Peer Group Games

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Tree-Connected Peer Group Situations and Peer Group Games jointly with R. Brânzei, S. Tijs

On the Computation of the Nucleolus of Line-Graph Peer Group Games jointly with R. Brânzei, S. Tijs

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Introduction

In many situations the social configuration of the organization influences the potential possibilities of all the groups of agents, i.e. economic and OR situations in which:

- the set of agents is (strict) hierarchically structured with a unique leader
- the potential individual economic possibilities interfere with the behavioristical rules induced by the organization structure

In a strict hierarchy every agent has a relationship with the leader either directly or indirectly with the help of one or more other agents	$\Rightarrow$	The important group for an agent consists of the leader, the agent himself and all the intermediate agents that exist in the given hierarchy (*peer group*)

The hierarchy may be described by a rooted directed tree with the leader in the root, each other agent in a distinct node and the peer group of each agent corresponds to the agents in the unique path connecting the agent with the leader

To each tree-connected peer group situation it is possible to associate a *peer group game*, with the agents as players and the characteristic function defined by pooling the individual economic possibilities of those members whose peer groups belong to the coalition

Peer groups are essentially the only coalitions that can generate a non-zero payoff in a peer group game

Peer group games form a cone, generated by unanimity games associated to peer groups, that lies in the intersection of the cones of convex games and monotonic veto-rich games (Arin, Feltkamp, 1997) with the leader as veto-player

Related classes:

- Games with communication structures - Myerson (1977, 1980), Owen (1986), Borm, van den Nouweland, Tijs (1994)
- Trading games - Deng, Papadimitriou (1994), Topkis (1998)
- Games with permission structures - Gilles, Owen, van der Brink (1992)

Notations

Given a set of agents $N = \{1, 2, ..., n\}$ with a strict hierarchy let T the associated tree

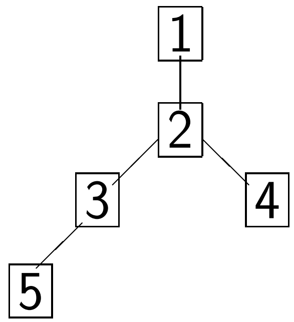
A T-*connected peer group situation* is a triplet $< N, P, a >$ where $P : N \longrightarrow 2^N$ is a mapping which associates to each agent $i \in N$ the T-*connected peer group of agent* i, $P(i) = [1, i]$ and $a \in \mathbb{R}_+^N$ is the vector of potential economic possibilities of each agent if all his superiors cooperate with him

The associated *peer group game* is $\langle N, v_{P,a} \rangle$, or $\langle N, v \rangle$, with

$$v(S) = \sum_{i:P(i) \subset S} a_i, \ \forall S \subset N; \ v(\emptyset) = 0$$

If $1 \notin S$ then $v(S) = 0$

Example 1. Given a set of agents $N = \{1, 2, 3, 4, 5\}$ with associated tree T:



the set of all the peer groups is:

$$[1,1] = \{1\},\ [1,2] = \{1,2\},\ [1,3] = \{1,2,3\},\ [1,4] = \{1,2,4\},\ [1,5] = \{1,2,3,5\}$$

and the peer group game is:

$$v(1) = v(1,3) = v(1,4) = v(1,5) = v(1,3,4) = v(1,3,5) = v(1,4,5) = v(1,3,4,5) = a_1$$
$$v(1,2) = v(1,2,5) = a_1 + a_2$$
$$v(1,2,3) = a_1 + a_2 + a_3$$
$$v(1,2,4) = v(1,2,4,5) = a_1 + a_2 + a_4$$
$$v(1,2,3,4) = a_1 + a_2 + a_3 + a_4$$
$$v(1,2,3,5) = a_1 + a_2 + a_3 + a_5$$
$$v(N) = a_1 + a_2 + a_3 + a_4 + a_5$$
$$v(S) = 0 \text{ otherwise}$$

Each peer group game v can be expressed as:

$$v = \sum_{i=1}^{n} a_i u_{[1,i]}$$

where a_i represents the Harsanyi dividend of the peer group $[1, i]$

Given a peer group structure P, peer group games form a cone $\{\langle N, v_{P,a}\rangle \mid a \in \mathbb{R}_+^N\}$, generated by the independent subset $\{u_{[1,i]} \mid i \in N\}$

Peer group games are related to convex games, monotonic games, veto-rich games and Γ-component additive games (Potters, Reijnierse, 1995):

- Peer group games are nonnegative combinations of convex games and are monotonic and agent 1 (the leader) is a veto player because $v(S) = 0$ for each $S \subset N$ with $1 \notin S$, so the cone of peer group games is a subcone both in the cone of convex games and in the cone of monotonic games with 1 as veto player
- From the convexity property it follows that peer group games are superadditive
- $w = v - a_1 u_{[1,1]}$ is zero-normalized and superadditive and is an element in the cone of T-component additive games

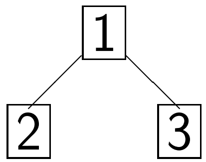
Peer group games have the following properties:

(i) *The cone of peer group games is in the intersection of the cones of convex games and monotonic veto rich games with* 1 *as veto player;*

(ii) *For each peer group game* v*, the zero-normalization* $v - a_1 u_{[1,1]}$ *is an element in the cone of* T*-component additive games.*

These properties are not characterizing for peer group games

Example 2. Consider a T-component additive game with associated tree T:



given by

$$v = 5u_{\{1,2\}} + 7u_{\{1,3\}} + 10u_{\{1,2,3\}}$$

v is a convex game, the T-component additive game, and also a monotonic game with 1 as veto player, but it is not a peer group game because $\{1, 2, 3\}$ is not a peer group

Economic and OR situations

Auctions

In a sealed bid second price auction the seller has a reservation price r, which is known to n potential bidders (players) $1, 2, ..., n$, each of them submitting one bid $b_1, b_2, ..., b_n$

The bidder with the highest bid obtains the object at the price of the second highest bid

The value w_i of the object for player i, with $w_1 > w_2 > w_3 > \cdots > w_n \geq r$ is not necessarily known by the other players

For player i a dominant strategy is $b_i = w_i \Rightarrow v(1) = w_1 - w_2, v(i) = 0$ if $i \neq 1$

For N a dominant strategy is $b_1 = w_1$ and $b_i = r$ if $i \neq 1 \Rightarrow v(N) = w_1 - r$

For $S \neq N$ a dominant strategy is $b_i(S) = w_{i(S)}$ and $b_i = r$ if $i \neq i(S) \Rightarrow v(S) = 0$ if $1 \notin S$ or $v(S) = w_1 - w_{k+1}$ if $[1, k] \subset S$ and $k + 1 \notin S$

Proposition 1. *$\langle N, v\rangle$ coincides with the peer group game corresponding to the T-connected peer group situation $< N, P, a >$ where $N = \{1, 2, ..., n\}$, T is a line-graph and $a_i = w_i - w_{i+1}$ for $i \in N$ with $w_{n+1} = r$.*

Example 3. In a sealed bid second price auction there are three bidders with $w = (100, 80, 50)$ and $r = 25$. The corresponding game is $v = 20u_1 + 30u_{1,2} + 25u_{1,2,3}$

In a first price sealed bid auction for which $w_1, w_2, ..., w_n$ and $r = w_{n+1}$ are common knowledge among the agents, let ε be the minimal increment, with $\varepsilon < w_i - w_{i+1},\ \forall i \in N$

The optimal strategy for player i is $b_i := w_{i+1} + \varepsilon$

For a subgroup S with $[1,i] \subset S$ and $i+1 \notin S$ the optimal strategy is $b_1 = w_{i+1} + \varepsilon$ and $b_i = r, i \in S \setminus \{1\}$

$\langle N, v \rangle$ is a peer group game, with $v = (w_1 - w_2 - \varepsilon)u_{\{1\}} + \displaystyle\sum_{2=1}^{n}(w_i - w_{i+1})u_{[1,i]}$

Graph-restricted binary communication

Consider a situation where gains a_i are made via binary interactions of a central agent 1 with each of the other agents $i \in N$, with restrictions on a tree T

Interactions may be information exchange between 1 and i, or import (export) of goods via harbour 1 for agent i, or approval by 1 of a planned action of player i

The communication restrictions lead to the peer group game $v = \sum_{i=1}^{n} a_i u_{[1,i]}$

Sequencing

A sequencing situation is a triplet (σ_0, p, α), where σ_0 is the initial order, p is the vector of processing times and α is the vector of the costs per unit of time

$u_i = p_i^{-1}\alpha_i, i \in N$ is the urgency index of agent i

It is optimal to serve the agents according to their urgency (Smith, 1956); this order can be obtained by neighbour switches

The corresponding sequencing game (Curiel, Pederzoli, Tijs, 1989) is a nonnegative combination of unanimity games on neighbours that switch

$$v = \sum_{(k,\ell),k<\ell} g_{k,\ell} u_{[k,\ell]}$$

where $g_{k,\ell}{=}(p_k\alpha_\ell{-}p_\ell\alpha_k)_+$

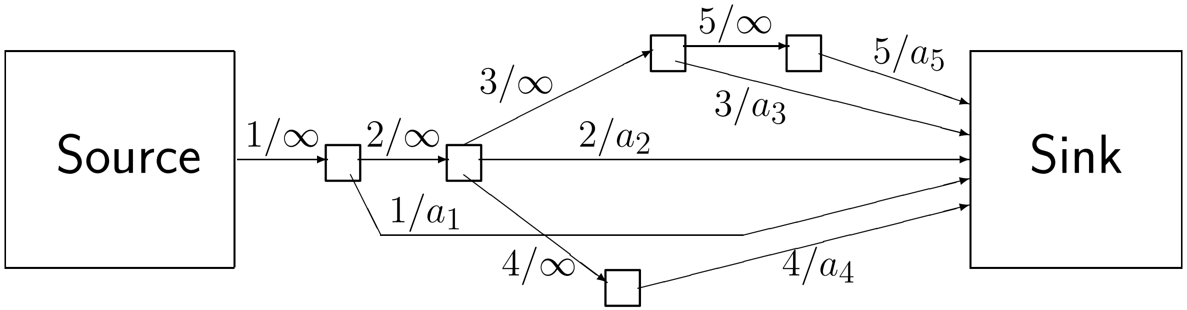
If the initial order σ_0 is such that $u_2 > u_3 > \cdots > u_n$, as $g_{k,\ell} = 0$ if $k \neq 1$, it results in the peer group game:

$$v = \sum_{i=2}^{n} g_{1,i} u_{[1,i]}$$

associated to a line-graph with $a_i = g_{1,i}, \forall i \in N \setminus \{1\}$ and $a_1 = 0$

Flow

Given the following flow situation, where the notation is owner/capacity:



The corresponding flow game (Kalai, Zemel, 1982) is equal to the peer group game given in Example 1

Landing

Planes of players $1, 2, ..., n$ need landing strips of length $\ell_1, \ell_2, ..., \ell_n$ with $\ell_1 > \ell_2 > \cdots > \ell_n$ and related costs $c_1 > c_2 > \cdots > c_n$

The corresponding airport game (Littlechild, Owen, 1977) $\langle N, c\rangle$ is:

$$c = c_n u^*_N + (c_{n-1} - c_n) u^*_{N\setminus\{n\}} + \cdots + (c_1 - c_2) u^*_{\{1\}},$$

where $\langle N, u^*_S\rangle$ is a game with $u^*_S(T) = 1$ if $S \cap T \neq \emptyset$ and $u^*_S(T) = 0$ otherwise

The dual game $\langle N, c^*\rangle$ corresponding to $\langle N, c\rangle$ is $c^*(S) = c(N) - c(N \setminus S), \forall S \subset N$ and can be written as:

$$c^* = c_n u_N + \sum_{i=1}^{n-1} (c_i - c_{i+1}) u_{\{1,2,...,i\}}$$

and is a peer group game associated to a line-graph

Solutions for peer group games

Proposition 2. *For peer group games the following properties of solution concepts hold:*

(i) *The bargaining set* $\mathcal{M}(v)$ *coincides with the core* $C(v)$*; [convexity]*

(ii) *The kernel* $\mathcal{K}(v)$ *coincides with the pre-kernel* $\mathcal{K}^*(v)$ *and the pre-kernel consists of a unique point which is the nucleolus of the game; [convexity]*

(iii) *The nucleolus* $Nu(v)$ *occupies a central position in the core and is the unique point satisfying*

$$Nu(v) = \{x \in C(v) \mid s_{ij}(x) = s_{ji}(x),\ \forall i, j\}$$

where $s_{ij}(x) = \max\{v(S) - x(S) \mid i \in S \subset N \setminus \{j\}\}$*; [convexity]*

(iv) *The core* $C(v)$ *coincides with the Weber set*

$$W(v) = \mathrm{conv}\{m^\sigma(v) \mid \sigma \textit{ permutation of } N\}; \textit{[Driessen (1988), Curiel (1997)]}$$

(v) *The Shapley value* $\Phi(v)$ *is the center of gravity of the extreme points of the core and is given by*

$$\Phi_i(v) = \sum_{j: i \in P(j)} \frac{a_j}{|P(j)|},\ i \in N; \textit{[convexity, Harsanyi dividend]}$$

(vi) *The* τ*-value is given by*

$$\tau(v) = \alpha(a_1, 0, ..., 0) + (1-\alpha)(M_1(v), M_2(v), ..., M_n(v)\},$$

where $M_i(v) = \sum_{j: i \in P(j)} a_j$; *[semiconvexity]*

(vii) *The core* $C(v)$ *coincides with the selectope*

$$S(v) = \mathrm{conv}\{m^\beta(v) \in \mathbb{R}^N | \beta : 2^N \backslash \{\emptyset\} \to N, \beta(S) \in S\}; \textit{[Derks, Haller, Peters (2000)]}$$

(viii) *There exist population monotonic allocation schemes* (*pmas*)*. [convexity]*

The nucleolus for line-graph peer group games

In this case $P = \{\{1\}, \{1,2\}, \{1,2,3\}, ..., \{1,2,...,n\}\}$, $a \in \mathbb{R}^n_+$, and z is the nucleolus of $\langle N, v\rangle$

Lemma 1. *For each $i \in \{1, ..., n-1\}$ the nucleolus satisfies the equation*

$$z_{i+1} = \min\{z_i, Z_i - A_i\}$$

where $Z_i = \sum_{k=1}^{i} z_k,\ A_i = \sum_{k=1}^{i} a_k$

Theorem 1. *The nucleolus of $\langle N, v\rangle$ is the unique solution of the n equations*

$$\begin{aligned} Z_n &= A_n \\ z_i &= \min\{z_{i-1}, Z_{i-1} - A_{i-1}\}, \quad i = 2, ..., n \end{aligned}$$

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