Embedding classical indices in the *FP* family

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Summary

Motivation

In various real-world situations a set of agents has to decide in favour of an issue or against it

Power indices allow evaluating the role played by each agent in the process that leads to the formation of a majority that may take a decision

Different indices account different features:

- ordering in the majority formation process (Shapley and Shubik, 1954)
- different majorities (Banzhaf, 1965 - Coleman, 1971)
- majorities with minimal number of agents (Deegan and Packel, 1978 - Holler, 1982)
- ...
- contiguity of the parties on a left-right axis (Fragnelli, Ottone and Sattanino, 2009)
  In the FP family, a coalition may form after a negotiation that includes all the intermediate parties

Exploiting the degrees of freedom in the choice of the FP family parameters (main contiguous majorities, their probabilities to form, role of each party in each majority) we want to select them in order to embed the classical power indices by Shapley-Shubik, Banzhaf-Coleman, Deegan-Packel and Holler

The modified indices may profit of some features of the classical ones, adding the relevance assigned to intermediate parties

Notation and Definitions

$[q; w_1, w_2, \ldots, w_n]$ is a *weighted majority situation*

where $N = \{1, 2, \ldots, n\}$ is the set of parties of a Parliament

q is the majority quota

$w = (w_1, w_2, \ldots, w_n)$ is a vector of weights (percentage of votes, number of seats, ...)

Given a weighted majority situation, (N, w) is the corresponding *weighted majority game*, where $w : 2^N \longrightarrow \{0, 1\}$ is the characteristic function defined as

$$w(S) = \begin{cases} 1 & \text{if } \sum_{j \in S} w_j \geq q \\ 0 & \text{otherwise} \end{cases}$$

$w(S) = 1 \longrightarrow S$ is a *winning coalition*

$w(S) = 0 \longrightarrow S$ is a *losing coalition*

Given a winning coalition S, a party $j \in S$ is *critical* for S if $S \setminus \{j\}$ is losing

$w(S) - w(S \setminus \{j\})$ is the *marginal contribution of player* j *w.r.t.* S

A winning coalition is *minimal* if each proper subcoalition is losing

(N, w) is *monotonic* if $S \subseteq T \Rightarrow w(S) \leq w(T)$

(N, w) is *simple* if it is monotonic, $w(S) \in \{0, 1\}, S \subset N$ and $w(N) = 1$

A simple game is *proper* if $w(S) = 1 \Rightarrow w(N \setminus S) = 0, S \subseteq N$

An *allocation* is a real vector $x = (x_1, x_2, \ldots, x_n)$ where x_i is the amount for $i \in N$

An allocation is *efficient* if $x(N) = \sum_{i \in N} x_i = w(N)$

A *solution* is a function ψ that assigns an efficient allocation $\psi(w)$ to every game (N, w)

A *power index* is a non negative solution $\psi(w)$ for a simple game

Power indices based on marginal contributions

$$\psi_j(w) = \sum_{S \in W, S \ni j} p(S)[w(S) - w(S \setminus \{j\})], \qquad j \in N$$

where W is the set of winning coalitions

- *Shapley-Shubik index* (Shapley and Shubik, 1954)

$$\phi_j(w) = \sum_{S \in W, S \ni j} \frac{(|S|-1)!(n-|S|)!}{n!}[w(S) - w(S \setminus \{j\})] \tag{1}$$

  where $|S|$ is the cardinality of the set S

- *normalized Banzhaf-Coleman index* (Banzhaf, 1965 - Coleman, 1971)

$$\beta_j(w) = \frac{\beta_j^*(w)}{\sum_{k \in N} \beta_k^*(w)}, \qquad j \in N \tag{2}$$

  where $\beta_j^*(w) = \sum_{S \in W, S \ni j} \frac{1}{2^{n-1}}[w(S) - w(S \setminus \{j\})], j \in N$

Power indices not based on marginal contributions

- *Deegan-Packel index* (Deegan and Packel, 1978)

$$\delta_j(w) = \sum_{S \in W^m, S \ni j} \frac{1}{|W^m|} \frac{1}{|S|}, \qquad j \in N \tag{3}$$

  where W^m is the set of minimal winning coalitions

- *Holler index* (Holler, 1982)

$$H_j(w) = \frac{h_j}{\sum_{k \in N} h_k}, \qquad j \in N \tag{4}$$

  where h_j is the number of minimal winning coalitions including $j \in N$

The FP family

The parties in N are ordered according to their ideological position in a *left-right axis*

Negotiations take place uniquely between adjacent parties $\longrightarrow$ feasible coalitions include only contiguous parties

Let W^c be the set of *contiguous winning coalitions*, where $S_i \in W^c$ if for all $k, h \in S_i$ if there exists $j \in N$ with $k < j < h$ then $j \in S_i$

$$FP_j = \sum_{S_i \in W^c, S_i \ni j} \alpha_i \beta_{ij}, \qquad j \in N \tag{5}$$

$\alpha_i \geq 0$ represents the relative probability of coalition S_i to form, with the condition

$$\sum_{S_i \in W^c} \alpha_i = 1 \tag{6}$$

$\beta_{ij} \geq 0$ is the power share assigned to player $j \in S_i$, with the condition

$$\sum_{j \in S_i} \beta_{ij} = 1, \qquad S_i \in W^c$$

The choice of α_i differentiates the power of the coalitions and the choice of β_{ij} differentiates the role of the parties inside coalitions

The parameters are exogenously given (analysis of historical data)

Embedding Procedure

To embed classical indices in the *FP* family, it is necessary to allow the winning but non-contiguous coalitions to have a probability to form

The extended family $\overline{FP}$

$$\overline{FP}_j = \sum_{S_i \in W, S_i \ni j} \alpha_i \beta_{ij}, \qquad j \in N \tag{7}$$

where $\alpha_i \geq 0$ and $\beta_{ij} \geq 0$ have the same interpretation as above, with the conditions

$$\sum_{S_i \in W} \alpha_i = 1 \tag{8}$$

and

$$\sum_{j \in S_i} \beta_{ij} = 1, \qquad S_i \in W \tag{9}$$

Given a generic index ψ, we ask for

$$\sum_{S_i \in W, S_i \ni j} \alpha_i \beta_{ij} = \psi_j(w), \qquad j \in N \tag{10}$$

There are several choices of the parameters as the system is overdetermined
A trivial solution is $\alpha_N = 1$, $\alpha_i, i \neq N$ and $\beta_{Nj} = \psi_j$

For the Shapley-Shubik index, we impose

$$\alpha_i \beta_{ij} = p(S_i)[w(S_i) - w(S_i \setminus \{j\})], \qquad S_i \in W, j \in S_i \tag{11}$$

Summing on $j \in S_i$ and because of the condition (9) this is equal to

$$\alpha_i = p(S_i) \sum_{j \in S_i} [w(S_i) - w(S_i \setminus \{j\})], \qquad S_i \in W$$

Denoting the set of the critical players of S_i as S_i^c and $c_i = |S_i^c|$

$$\alpha_i = p(S_i) c_i, \qquad S_i \in W \tag{12}$$

Condition (8) holds if

$$\sum_{S_i \in W} \sum_{j \in S_i} p(S_i)[w(S_i) - w(S_i \setminus \{j\})] = 1$$

which is obviously true

By relations (11) and (12) we obtain

$$\beta_{ij} p(S_i) c_i = p(S_i)[w(S_i) - w(S_i \setminus \{j\})], \qquad S_i \in W, j \in S_i$$

from which we get

$$\beta_{ij} = \begin{cases} \dfrac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}, \qquad S_i \in W \tag{13}$$

For each coalition $S_i \in W$, we introduce a sequence of parameters $((\gamma_i)_t)_{t\in\mathbb{N}_>}$

$$(\gamma_i)_t = \begin{cases} p(S_i)c_i & \text{if } S_i \in W^c \\ (p(S_i)c_i)^t & \text{if } S_i \in W \setminus W^c \end{cases}$$

normalized as

$$(\alpha_i)_t = \frac{(\gamma_i)_t}{\sum_{S_k\in W}(\gamma_k)_t}, \qquad S_i \in W$$

Take the limit for $t \to +\infty$

$$\gamma_i^* = \begin{cases} p(S_i)c_i & \text{if } S_i \in W^c \\ 0 & \text{if } S_i \in W \setminus W^c \end{cases}$$

as $p(S_i)c_i < 1$ if $S_i \neq N$, while $N \in W^c$

Consequently, $(\alpha_i)_t$ converges to

$$\alpha_i^* = \begin{cases} \dfrac{p(S_i)c_i}{\sum_{S_k\in W^c} p(S_k)c_k} & \text{if } S_i \in W^c \\ 0 & \text{if } S_i \in W \setminus W^c \end{cases} \tag{14}$$

We can assume β_{ij} does not depend on t, so $\beta_{ij}^* = (\beta_{ij})_t = \beta_{ij}$ for each $t \geq 1$

The embeded Shapley-Shubik index is

$$\phi_j^{FP}(w) = \sum_{S_i \in W^c, S_i^c \ni j} \left(\frac{p(S_i)c_i}{\sum_{S_k \in W^c} p(S_k)c_k} \frac{1}{c_i} \right), \qquad j \in N \tag{15}$$

The same procedure may be applied to any power index in the family $\overline{FP}$

Proposition 1. *For each power index* $\overline{FP}$ *and for each* $t \in \mathbb{N}_>$ *we have that* $(\overline{FP})_t = ((\overline{FP}_1)_t, \ldots, (\overline{FP}_n)_t)$ *is a power index*

To embed the other classical power indices of Banzhaf-Coleman, Deegan-Packel and Holler in the $\overline{FP}$ family and, consequently, to obtain the corresponding FP indices, we need suitable α_i and β_{ij}

Parameters	α_i	β_{ij}
Banzhaf-Coleman	$\frac{c_i}{\sum_{S_k \in W} c_k}, \quad S_i \in W$	$\begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}$
Deegan-Packel	$\begin{cases} \frac{1}{\lvert W^m \rvert} & \text{if } S_i \in W^m \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}$
Holler	$\begin{cases} \frac{c_i}{\sum_{S_k \in W^m} c_k} & \text{if } S_i \in W^m \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}$

The embedded indices are

- $$\beta_j^{FP}(w) = \frac{h_j^c}{\sum_{k \in N} h_k^c}, \qquad j \in N \tag{16}$$

  where h_j^c counts how many times a player is critical in a contiguous winning coalition

- $$\delta_j^{FP}(w) = \sum_{S_i \in W^{mc}} \frac{1}{|W^{mc}|} \frac{1}{|S_i|}, \qquad j \in N \tag{17}$$

  where W^{mc} is the set of contiguous minimal winning coalition

- $$H_j^{FP}(w) = \frac{h_j^{mc}}{\sum_{k \in N} h_k^{mc}}, \qquad j \in N \tag{18}$$

  where h_j^{mc} counts how many times a player belongs to a contiguous minimal winning coalition

Example

Italian *Camera dei Deputati*

It includes 630 seats and the majority quota is $\lfloor \frac{v}{2} + 1 \rfloor$, where v is the number of voters, excluding absences and abstentions

The data are taken from the general election of April 2008; for sake of simplicity we do not consider 18 seats belonging to very small parties which, historically, have no practical influence on the decisions of the Camera even if, in theory, they could change the outcome

The remaining 612 seats are assigned as follows

$Parties$	IdV	PD	UDC	PDL	LN
$Seats$	28	218	34	272	60

The ordering of the parties is assigned according to their willingness to form a coalition in the recent political history

Supposing that all the 612 Deputies vote, we have the weighted majority situation [307; 28, 218, 34, 272, 60]

In order to compute the Shapley-Shubik index, we need the following data

S_i	$\underline{24}$	$\underline{45}$	$1\underline{24}$	$\underline{134}$	$1\underline{45}$	$\underline{2}3\underline{4}$	$\underline{235}$	$2\underline{4}5$	$3\underline{45}$	$123\underline{4}$	$1\underline{235}$	$12\underline{4}5$	$13\underline{4}5$	2345	12345
$p(S_i)$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$
α_i	$\frac{6}{60}$	$\frac{6}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{4}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{2}{60}$	$\frac{4}{60}$	$\frac{3}{60}$	$\frac{9}{60}$	$\frac{3}{60}$	$\frac{3}{60}$	0	0
β_{ij}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{2}$	1	$\frac{1}{3}$	1	1		

critical parties are underlined, $\alpha_i = p(S_i)c_i$, $\beta_{ij} = \frac{1}{c_i}$

Coalition $\{1, 2, 3, 5\}$ has the highest probability to form and includes the leftmost and rightmost parties (IdV and LN) and excludes the relative majority party (PDL)

The actual majority coalition $\{4, 5\}$ has a lower probability to form

Parties	IdV	PD	UDC	PDL	LN
$\phi(w)$	$\frac{2}{60}$	$\frac{12}{60}$	$\frac{7}{60}$	$\frac{27}{60}$	$\frac{12}{60}$
$\beta(w)$	$\frac{1}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{11}{25}$	$\frac{5}{25}$
$\delta(w)$	$\frac{2}{24}$	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{8}{24}$	$\frac{5}{24}$
$H(w)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

Parties	IdV	PD	UDC	PDL	LN
$\phi^{FP}(w)$	0	$\frac{2}{17}$	0	$\frac{10}{17}$	$\frac{5}{17}$
$\beta^{FP}(w)$	0	$\frac{1}{7}$	0	$\frac{4}{7}$	$\frac{2}{7}$
$\delta^{FP}(w)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$H^{FP}(w)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

$\phi(w)$ and $\beta(w)$ assign a positive power to the small parties, IdV and UDC, as they are critical for some winning coalitions, but they are never critical for a contiguous winning coalition, so they have null power with $\phi^{FP}(w)$ and $\beta^{FP}(w)$, even if UDC has an intermediate position on the left-right axis

$\phi^{FP}(w)$ and $\beta^{FP}(w)$ give a higher power to PDL and LN (the actual majority coalition), favouring PDL that has the relative majority, guaranteeing a positive power to PD which is the second party of the Camera and remains critical for some contiguous winning coalitions

$\delta(w)$ and $H(w)$ assign a positive power to each party because all of them belong to at least one minimal winning coalition

The unique contiguous minimal winning coalition is $\{PDL, LN\}$ so $\delta^{FP}(w)$ and $H^{FP}(w)$ equally share the power between them, as they are in a symmetric position

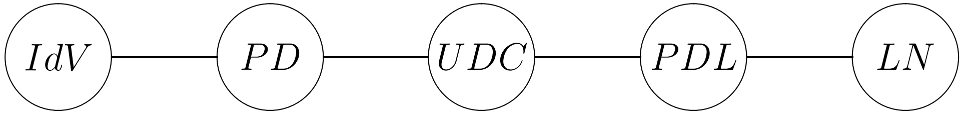
Cooperation structures

We relax the hypothesis of contiguity of the parties, using a graph as in the cooperation structure by Myerson (1977)
FEATURES
Non-oriented graph
Vertices are the parties
Edges represent the willingness of pairs of parties to reach an agreement; an edge between parties k and h is denoted by $k : h$

Let $g^N = \{k : h | k \in N, h \in N, k \neq h\}$ be the complete graph and let $G^N = \{g | g \subseteq g^N\}$ be the set of all graphs on N
Given $S \subseteq N$ and $g \in G^N$, $k, h \in S$ are *connected in S by g* if there exists a path in g from k to h
$S \subseteq N$ is *connected by g* if all pairs $k, h \in S$ are connected in S by g

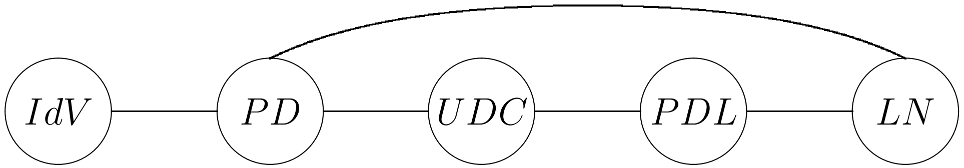
The central role played by the contiguous coalitions is now assigned to the connected ones

The model of the left-right axis is a coalition structure represented by a line-graph g'



The FP family is extended to a new family, denoted by $\widetilde{FP}$, based on the coalitions connected by a given graph $g \in G^N$, their relative probability to form and a rule for sharing the power inside each coalition

Consider the previous example, adding the edge PD:LN (such an agreement took place in 1996) the new graph g'' is



$Parties$	IdV	PD	UDC	PDL	LN
$\phi^{\widetilde{FP}}$	0	$\frac{7}{40}$	$\frac{5}{40}$	$\frac{18}{40}$	$\frac{10}{40}$
$\beta^{\widetilde{FP}}$	0	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{7}{16}$	$\frac{4}{16}$
$\delta^{\widetilde{FP}}$	0	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{5}{12}$
$H^{\widetilde{FP}}$	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

All the indices related to g'' reduce the power of the main party, PDL, as coalitions not including it may form

The Myerson index (see Myerson, 1977) in the situations g' and g'' is

$Parties$	IdV	PD	UDC	PDL	LN
$M(w, g')$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{3}{12}$
$M(w, g'')$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$

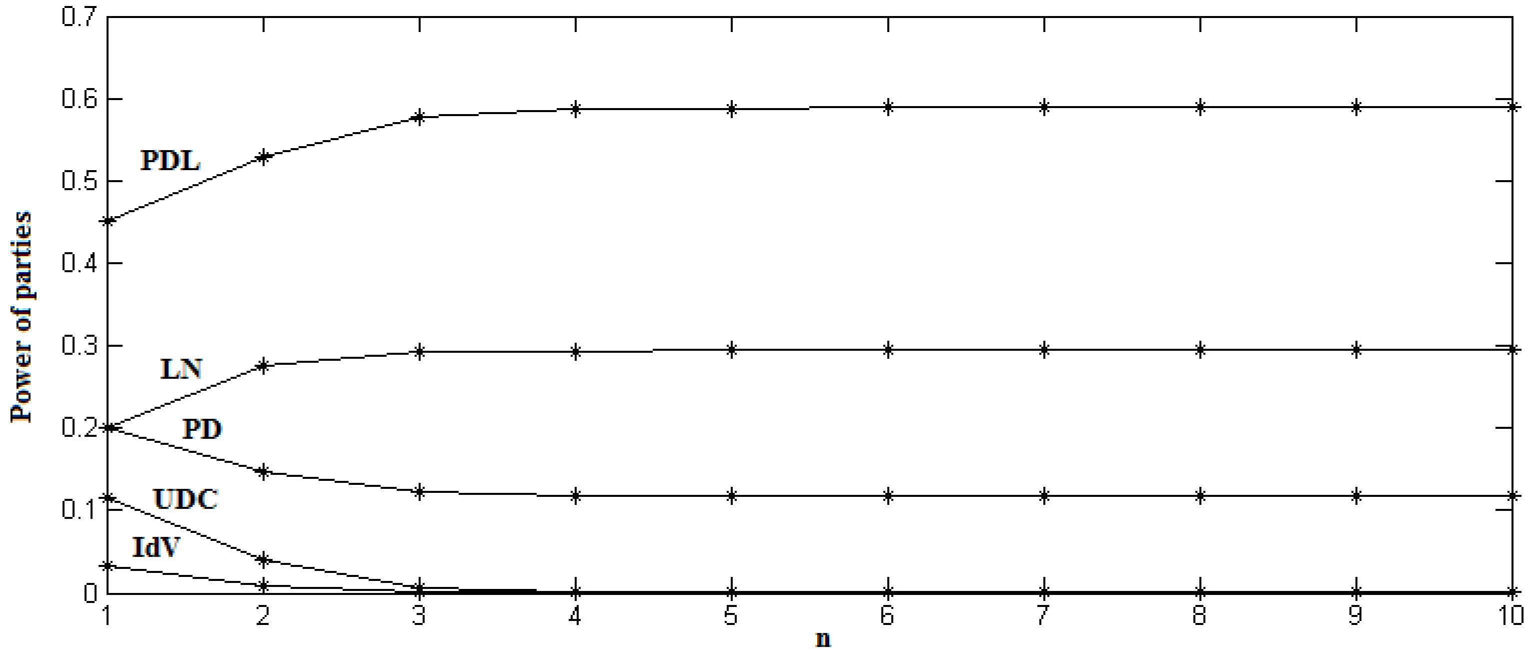
Myerson index satisfies the properties of *equity* (both the parties of an edge have the same variation of power) and *total stability*(introducing an edge, both the parties have a non negative variation of power)

In our model PD has a higher power with g'', while LN prefers the g' situation

Concluding Remarks

- FP family of power indices is quite far from the classical indices
  The embedding procedure allows profiting of some properties of the classical indices and of the relevance of the contiguous coalitions
- The idea of giving zero probability to form to the non-contiguous coalitions can be strong, but in some particular situations parties with quite different political ideologies may have the necessity to negotiate and make an agreement
- It is possible to use any sequence that reduces to zero the probability of non-contiguous coalitions, leaving a positive probability to some contiguous coalitions
- It is possible to directly assign null probabilities to non-contiguous coalitions

- The sequence $((\overline{FP})_t)_{t \in \mathbb{N}_>}$ provides a power index for each $t \in \mathbb{N}_>$, where the non-contiguous coalitions have a reduced, but positive, probability



- After a suitable analysis of real data, we can choose any vector $(\overline{FP})_t$ selecting an appropriate value for t

- Defining a new game (N, w') with $w'(S) = w(S), S \in W^c$ and $w'(S) = 0$ otherwise, monotonicity no longer holds

  The Shapley value (Shapley, 1953) may give negative values to some parties

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Thank you
for your attention!