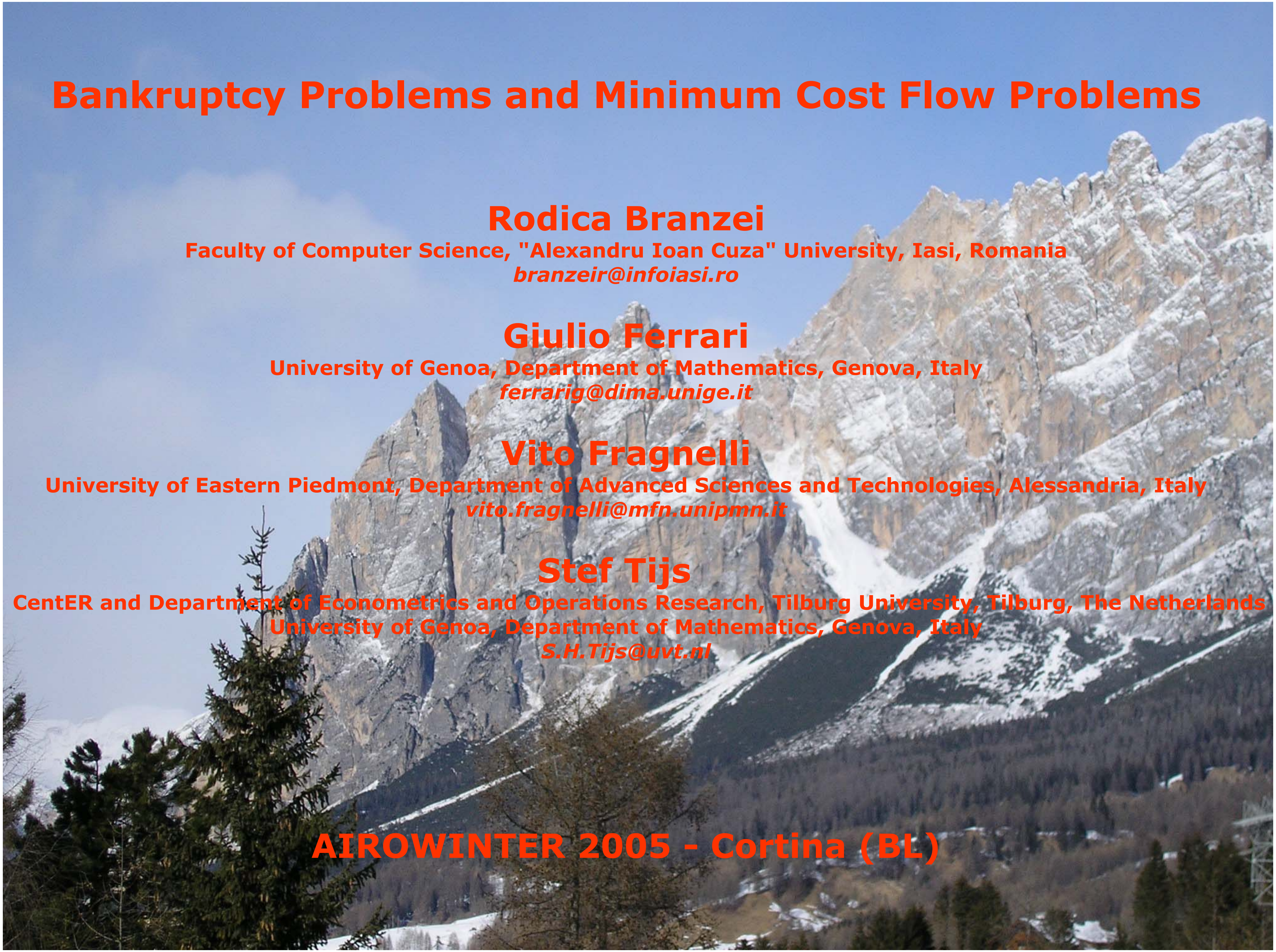

Bankruptcy Problems and Minimum Cost Flow Problems

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Outline

Joint Projects

Bankruptcy Problems and Related Flow Problems

Two Generalized Bankruptcy Problems

Bankruptcy Rules and Min Cost Flow Problems

Concluding Remarks

Joint Projects

Projects whose activities are carried out by different firms

Game theoretic approach of rationing problems arising from joint project management:

Bergantiños and Sánchez (2002): NTU games for allocating time

Branzei, Ferrari, Fragnelli and Tijs (2002): TU games for allocating costs

Castro and Tejada (2004): allocating time depending on "utility" (ground operations on aircrafts)

Branzei, Ferrari, Fragnelli and Tijs (2002): Divide the cost of the delay of a project among the activities that have a delay via bankruptcy approach

Two new questions:

- What to do if some activities recover part of the delay?
- What to do if an extra reward arises from an early termination of the project?

Bankruptcy Problems and Related Flow Problems

Classical bankruptcy problem

(N, E, c)

where	$N = \{1, ..., n\}$	set of claimants
	$E \in \mathbb{R}_+$	estate to be divided among claimants
	$c = (c_1, ..., c_n) \in \mathbb{R}^n_+$	claims, with $0 \leq E \leq C = c_1 + c_2 + ... + c_n$

A solution is a real vector $x = (x_1, ..., x_n)$ s.t. $\sum_{i \in N} x_i = E; 0 \leq x_i \leq c_i, \forall\ i \in N$

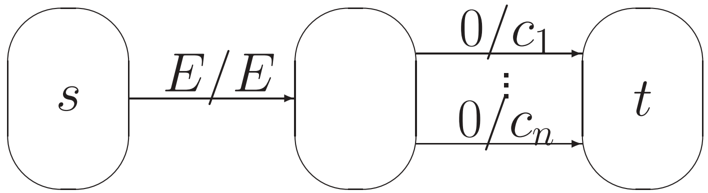
Standard flow problem

Network $G(N, A)$ with two nodes, the source s with no entering arcs and the sink t with no outgoing arcs; arcs have minimal and maximal capacity constraints

A flow is a function $x : A \rightarrow \mathbb{R}_+$ that respects the capacity constraints and such that

$$\sum_{j \in N} x_{ij} = \sum_{j \in N} x_{ji}, \forall\ i \in N \setminus \{s, t\}$$

A classical bankruptcy problem can be represented as a standard flow problem



Each feasible flow corresponds to a solution of the bankruptcy problem

Two Generalized Bankruptcy Problems

Claimants and debtors

- Generalized bankruptcy problem A - *The bank has an estate* E

(N_c, N_d, E, c, d)

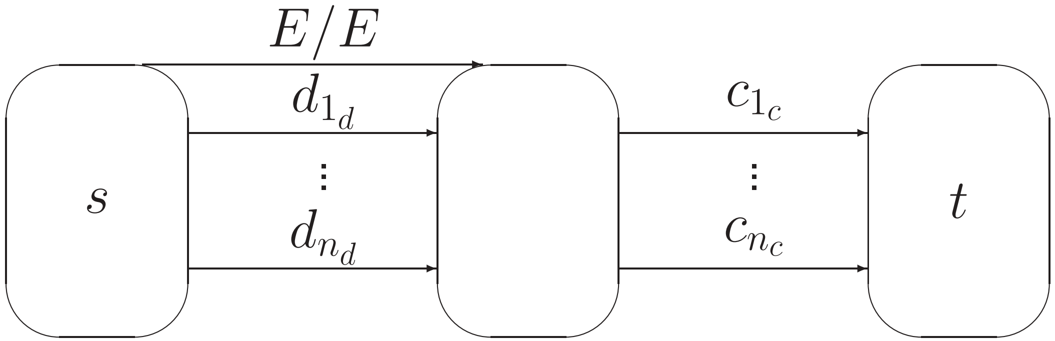
where $N_c = \{1_c, ..., n_c\}$ set of claimants

$N_d = \{1_d, ..., n_d\}$ set of debtors

$E \in \mathbb{R}_+$ estate

$c = (c_{1_c}, ..., c_{n_c}) \in \mathbb{R}_+^{n_c}$ claims, with $0 \leq E \leq C = c_{1_c} + ... + c_{n_c}$

$d = (d_{1_d}, ..., d_{n_d}) \in \mathbb{R}_+^{n_d}$ debts



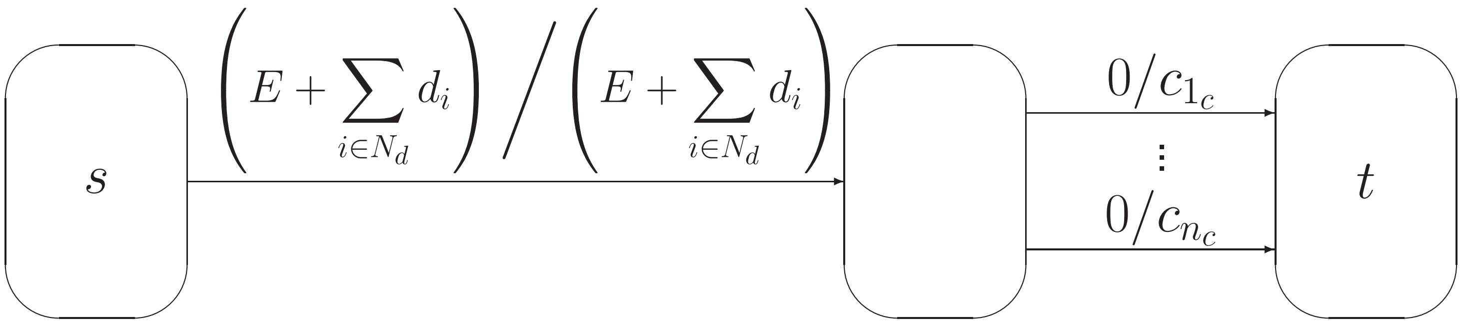
The values of E, c and d identify the following two subcases:

A1. $\sum_{i \in N_c} c_i \geq E + \sum_{i \in N_d} d_i$

A2. $E \leq \sum_{i \in N_c} c_i < E + \sum_{i \in N_d} d_i$

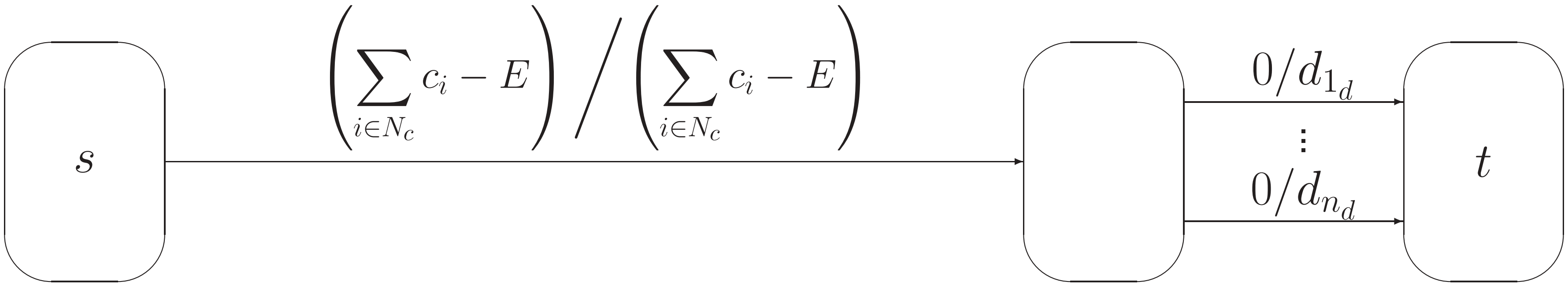
A1. (easy)

(N', E', c') with $N' = N_c$, $E' = E + \sum_{i \in N_d} d_i$, $c'_i = c_i, \forall\ i \in N_c$



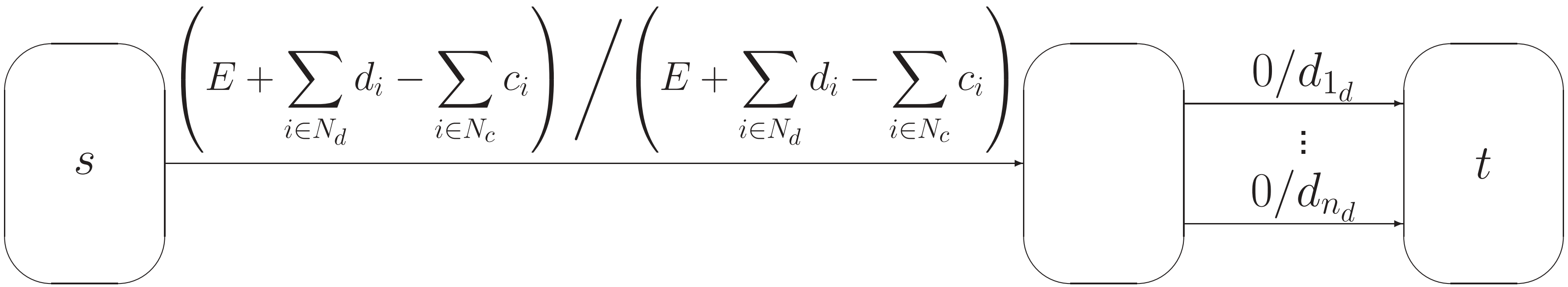
A2.i Allocate the quotas

$E' = \sum_{i \in N_c} c_i - E, N' = N_d, c'_i = d_i, \forall\ i \in N_d$



A2.ii Allocate the savings

$E' = E + \sum_{i \in N_d} d_i - \sum_{i \in N_c} c_i, N' = N_d, c'_i = d_i, \forall\ i \in N_d$

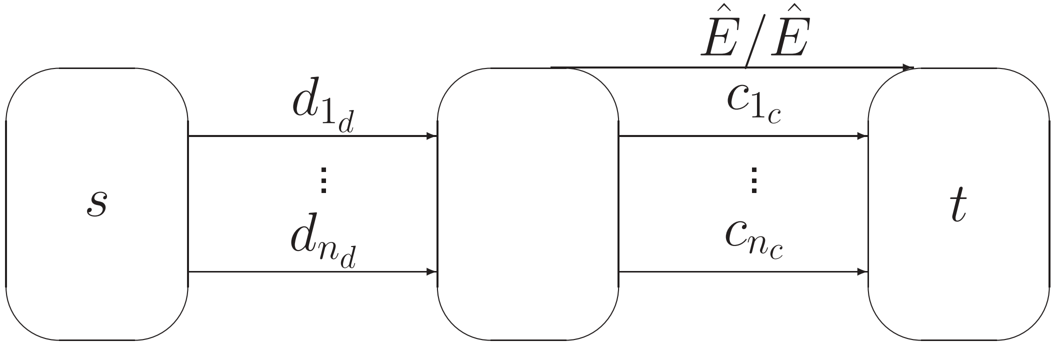


Example 1 *Consider a situation in which a bank has an estate* $E = 10$*; the set of claimants is* $N_c = \{1_c, 2_c, 3_c\}$ *with claim vector* $c = (5, 6, 7)$ *and the set of debtors is* $N_d = \{1_d, 2_d\}$ *with debt vector* $d = (1, 2)$*. This situation ends in subcase A1 and the corresponding classical bankruptcy problem is* (N', E', c') *with* $N' = \{1_c, 2_c, 3_c\}, E' = 13, c' = (5, 6, 7)$
If we consider an estate $E = 16$ *we are in subcase A2 and the corresponding classical bankruptcy problem is* (N', E', c') *with* $N' = \{1_d, 2_d\}, E' = 2, c' = (1, 2)$ *according to the approach A2.i or* (N', E', c') *with* $N' = \{1_d, 2_d\}, E' = 1, c' = (1, 2)$ *according to the approach A2.ii*

- Generalized bankruptcy problem B - *The bank has a debt* $\hat{E}$

$(N_c, N_d, \hat{E}, c, d)$

where	$N_c = \{1_c, ..., n_c\}$	set of claimants
	$N_d = \{1_d, ..., n_d\}$	set of debtors
	$\hat{E} \in \mathbb{R}_+$	debt of the bank
	$c = (c_{1_c}, ..., c_{n_c}) \in \mathbb{R}_+^{n_c}$	claims
	$d = (d_{1_d}, ..., d_{n_d}) \in \mathbb{R}_+^{n_d}$	debts, with $0 \leq \hat{E} \leq D = d_{1_d} + ... + d_{n_d}$



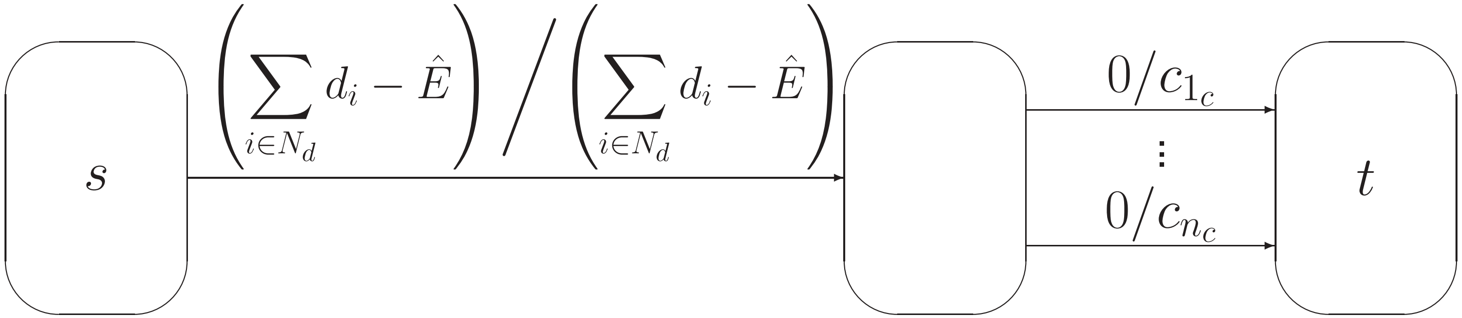
Again, the values of $\hat{E}$, c and d identify the following two subcases:

B1. if $\hat{E} \leq \sum_{i \in N_d} d_i < \hat{E} + \sum_{i \in N_c} c_i$

B2. if $\sum_{i \in N_d} d_i \geq \hat{E} + \sum_{i \in N_c} c_i$

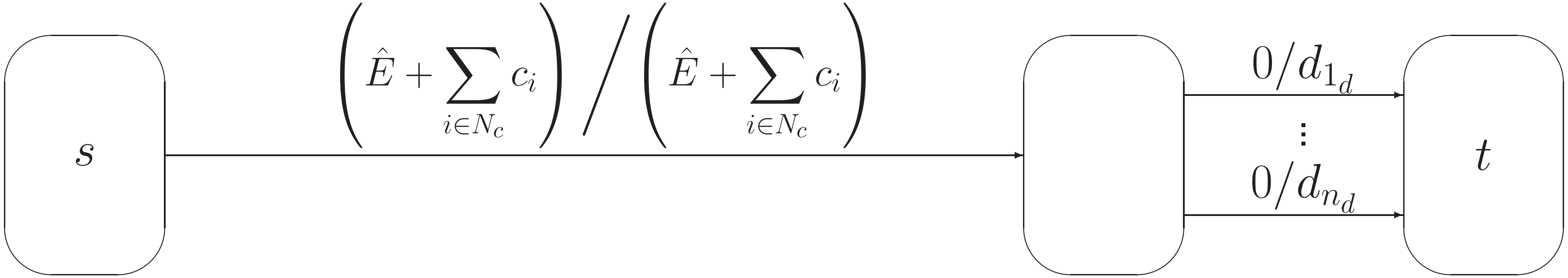
B1. (easy)

(N', E', c') with $N' = N_c$, $E' = \sum_{i \in N_d} d_i - \hat{E}$, $c'_i = c_i, \forall\, i \in N_c$



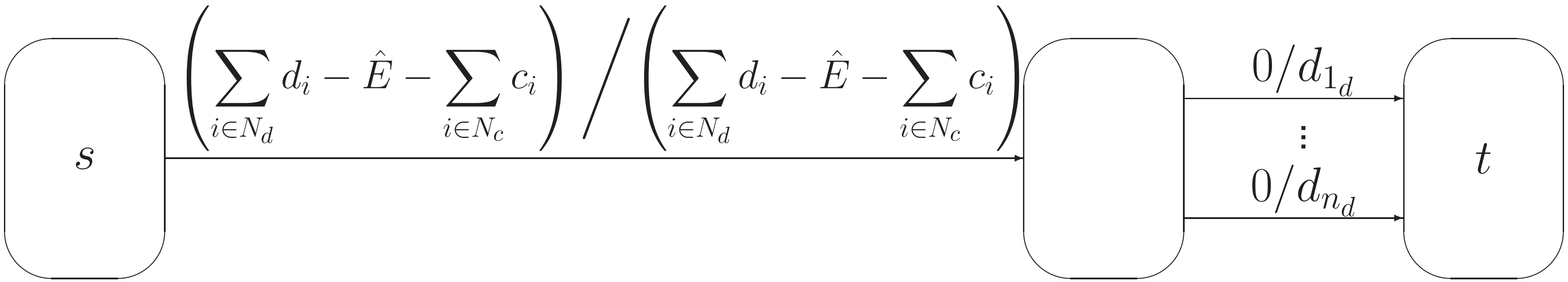
B2.i Allocate the quotas

(N', E', c') with $N' = N_d, E' = \hat{E} + \sum_{i \in N_c} c_i, c'_i = d_i, \forall\ i \in N_d$



B2.ii Allocate the savings

(N', E', c') with $N' = N_d, E' = \sum_{i \in N_d} d_i - (\hat{E} + \sum_{i \in N_c} c_i), c'_i = d_i, \forall\ i \in N_d$



Example 2 *Consider a situation in which a bank has a debt* $\hat{E} = 12$*; the set of claimants is* $N_c = \{1_c, 2_c, 3_c\}$ *with claim vector* $c = (3, 4, 5)$ *and the set of debtors is* $N_d = \{1_d, 2_d\}$ *with debt vector* $d = (8, 9)$*. This situation ends in subcase B1 and the corresponding classical bankruptcy problem is* (N', E', c') *with* $N' = \{1_c, 2_c, 3_c\}, E' = 5, c' = (3, 4, 5)$
If we consider a debt $\hat{E} = 2$ *we are in subcase B2 and the corresponding classical bankruptcy problem is* (N', E', c') *with* $N' = \{1_d, 2_d\}, E' = 14, c' = (8, 9)$ *according to the approach B2.i or* (N', E', c') *with* $N' = \{1_d, 2_d\}, E' = 3, c' = (8, 9)$ *according to the approach B2.ii*

Bankruptcy Rules and Min Cost Flow Problems

A division rule is a function f which assigns to any bankruptcy problem (N, E, c) a vector $f(N, E, c) \in \mathbb{R}^n$ such that:

$$0 \leq f_i(N, E, c) \leq c_i, \quad \text{for each } i \in N$$
$$\sum_{i \in N} f_i(N, E, c) = E$$

Well known division rules are the proportional rule (*PROP*), the constrained equal award rule (*CEA*), the constrained equal loss rule (*CEL*), the talmudic rule (*TAL*) and the adjusted proportional rule (*APROP*)

- $PROP_i(N,E,c) = \left(\sum_{j\in N} c_j\right)^{-1} c_i E,\ i \in N$
- $CEA_i(N,E,c) = \min\{c_i, \alpha\},\ i \in N$
  where α is the unique real number so that $\sum_{i\in N} CEA_i(N,E,c) = E$
- $CEL_i(N,E,c) = \max\{c_i - \beta, 0\},\ i \in N$
  where β is the unique real number so that $\sum_{i\in N} CEL_i(N,E,c) = E$
- $TAL_i(N,E,c) = \begin{cases} CEA_i(N,E,\frac{1}{2}c) & \text{if } E < \frac{1}{2}\sum_{j\in N} c_j \\ c_i - CEA_i(N,E',\frac{1}{2}c) & \text{if } E \geq \frac{1}{2}\sum_{j\in N} c_j \end{cases},\ i \in N$
  where $E' = \sum_{j\in N} c_j - E$
- The minimal right of each player $i \in N$ is $m_i(N,E,c) = \max\{0, E - \sum_{j\in N\setminus\{i\}} c_j\},\ i \in N$
  $APROP_i(N,E,c) = m_i(N,E,c) + PROP_i(N,E',c'),\ i \in N$
  where $E' = E - \sum_{j\in N} m_j$ and $c_i' = min\{E', c_i - m_i(N,E,c)\},\ i \in N$

A classical bankruptcy problem (N,E,c) and a division rule f can be related to a min cost flow problem

Theorem 1 *Let (N, E, c) be a classical bankruptcy problem. The division rules PROP, CEA, CEL and TAL can be implemented via a min cost flow problem.*

Proof. We only give for each rule a set of suitable cost functions

- $PROP$: $k_i(x_i) = \dfrac{x_i^2}{c_i}$
- CEA: $k_i(x_i) = x_i^2$
- CEL: $k_i(x_i) = (c_i - x_i)^2$
- For the Talmudic rule if $E < \frac{1}{2}\sum_{i \in N} c_i$:

$$TAL\text{: } k_i(x_i) = x_i^2,\ i \in N$$

  setting the maximal capacity of the arcs corresponding to the claimants to $\frac{1}{2}c_i$
  If $E \geq \frac{1}{2}\sum_{i \in N} c_i$:

$$TAL\text{: } k_i(x_i) = (c_i - x_i)^2,\ i \in N$$

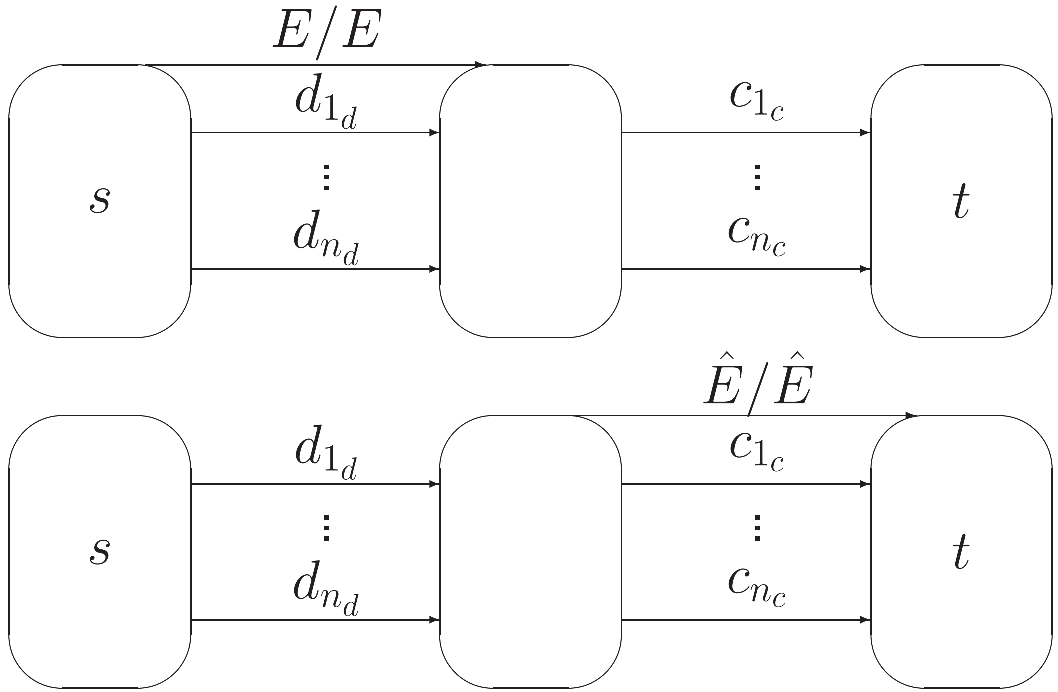
  setting the minimal capacity of the arcs corresponding to the claimants to $\frac{1}{2}c_i$ □

Remark 1 *Let* (N, E, c) *be a classical bankruptcy problem. The solution APROP*(N, E, c) *can be obtained via a min cost flow problem with set of cost functions:*

$$k_i(x_i) = \begin{cases} 0 & \text{if } x_i \leq m_i \\ \dfrac{(x_i - m_i)^2}{c_i' - m_i} & \text{if } x_i > m_i \end{cases}, \quad i \in N$$

Concluding Remarks

- Particular choices of cost functions in a min cost flow problem can lead to other rules from the bankruptcy literature, or suggest new divisions according to different fairness criteria, tailoring the amount on each agent



Bankruptcy approach “saturates” arcs related to claimants or debtors
Min cost flow approach may “drive” the solution towards non-saturation of these arcs

- A multi-claim bankruptcy problem and/or an interval bankruptcy situation can be represented by an appropriate flow problem

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Thanks!