Operations Research Games

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Outline

Operations Research and Game Theory

Operations Research, mainly Optimization, tackles decision problems looking for the most advantageous alternative (or alternatives), with respect to given objectives, satisfying the constraints arising from the situations at hand
Applications: business, politics, social sciences, economics ...

Operations Research tools may not catch all the features of those situations with multiple decision-makers
The different decision-makers may have different objectives (multi-objective programming)

If a suitable coordination of the choices allows improving the results that the agents may guarantee themselves acting separately, they have to share in a reasonable (fair) way their total gain

Example 1

A production process makes use of two resources, A and B; two agents, I e II, may both obtain the resources they need but at different unitary prices, as in the following table:

	cost for A (in euros)	*cost for B (in euros)*
I	11	6
II	4	10

Supposing that each unit of finite product is sold at the price of 20 euros, agent I and II get a unitary gain of 3 euros and 6 euros, respectively. If the two agents make an agreement they can use resource A at the cost of agent II and resource B at the cost of agent I, with a unitary gain of 10 euros □

The two agents obtain a larger profit acting jointly, but the profit should be shared in a satisfactory way for both of them
A trivial equal share of 5 euros each will be disadvantageous for agent II
How to divide the profit in order that all the agents accept to put in common their resources is the natural habitat for Game Theory

Summarizing, single-objective optimization has its natural application to those situations in which there is a single decision-maker, that looks for the best solution for himself, while Game Theory deals with those situations in which several decision-makers, the *players*, are able to influence the final outcome

The class of *Operations Research Games* arising from classical Operations Research problems with several decision-makers has gained an increasing interest among the scholars, due to the large number of real-world situations which it can be applied to

Recalls on TU Games

A TU Game in *characteristic function form* is a couple $G = (N, v)$ where $N = \{1, 2, ..., n\}$ is the *set of players*, $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$ is the *characteristic function*

$S \subset N$ is a *coalition* and N is the *grand coalition*
The characteristic function assigns to each coalition S the worth $v(S)$ that represents the utility that the players in S may obtain without the cooperation of the other players

An *allocation* or *payoff vector* is a vector $(x_i)_{i \in N} \in \mathbb{R}^n$ s.t. $\sum_{i \in N} x_i = v(N)$; this condition is called *efficiency*

The Core

The simplest set solution is the *imputation set*, that includes the allocations that satisfy the *individual rationality*: $x_i \geq v(\{i\}),\ \forall\ i \in N$

This solution concept accounts the opportunities that a player has when he does not coordinate with others but disregard the possibilities he has from cooperating with a subcoalition

In 1953 Gillies proposed a solution concept, the *core*, that starts from this remark

The core is the subset of the imputation set whose allocations satisfy the *coalitional rationality* or *collective rationality*, so:

$$core(v) = \left\{ (x_i)_{i \in N} \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq v(S),\ \forall\ S \subset N \text{ and } \sum_{i \in N} x_i = v(N) \right\}$$

The Shapley Value

The most popular point solution is the *Shapley value* (Shapley, 1953)
It is based on the *average marginal contribution* of each player with respect to all the possible orderings (permutations)
The marginal contribution of player i with respect to coalition S is $v(S \cup \{i\}) - v(S)$

The Shapley Value is:

$$\phi_i = \frac{1}{n!} \sum_{\pi \in \Pi} (v(P(\pi, i) \cup \{i\}) - v(P(\pi, i))), \quad \forall\, i \in N$$

where Π is the set of all possible permutations and $P(\pi, i)$ are the players preceding i in permutation π

The previous formula is computationally very complex
Sometimes, it is possible to exploit the structure of the game for reducing the complexity
The most important case is the airport game for which Littlechild and Owen (1973) provided a linear formula

Axiomatization of the Shapley Value

1. Symmetry
   If two players i, j are symmetric, i.e. $v(S \cup \{i\}) = v(S \cup \{j\}), \forall\ S \subseteq N \setminus \{i, j\}$ then $\psi_i(v) = \psi_j(v)$

2. Null player
   Let i be a null player, i.e. $v(S \cup \{i\}) = v(S), \quad S \subseteq N \setminus \{i\}$ then $\psi_i(v) = 0$

3. Additivity or independence (Debatable axiom)
   Given two games with player set N and characteristic functions v and u, respectively, let $(u + v)$ the sum game defined as $(u + v)(S) = u(S) + v(S),\ \forall\ S \subseteq N$ then $\psi_i(u + v) = \psi_i(u) + \psi_i(v), \forall\ i \in N$

ϕ is the unique efficient vector satisfying the previous axioms

There exist several set and point solutions:

- the *stable sets* (von Neumann and Morgenstern, 1944)
- the *bargaining set* (Aumann and Maschler, 1964)
- the *kernel* (Davis and Maschler, 1965)
- the *nucleolus* (Schmeidler, 1969)
- the solutions *ECA*, *ACA* and *CGA* (see Straffin and Heaney, 1981)
- the *power indices*: Shapley-Shubik (1954), Banzhaf (1965), Deegan-Packel (1978), Holler (1982)

Operations Research Games

Linear Production Games

In a linear production problem, a set of production processes $M = \{1, ..., m\}$ allows obtaining m goods, using k resources, each one available in a given amount $b_j, j = 1, ..., k$
Production processes are described through a matrix A called the *technology matrix*, whose entry $A_{lj}, l = 1, ..., m, j = 1, ..., k$ represent the quantity of units of resource j that are necessary for producing one unit of good l
Supposing that one unit of good l has a selling price $c_l, l = 1, ..., m$ we look for how many units of each good have to be produced in order to maximize the income, respecting the available resources
Let $y_l, l = 1, ..., m$ be the units of good l to be produced, so the problem can be written as:

$$\begin{array}{lll} max & \sum_{l=1,...,m} c_l y_l & \\ s.t. & \sum_{l=1,...,m} A_{lj} y_l \leq b_j & j = 1, ..., k \\ & y_l \geq 0 & l = 1, ..., m \end{array}$$

According to Owen (1975), it is possible to define a TU game in which the resources belong to the players of the set $N = \{1, ..., n\}$, each one endowed with an amount $b_j^i, i = 1, ..., n$ of resource $b_j, j = 1, ..., k$, with the condition $\sum_{i \in N} b_j^i = b_j, j = 1, ..., k$
The players of a coalition S can use only their own resources $b^S = \sum_{i \in S} b^i$

$$v(S) = max\ \{c^T y | Ay \leq b^S, y \geq 0\}, \quad \forall\ S \subseteq N$$

In 1975 Owen stated the following important theorem:

Theorem 1 *All the games arising from a linear programming problem have non empty core and a core allocation x may be computed from an optimal solution u^* of the dual problem $min\ \{(b^N)^T u | A^T u \geq c, u \geq 0\}$ and it is:*

$$x_i = {b^i}^T u^*, \quad \forall\ i \in N$$

Note that the resulting allocations satisfy no fairness property

Connection Games

In a connection problem, a set of agents $N = \{1, ..., n\}$ have to be connected to a common-source service, as phone, electricity, water and so on
The situation is represented by an undirected complete simple graph with $n+1$ vertices, where v_0 is associated to the service provider, each vertex $v_i, i = 1, ..., n$ corresponds to agent i and the arc $a_{ij} = (v_i, v_j)$ represents a connection between vertices v_i and v_j whose cost is c_{ij}

This problem is solved determining a *minimum cost spanning tree* using one of the existing algorithms (Kruskal, 1956 or Prim, 1957)

If the agents may decide to join the service or not (see Granot and Huberman, 1981), the cost of coalition S represents the cost of connecting with the source only the vertices corresponding to players in S:

$$c(S) = \min_{T \supseteq S \cup \{v_0\}} \left\{ \sum_{v_i \in T} c_i \right\} \qquad S \subseteq N$$

where c_i is the cost of the unique arc entering the vertex v_i in the oriented tree T

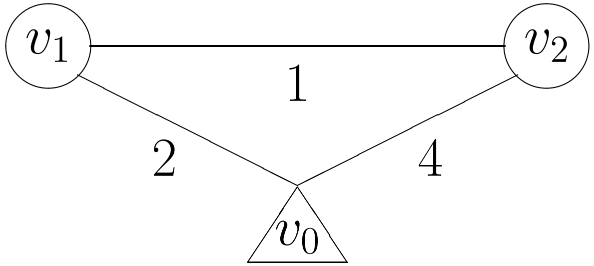
There are two alternative definitions of the characteristic function, each suitable for different real-world situations, depending on the matter that $T = S \cup \{v_0\}$ or not

In the second case the game is monotonic

The monotonic game is suitable for modeling situations like phone connections in which it is allowed using the minimum cost connection, while the non monotonic game is more appropriate for water connections situations, when it is not possible to establish the minimum cost connection without permission

Example 2

Consider the following connection situation:



In this case the monotonic connection game is given by:

$$c(\emptyset) = 0, c(\{1\}) = 2, c(\{2\}) = 3, c(\{1,2\}) = 3$$

while the non monotonic game is:

$$c(\emptyset) = 0, c(\{1\}) = 2, c(\{2\}) = 4, c(\{1,2\}) = 3$$

because coalition $\{2\}$ *cannot use the arcs* a_{01} *and* a_{12} $\square$

Connection games have non empty core and a core allocation may be obtained using the Bird's rule (Bird, 1976), where each player i pays the cost c_i in optimal tree for the grand coalition
In Example 2 the Bird allocation assigns 2 to player 1 and 1 to player 2

Also these core allocations are usually unfair

Sequencing Games

A classical sequencing problem is represented by a set $N = \{1, ..., n\}$ of machines that have to be repaired

For each machine it is known the deterministic repairing time $t_i > 0, i = 1, ..., n$ and the cost per unit of time before it is made available again $\alpha_i \geq 0, i = 1, ..., n$

Given an ordering σ, machine i is repaired after all its predecessors, whose set is denoted by $P(\sigma, i)$, so its cost is $\alpha_i \left(t_i + \sum_{j \in P(\sigma,i)} t_j\right)$ and the cost of the ordering σ may be denoted by:

$$C_\sigma = \sum_{i \in N} \left(\alpha_i \left(t_i + \sum_{j \in P(\sigma,i)} t_j \right) \right)$$

The problem consists in determining a minimal cost ordering

According a well-known result by Smith (1956), the machines have to be ordered according to non increasing urgency indices, where the urgency index of machine $i = 1, ..., n$ is $u_i = \frac{\alpha_i}{t_i}$

When the machines belongs to different agents, the reordering is possible only after an agreement
The situation is represented by a TU game (Curiel, Pederzoli and Tijs, 1989) that assigns to each coalition S the maximal saving that the players may obtain with a new ordering σ in the set Σ^S of the feasible orderings for the players in S, i.e. those preserving contiguous subcoalitions
Let σ_0 be the initial order, then:

$$v(S) = C_{\sigma_0} - \min_{\sigma \in \Sigma^S} \{C_\sigma\}, \quad S \subseteq N$$

Sequencing games have non empty core and a core allocation may be obtained by the so-called *Equal Gain Splitting Rule* (EGS) (Curiel, Pederzoli and Tijs, 1989)
The optimal ordering may be obtained from the initial one by a series of switches among neighbor players; let $g_{ij} = max \ \{\alpha_i t_j - \alpha_j t_i, 0\}$ be the gain of the switch of players i and j, the EGS assigns to each of the two players i and j one half of the gain:

$$EGS_i = \frac{1}{2} \sum_{j \in P(\sigma,i)} g_{ji} + \frac{1}{2} \sum_{j \in S(\sigma,i)} g_{ij}, \quad i \in N$$

where $S(\sigma, i)$ the set of successors of player i in the ordering σ

Note that the EGS may be not symmetric

Example 3

Consider the sequencing situation defined by $N = \{1,2,3\}, \alpha = (5,9,8), t = (5,3,4)$

The cost associated to $\sigma_0 = (1,2,3)$ *is* $C_{\sigma_0} = 25 + 72 + 96 = 193$ *and the urgency indices are* $u = (1,3,2)$*, so* $\sigma^* = (2,3,1)$ *and the associated cost is* $C_{\sigma^*} = 27 + 56 + 60 = 143$

The corresponding TU game is:

S	1	2	3	12	13	23	123
$v(S)$	0	0	0	30	0	0	50

The gains g_{ij} *are:*

ij	12	13	21	23	31	32
g_{ij}	30	20	0	0	0	12

so

$$EGS_1 = \frac{1}{2}(g_{12} + g_{13}) = 25$$

$$EGS_2 = \frac{1}{2}g_{12} + \frac{1}{2}g_{23} = 15$$

$$EGS_3 = \frac{1}{2}(g_{13} + g_{23}) = 10$$

□

All the allocations proposed for the three games presented does not require the knowledge of the characteristic function of the game
This aspect is very important as a TU game has 2^n coalitions and the computational effort is relevant

OR games may profit of the algorithms available for the corresponding OR problems

OR games includes:

- flow games (Kalai and Zemel, 1982)
- standard fixed tree games (Bjørndal, Koster and Tijs, 1999)
- forest games (Kuipers, 1997)
- shortest path games (Fragnelli, Garcia-Jurado and Mendez-Naya, 2000)
- transportation games (Sanchez-Soriano, Lopez and Garcia-Jurado, 2001)
- assignment games (Shapley and Shubik, 1972)
- permutation games (Tijs, Parthasarathy, Potters and Prasad, 1984)
- travelling salesman games (Potters, Curiel and Tijs, 1992)
- chinese postman game (Hamers, Borm, van de Leensel and Tijs, 1999)
- PERT games (Branzei, Ferrari, Fragnelli and Tijs, 2002)
- inventory games (Meca, Timmer, Garca-Jurado and Borm, 2004), (Tijs, Meca, Lopez, 2005)
- resource levelling games (Fragnelli, Garcia-Jurado and Mendez-Naya, 2004)

For a wider survey we address to Curiel (1997) and Borm, Hamers and Hendrickx (2001)

Concluding Remarks

There exist other classes of TU games that allow representing important and common economic situations, that do not belong to the class of OR games:

- bankruptcy games (O'Neill, 1984)
- communication games (Pederzoli, 1991)
- market games (Shubik, 1959)