The Owen Set of Linear Programming Games and Nash Equilibria

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Outline

The Setting

Some agents own a bundle of resources that can be used for a common project

If a subset of agents agree on a fair division of the income they develop the project using their resources

Linear Production Games - Owen, 1975

An external monopolistic company offers to purchase the resources

Each agent, separately, communicates the price he wants for his whole bundle of resources

After that the company decides which bundles to buy, trying to maximize the quantity of resources rather then the gain

The company will not accept a loss, but is indifferent among different amounts of gain

The company $\rightarrow$ buys those bundles that allow producing goods with no loss

Each agent $\rightarrow$ decides the price to ask in order to maximize the utility: *not too high, but as large as possible*

A unit of good, sold at 4, requires 1 unit of R_1 and 2 units of R_2

	R_1	R_2	price
$Agent\ 1$	2	2	sold if ≤ 4
$Agent\ 2$	0	2	sold if $= 0$

Jointly the bundles can be sold up to 8 $(e.g.\ 6+2)$

Game Theoretical Approach

Cooperative game $GC = (N, v)$

where $N = \{1, ..., n\}$ set of players

$v : 2^N \rightarrow \mathbb{R}$ characteristic function

$$core(v) = \left\{ x \in \mathbb{R}^n \ s.t. \ \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \subset N \right\}$$

Non-cooperative game $GNC = (N, (\Sigma_i)_{i \in N}, (\pi_i)_{i \in N})$

where $N = \{1, ..., n\}$ set of players

Σ_i non-empty set of strategies of player $i \in N$

$\pi_i : \prod_{j \in N} \Sigma_j \rightarrow \mathbb{R}$ payoff function of player $i \in N$

A Nash equilibrium is a strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ s.t.

$$\pi_i(\sigma_1^*, ..., \sigma_i^*, ..., \sigma_n^*) \geq \pi_i(\sigma_1^*, ..., \sigma_i, ..., \sigma_n^*), \forall \sigma_i \in \Sigma_i, \forall i \in N$$

Our case

Starting situation: cooperative game

Players	the agents
Worth of coalition S	the value of the goods that the players in S may obtain using just their own resources

Final situation: non-cooperative game

Players	the agents
Strategies	the possible selling prices
Payoff	the asked price if the bundle is purchased, 0 otherwise

Two questions

How to associate a non-cooperative game to a given cooperative game with the same set of players?

Is it possible to exploit the characteristics of the cooperative game in order to solve the non-cooperative one?

Previous approaches

- *Von Neumann and Morgenstern, 1944* : transforming a game in characteristic form into a game in strategic form, in a general framework
- *Borm and Tijs, 1992* : cooperative games with non transferable utility
- *Gambarelli, 2000* : general transformation procedure taking into account the probability that a coalition in which he takes part forms
- *Feltkamp, Tijs and Muto, 1999* : in a minimum cost spanning tree situation, construct edges minimizing cost but looking for connection to the source

Linear programming games (*Samet and Zemel, 1984*)

The players control a bundle of resources and the worth of a coalition is the best output that the players can obtain using just their resources:

$$v(S) = max\{c^T y \ s.t. \ Ay \leq b^S\}, \forall S \subseteq N$$

where	$c = \{c_1, ..., c_m\}$	vector of the prices of the goods $y = \{y_1, ..., y_m\}$
	$A = (a_{lh})_{l=1,...,m;h=1,...,k}$	matrix of the coefficients of the constraints
	$b^S = \sum_{i \in S} b^i, S \subseteq N$	vector of the resources available for coalition S,
		where $b^i = \{b^i_1, ..., b^i_k\}$ is the vector of the resources owned by player $i \in N$

Owen, 1975: linear programming games are balanced and a core allocation x can be obtained starting from an optimal solution z^* of the dual program

$$min\{(b^N)^T z \ s.t. \ A^T z = c; z \geq 0\}$$

as

$$x_i = (b^i)^T z^*, \forall i \in N$$

van Gellekom et al., 2000: Owen set = subset of core allocations associated to dual optimal solutions

The Intertwined Games

Our cooperative game is a linear programming game

In the associated non-cooperative game player $i \in N$ may ask for a price $p(i)$ s.t.

$$v(i) \leq p(i) \leq max\left\{v(S) - \sum_{j \in S \setminus \{i\}} v(j), S \ni i\right\}, \forall i \in N$$

or, by balancedness:

$$v(i) \leq p(i) \leq v(N) - \sum_{j \in N \setminus \{i\}} v(j), \forall i \in N$$

Example 1 (Production situation) *Technology matrix:*

	units of resource R1	*units of resource R2*	*value*
$itemA$	3	1	4
$itemB$	2	1	3

Let $N = \{I, II\}$*; the game is described in the following table:*

S	$R1$	$R2$	A	B	$v(S)$
I	2	2	0	1	3
II	5	1	1	0	4
I, II	7	3	1	2	10

The selling prices have to satisfy the following constraints:

$$3 \leq p(I) \leq 6$$

$$4 \leq p(II) \leq 7$$

The non-cooperative game is (only integer values of prices appear):

I/II	4	5	6	7
3	$(3, 4)$	$(3, 5)$	$(3, 6)$	$(3, 7)$
4	$(4, 4)$	$(4, 5)$	$(4, 6)$	$(0, 0)$
5	$(5, 4)$	$(5, 5)$	$(0, 0)$	$(0, 0)$
6	$(6, 4)$	$(0, 0)$	$(0, 0)$	$(0, 0)$

The Core and the Nash Equilibria

Core allocations are Nash equilibria: if a player unilaterally reduces his price, then his payoff decreases

if a player unilaterally increases his price then his bundle is not sold

The Owen set allows to find very easily core allocations, ending in efficient Nash equilibria

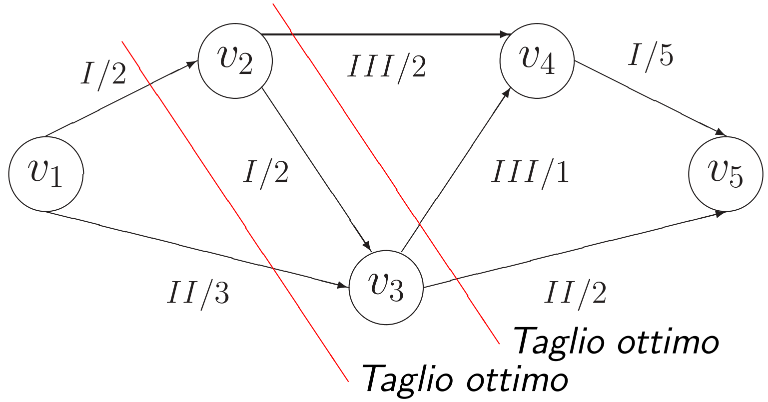
In Example 1 $core(v) = \{(\alpha, 10-\alpha) \in \mathbb{R}^2 \; s.t. \; 4 \leq \alpha \leq 7\}$ coincides with the set of Nash equilibria

The equilibrium profile $(4, 6)$ corresponds to the optimal dual solution $(1, 1)$

In general the core and the set of Nash equilibria do not coincide

Example 2 (Flow situation)

$N = \{I, II, III\}$ *(the notation is owner/capacity):*



The associated game is:

S	$\{I\}$	$\{II\}$	$\{III\}$	$\{I, II\}$	$\{I, III\}$	$\{II, III\}$	N
$v(S)$	0	2	0	2	2	2	5

The selling prices that players may ask for have to satisfy:

$$0 \leq p(I) \leq 3$$

$$2 \leq p(II) \leq 5$$

$$0 \leq p(III) \leq 3$$

The non-cooperative game is (only integer values of prices appear):

$III=0$	I/II	2	3	4	5
	0	$(0,2,0)$	$(0,3,0)$	$(0,4,0)$	$(0,5,0)$
	1	$(1,2,0)$	$(1,3,0)$	$(1,4,0)$	$(1,0,0)$
	2	$(2,2,0)$	$(2,3,0)$	$(2,0,0)$	$(2,0,0)$
	3	$(3,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

$III=2$	I/II	2	3	4	5
	0	$(0,2,2)$	$(0,3,2)$	$(0,0,2)$	$(0,0,2)$
	1	$(1,2,2)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	2	$(0,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	3	$(0,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

$III=1$	I/II	2	3	4	5
	0	$(0,2,1)$	$(0,3,1)$	$(0,4,1)$	$(0,0,1)$
	1	$(1,2,1)$	$(1,3,1)$	$(1,0,1)$	$(1,0,1)$
	2	$(2,2,1)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	3	$(0,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

$III=3$	I/II	2	3	4	5
	0	$(0,2,3)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	1	$(0,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	2	$(0,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	3	$(0,2,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Equilibrium profiles $(2,3,0)$ *and* $(0,2,3)$ *correspond to the optimal cuts,* $\{(v_1,v_2),(v_1,v_3)\}$ *and* $\{(v_2,v_4),(v_3,v_4),(v_3,v_5)\}$, *that are core allocations, according to the result of* Kalai-Zemel, 1982

The Nash equilibrium $(0,2,0)$ *is not efficient and does not correspond to a core allocation*

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Thanks!