Communication Structures and Incompatible Agents

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Summary

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The Setting

0-1 decision problems: the result is simply against or in favor of a proposal, with no intermediate position

Weighted majority situation $[q; w_1, w_2, ..., w_n]$

where $N = \{1, 2, ..., n\}$ set of decision-makers

$w_1, w_2, ..., w_n$ weights of decision-makers

q majority quota

Weighted majority game (N, v):

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}, S \subseteq N$$

$S \subseteq N$ is called *winning* if $v(S) = 1$ and *losing* if $v(S) = 0$

An agent $i \in S \subseteq N$ is called *critical for* S if $v(S) = 1$ and $v(S \setminus \{i\}) = 0$

Question: how to evaluate the influence of each member on the final decision? mainly when the members are not equivalent?

This question may be answered, inter alia, by using power indices

Survey

Each existing power index emphasizes different features of the problem, making it particularly suitable for specific situations

- *First indices*

Penrose (1946), Shapley and Shubik (1954), Banzhaf (1965), Coleman (1971)
Ability of a decision-maker to switch the result of the voting from rejection to approval by joining a set of other decision-makers that are in favor of the proposal
The indices of Penrose, Banzhaf and Coleman tally the switches w.r.t. the possible coalitions, while in the Shapley-Shubik index also the order agents form a coalition plays a role

• *Relations among agents*

Myerson (1977), Owen (1977)

Myerson proposes to use an undirected graph, called *communication structure*, for representing the relationships among the decision-makers and considered a restricted game

Owen introduces the *a priori unions*, or *coalition structures*, that account for existing agreements, not necessarily binding, among some decision-makers

Owen (1986) studied the relationship among the power indices, mainly Shapley-Shubik and Banzhaf, in the original game and in the restricted game á la Myerson

Winter (1989) requires that the different unions may join only according to a predefined scheme, called *levels structure*

Khmelnitskaya (2007) combines communication structures by Myerson and coalition structures by Owen

• *Power sharing*

Deegan and Packel (1978), Johnston (1978), Holler (1982)

Deegan and Packel accounted only the *minimal winning coalitions* (each agent is critical), while Johnston included the *quasi-minimal winning coalitions* (at least one agent is critical)

Both indices divide the unitary power among the minimal or quasi-minimal winning coalitions, respectively; then the power assigned to each coalition is equally shared among its critical agents

Holler introduced the Public Good index, supposing that the worth of a coalition is a *public good*, so the members of the winning decisive sets, i.e. the minimal winning coalitions, have to enjoy the same value; the power of an agent is proportional to the number of minimal winning coalitions s/he belongs to

• *Weights*

Kalai and Samet (1987), Haeringer (1999)

Kalai and Samet added a *weight* to the elements characterizing each agent, modifying the Shapley value; Haeringer combined weights and communication structure (weighted Myerson value)

• *Restricted cooperation - Permission structures*

Gillies et al. (1992), Van den Brink and Gillies (1996) and Van den Brink (1997)

The papers introduce the conjunctive and the disjunctive permission value for games with a permission structure

• *Restricted cooperation - Feasible coalitions*

Bilbao et al. (1998), Bilbao and Edelmann (2000), Algaba et al. (2003, 2004)

The first two papers consider the Banzhaf index and the Shapley value on convex geometries, respectively; the last two papers study the Shapley value and the Banzhaf value on antimatroids, respectively

Katsev (2010) surveys values for games with restricted cooperation

• *Contiguity and connection*

Fragnelli, Ottone and Sattanino (2009), Chessa and Fragnelli (2011)

The first paper introduces a new family of power indices, called *FP*, that account the issue of *contiguity* in a monodimensional voting space; the second paper extends to the idea of *connection* in a possibly multidimensional voting space

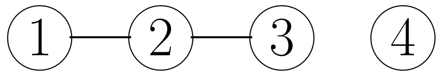
In both cases, non-contiguous and non-connected coalitions are ignored

The idea of monodimensionality was already considered in Amer and Carreras (2001)

The Issue of Incompatibility

In Myerson (1977) compatibility is represented by an undirected graph whose vertices correspond to the agents and the edges connect pairs of agents that may communicate
In the *restricted game*, a coalition is feasible and its worth is "effective" if the vertices associated to its players are connected, otherwise the worth of the coalition is the sum of the worths of the subcoalitions in the partition in connected components induced by the communication graph

Example 1 *Consider the weighted majority situation* $[51; 35, 30, 25, 10]$
The winning coalitions are $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$
Suppose that the communication structure is represented by the graph G*:*



Player 4 *is not connected and therefore will not be in a feasible coalition with one of the other players; also players* 1 *and* 3 *are not directly connected*
In the restricted game (N, v_G) *induced by the graph* G*, the winning coalitions reduce to* $\{1, 2\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$*; for instance,* $v_G(\{1, 3\}) = v(\{1\}) + v(\{3\}) = 0$ *while* $v_G(\{1, 2, 4\}) = v(\{1, 2\}) + v(\{4\}) = 1$

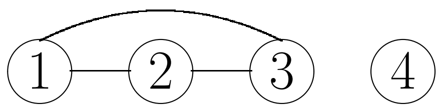
Comments

i) The games v and v_G are monotonic; in fact, $v_G(\{1,2\}) = 1$ and $v_G(\{1,2,4\}) = 1$, even if the second coalition is not feasible

ii) Dealing with infeasible coalitions, the concept of marginal contribution should be revised
Consider the weighted majority situation $[51; 24, 25, 51]$ in which coalitions have to be contiguous; feasible winning coalitions are $\{3\}$, $\{2,3\}$, $\{1,2,3\}$ and party 3 should have all the power
Assigning value 0 to the infeasible coalition $\{1,3\}$, then party 1 enters the winning coalition $\{3\}$ generating the losing coalition $\{1,3\}$ (negative marginal contribution) and party 2 results to be critical for coalition $\{1,2,3\}$ (positive marginal contribution)
A possible solution is to account only the marginal contributions that involve pairs of feasible coalitions, including losing ones, and to modify the definition of the indices accordingly
This problem does not show up with the indices which do not compare pairs of coalitions

iii) In Example 1, the graph G indicates that coalitions $\{1,2\}$, $\{2,3\}$, $\{1,2,3\}$ are feasible while coalition $\{1,3\}$ is infeasible
Suppose that parties 1 and 3 are both available for forming a two-party majority with party 2 but they never want to stay in the same coalition, so that also coalition $\{1,2,3\}$ is infeasible; the previous approach does not enable us to represent this situation

iv) Referring to the previous point, it is possible to modify the definition of feasibility requiring that the corresponding subgraph is complete (*clique*)
The graph G represents the situation in which coalitions $\{1,2\}$, $\{2,3\}$ are feasible and coalitions $\{1,3\}$, $\{1,2,3\}$ are infeasible
If coalition $\{1,2,3\}$ is feasible, it is necessary to consider the graph G', where an edge between parties 1 and 3 is added:



The new graph G' indicates that parties 1 and 3 accept forming a majority also without party 2
The problem cannot be solved neither introducing oriented arcs instead of unoriented edges

v) How to evaluate the probability that a coalition forms?

Starting from Calvo et al. (1999), a real number $0 \leq p_{ij} \leq 1$ may be associated to each edge $(i, j), i, j \in N$; p_{ij} can be viewed as the probability parties i and j enter the same coalition; how to combine the values p_{ij} for computing the probability p_S of each feasible coalition $S \subseteq N$?

• Let p_S be the product of the weights of the edges of the subgraph associated to S, i.e. the events that two parties join are independent

REMARK: $S \subset T \Rightarrow p_S \geq p_T$: a larger coalition has a lower probability (negotiation cost), unless only minimal winning coalitions are considered

Computing $p_{\{1,2,3\}}$ with G it is equal to $p_{12}p_{23}$ that is greater than or equal to $p_{12}p_{23}p_{13}$ that is the value obtained with G', where the three parties have stronger connections

• Complete graph, used in Calvo et al. (1999) for introducing the *probabilistic extension* of the Myerson value

All pairs of agents may join according to a probability of direct communication (degree of co-operation) and all the subgraphs associated to each coalition play a role

• Let p_S be the sum of the probabilities of the edges of the subgraph associated to S, i.e. the events that two parties enter the same coalition are disjoint, otherwise p_S could be larger than 1

Proposal

The probability to form for a coalition cannot be based on the same hypotheses whatever the parties entering the coalition and the parties in the coalition (contiguity/connection), so:

1. Define a graph to represent compatibilities among pairs of agents; then identify a set of feasible coalitions (or "relevant" coalitions) among the connected coalitions

2. Assign a probability to each feasible coalition, referring to opinions of a panel of experts, or to the majorities formed in the past

This approach does not require the estimation of the probabilities of all the pairs of agents (e.g., see Remark 5.3 in Calvo et al., 1999), and does not need hypotheses for combining the probabilities of the pairs

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