GAME THEORY AND POLITICS
Power Indices and Parliamentary Indices

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1 Introduction

Game Theory deals with situations whose final result depends on the choices of several decision-makers, the *players*; their targets may be

- common, even if not identical
- different
- opposite

Random elements are allowed

The name is after *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944)

Harsanyi Classification (1966):

- **Non-cooperative games**: Binding agreements are not allowed
- **Cooperative games**: Binding agreements are allowed
1.1 Representation of a Game

The three most important forms for representing a game are:

- Extended form - von Neumann (1928) and Kuhn (1953)
- Strategic form - Shubik (1982); normal form - von Neumann and Morgenstern (1944)
- Characteristic form - von Neumann and Morgenstern (1944); for cooperative games only

**Definition 1**

- A strategy is a function that assigns a move to a player for each possible situation of the game in which he is the decision-maker
- A payoff function assigns a value to each player for each possible termination of the game

A strategy is an “action plan” of a player that allows him selecting a “move” for each situation of a game
1.2 Characteristic Form

This form is allowed only for cooperative games

**Definition 2**

- **Given a set of players** \( N \), each subset \( S \subseteq N \) is a coalition. \( N \) is the grand coalition
- **The characteristic function is a real valued function** \( v : 2^N \rightarrow \mathbb{R} \) s.t. \( v(\emptyset) = 0 \)
- \( v \) assigns to \( S \) the best result that the players in \( S \) may obtain *independently* from the choices of the other players

The representation via the characteristic function is called *characteristic form* or *coalitional form*

**Example 1 (Simple Majority)**

*Three players have a target; they succeed when at least two players coordinate their efforts*

*This situation may be represented as follows:*

\[
N = \{1, 2, 3\}
\]

\[
v(\emptyset) = v(1) = v(2) = v(3) = 0; v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1
\]

This description is very “poor”, as it takes into account only the worth of each coalition
There exist other representations, among them the *partition function form* (Thrall and Lucas, 1963), that is applicable to cooperative games only, and focuses on the behavior of the players outside a coalition, accounting that the payoff of a coalition may vary depending on the coalitions that the other players form.

**Definition 3** A game in partition function form is defined by a triple \((N, \mathcal{K}, \{v_K\}_{K \in \mathcal{K}})\) where \(N\) is the set of players, \(\mathcal{K}\) is the set of all coalition structures on \(N\), i.e. partitions of the players, and, for each \(K \in \mathcal{K}\), \(v_K\) is a partition function that assigns to each coalition \(S \in K\) its worth \(v_K(S)\)

**Example 2 (Plurality Vote)**

Five agents have to elect a representative among them; who obtains the highest number of votes is elected.

A coalition of two agents may succeed or not, depending on the decisions of the other agents; let:

\[
\mathcal{K}_1 = \{\{1, 2\}, \{3\}, \{4\}, \{5\} \}; \quad v_K(\mathcal{K}_1) = (1, 0, 0, 0)
\]

\[
\mathcal{K}_2 = \{\{1, 2\}, \{3, 4, 5\}\}; \quad v_K(\mathcal{K}_2) = (0, 1)
\]
2 Cooperative Games

Agents may cooperate in order to improve their utility

The cooperation requires:

- the possibility of agreements, i.e. there do not exist antitrust rules
- the possibility of forcing the respect of the agreements, i.e. there exists a superpartes authority, accepted by all the agents

Cooperative games are divided into two classes:

- Cooperative games without transferable utility (NTU-Games): the players receive the payoff according to the strategy profile they agreed upon
- Cooperative games with transferable utility (TU-Games): the players may share the total payoff generated by the strategy profile they agreed upon

TU-Games are a special case of NTU-Games
A TU-Game has to satisfy the following three additional hypotheses:

- it is possible to transfer the utility (in a normative sense)
- there exists a common exchange tool, e.g. the money, which allows transferring the utility (in a material sense)
- the utility functions of the players must be equivalent, in order to add the utilities of the players of a coalition

Note that the decision on the sharing of the total payoff in a TU-Game is part of the binding agreement
2.1 Simple Games

**Definition 4** A game is simple when $v : 2^N \rightarrow \{0, 1\}$, with $S \subset T \subseteq N \Rightarrow v(S) \leq v(T)$ and $v(N) = 1$

- $v(S) = 0 \Rightarrow S$ is losing, $v(S) = 1 \Rightarrow S$ is winning
- $i$ is a critical player for $S$ if $S$ is winning and $S \setminus \{i\}$ is losing
- $S$ is a minimal winning coalition if all $T \subset S$ are losing
- $S$ is a quasi-minimal winning coalition if includes at least one critical player

**Definition 5** Given a simple game $(N, v)$, then

- A player $i \in N$ is a veto player if $v(S) = 0$ when $i \notin S$
- A player $i \in N$ is a dictator when $v(S) = 1$ iff $i \in S$
- A simple game is proper if for each $S \subseteq N$ then $v(S) = 1$ implies $v(N \setminus S) = 0$
2.2 Weighted Majority Games

*Weighted majority games* are an important class of simple games; they are used for representing voting situations, e.g. electoral systems.

A weighted majority situation is a tuple:

$$\mathcal{W} = (N, w, q) = [q; w_1, \ldots, w_n]$$

where

- $N = \{1, \ldots, n\}$ set of agents
- $w = (w_1, \ldots, w_n) \in \mathbb{R}_n^+$ weights of the agents
- $q \in \mathbb{R}_+$ majority quota, with $q \leq \sum_{i \in N} w_i$

The corresponding TU-game $(N, v)$ may be defined as follows:

- the set of players is $N$
- each member of each coalition $S \subseteq N$ is supposed to cast a positive vote, so:

$$v(S) = \begin{cases} 
1 & \text{if } \sum_{i \in S} w_i \geq q \ (S \text{ winning}) \\
0 & \text{if } \sum_{i \in S} w_i < q \ (S \text{ losing})
\end{cases}$$

Weighted majority games are always monotonic and proper if $q > \frac{1}{2} \sum_{i \in N} w_i$
Example 3 (UN Security Council) It includes five permanent members and ten rotating members. Approval of substantive matters require 9 votes, including the concurring votes of all five permanent members. This rule is called great power unanimity or simply veto power.

This situation may be represented as \([39; 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \)
3 Point Solutions for TU Games

These solutions include the so-called values and power indices. Values are widely used as allocation rules.

3.1 Shapley Value

It is one of the most commonly used point solutions (Shapley, 1953) and it is rooted in the concept of marginal contribution.

**Definition 6** Given a game $(N,v)$, the Shapley value is the vector $\phi(v)$ whose component $\phi_i(v)$ is the average marginal contribution of player $i \in N$ w.r.t. all the permutations of the players:

$$
\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \left[ v(P(\pi, i)) \cup \{i\} - v(P(\pi, i)) \right]
$$

where $n = |N|$, $\Pi$ is the set of permutations of $N$ and $P(\pi, i)$ is the set of players preceding $i$ in the permutation $\pi$.

The Shapley value always exists and it is unique.
Example 4

Consider the game with player set \( N = \{1, 2, 3\} \) and characteristic function defined by \( v(1) = v(2) = v(3) = v(23) = 0; v(12) = 2; v(13) = v(123) = 5 \) the Shapley value is given by:

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Player 1</th>
<th>Player 2</th>
<th>Player 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3</td>
<td>( v(1) - v(\emptyset) = 0 )</td>
<td>( v(12) - v(1) = 2 )</td>
<td>( v(123) - v(12) = 3 )</td>
</tr>
<tr>
<td>1 3 2</td>
<td>( v(1) - v(\emptyset) = 0 )</td>
<td>( v(123) - v(13) = 0 )</td>
<td>( v(13) - v(1) = 5 )</td>
</tr>
<tr>
<td>2 1 3</td>
<td>( v(12) - v(2) = 2 )</td>
<td>( v(2) - v(\emptyset) = 0 )</td>
<td>( v(123) - v(12) = 3 )</td>
</tr>
<tr>
<td>2 3 1</td>
<td>( v(123) - v(23) = 5 )</td>
<td>( v(2) - v(\emptyset) = 0 )</td>
<td>( v(23) - v(2) = 0 )</td>
</tr>
<tr>
<td>3 1 2</td>
<td>( v(13) - v(3) = 5 )</td>
<td>( v(123) - v(13) = 0 )</td>
<td>( v(3) - v(\emptyset) = 0 )</td>
</tr>
<tr>
<td>3 2 1</td>
<td>( v(123) - v(23) = 5 )</td>
<td>( v(23) - v(3) = 0 )</td>
<td>( v(3) - v(\emptyset) = 0 )</td>
</tr>
</tbody>
</table>

\[ \phi_i(v) = \frac{1}{6} \cdot 17 \quad \frac{2}{6} \quad \frac{11}{6} \]

In general, the Shapley value is very complex to compute. A simpler formula is:

\[ \phi_i(v) = \sum_{S \in N, S \ni i} \frac{(s - 1)! (n - s)!}{n!} [v(S) - v(S \setminus \{i\})], \quad \forall i \in N \]

where \( s \) is the cardinality of \( S \)
3.1.1 Axiomatization of the Shapley Value

Given a rule $\psi$ that assigns a $n$-dimensional vector in $\mathbb{R}^N$ to each TU-Game $(N, v)$

1. Symmetry
   If two players $i, j$ are symmetric, i.e. $v(S \cup \{i\}) = v(S \cup \{j\})$ for each $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) = \psi_j(v)$

2. Dummy player
   Let $i$ be a dummy player, i.e. $v(S \cup \{i\}) = v(S) + v(i)$ for each $S \subseteq N \setminus \{i\}$, then $\psi_i(v) = v(i)$

3. Additivity or independence (Debatable axiom)
   Given two games with player set $N$ and characteristic functions $v$ and $u$, respectively, let $(N, (u + v))$ be the sum game defined as $(u + v)(S) = u(S) + v(S)$ for each $S \subseteq N$, then $\psi_i(u + v) = \psi_i(u) + \psi_i(v)$ for each $i \in N$

$\phi$ is the unique efficient rule satisfying the previous axioms
• The axiom of symmetry can be replaced by the axiom of *anonymity*:
  Given a game \((N, v)\) and a permutation \(\pi\) of the players, let \((N, u)\) be the game where
  \(u(\pi(S)) = v(S)\) for each \(S \subseteq N\), then \(\psi_{\pi(i)}(u) = \psi_i(v)\)

• The axiom of dummy player can be replaced by the axiom of *null player*:
  Let \(i\) be a null player, i.e. \(v(S \cup \{i\}) = v(S)\) for each \(S \subseteq N \setminus \{i\}\), then \(\psi_i(v) = 0\)
3.2 Power Indices

*Power indices* represent a wide class of point solutions used in simple games, in order to evaluate the relevance or power of each player.

3.2.1 Shapley-Shubik Index

It is a modification of the Shapley value, proposed by Shapley and Shubik (1954) for simple games.

**Definition 7** A *swing for player* $i \in N$ *is a coalition* $S \subseteq N$ *such that* $S$ *is winning and* $S \setminus \{i\}$ *is losing.*

*Player* $i$ *is critical for* $S$.

**Definition 8** The *Shapley-Shubik Index* assigns to each player $i \in N$ the power:

$$
\varphi_i(v) = \sum_{S \text{ swing for } i} \frac{(s - 1)!(n - s)!}{n!}
$$
Example 5 (EU Council 1958-1973)

The Shapley-Shubik Index points out a worm in the weights assigned to the EU countries in the Council. In 1958, the majority quota was 12 on 17 (≈ 70%) while in 1973 it was 41 on 58 (≈ 70%)

<table>
<thead>
<tr>
<th>Countries</th>
<th>Weight</th>
<th>%</th>
<th>SS index</th>
<th>Weight</th>
<th>%</th>
<th>SS index</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>4</td>
<td>23.53</td>
<td>0.233</td>
<td>10</td>
<td>17.24</td>
<td>0.179</td>
</tr>
<tr>
<td>Germany</td>
<td>4</td>
<td>23.53</td>
<td>0.233</td>
<td>10</td>
<td>17.24</td>
<td>0.179</td>
</tr>
<tr>
<td>Italy</td>
<td>4</td>
<td>23.53</td>
<td>0.233</td>
<td>10</td>
<td>17.24</td>
<td>0.179</td>
</tr>
<tr>
<td>Belgium</td>
<td>2</td>
<td>11.76</td>
<td>0.150</td>
<td>5</td>
<td>8.62</td>
<td>0.081</td>
</tr>
<tr>
<td>The Netherlands</td>
<td>2</td>
<td>11.76</td>
<td>0.150</td>
<td>5</td>
<td>8.62</td>
<td>0.081</td>
</tr>
<tr>
<td>Luxemburg</td>
<td>1</td>
<td>5.88</td>
<td>0.000</td>
<td>2</td>
<td>3.45</td>
<td>0.010</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>10</td>
<td>17.24</td>
<td>0.179</td>
</tr>
<tr>
<td>Denmark</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>5.17</td>
<td>0.057</td>
</tr>
<tr>
<td>Ireland</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>5.17</td>
<td>0.057</td>
</tr>
<tr>
<td>Total</td>
<td>17</td>
<td>100.00</td>
<td>1.000</td>
<td>58</td>
<td>100.00</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Luxemburg reduced its weight in percentage, but was no longer a dummy player.
3.2.2 Normalized Banzhaf-Coleman Index

It was introduced by Banzhaf (1965) and Coleman (1971), following an idea of Penrose (1946).

**Definition 9** The Banzhaf-Coleman Index assigns to each player $i \in N$ the power:

$$\vartheta_i(v) = \sum_{S \text{ swing for } i} \frac{1}{2^n-1}$$

This index is not efficient. It may be normalized to 1; otherwise let $\beta_i^*(v)$ be the number of swings of player $i \in N$.

**Definition 10** The normalized Banzhaf-Coleman Index assigns to each player $i \in N$ the power:

$$\beta_i(v) = \frac{\beta_i^*(v)}{\sum_{j \in N} \beta_j^*(v)}$$
3.2.3 Deegan-Packel Index

It was introduced by Deegan and Packel (1978)
Let $W^m$ be the set of *minimal winning coalitions*, i.e. the winning coalitions in which all players
are critical or each proper subcoalition is losing

The unitary power is equally shared among the minimal winning coalitions and the quota of
each coalition is equally divided among its members

**Definition 11** The Deegan-Packel Index assigns to each player $i \in N$ the power:

$$\delta_i(v) = \sum_{S_j \ni i; S_j \in W^m} \frac{1}{m} \frac{1}{s_j}$$

where $m$ and $s_j$ are the cardinalities of $W^m$ and $S_j$, respectively

This index is non-monotonic w.r.t. the weights
Example 6 (Non-monotonicity) Consider the game defined by the weighted majority situation $[51; 26, 25, 25, 23, 1]$; the minimal winning coalitions are:

$$W^m = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{2, 3, 5\}\}$$

$$\delta(v) = \left(\frac{6}{24}', \frac{7}{24}', \frac{7}{24}', \frac{2}{24}, \frac{2}{24}\right)$$

Players 2 and 3 have a power larger than player 1, that has a larger weight

As reference, note that:

$$\varphi(v) = \left(\frac{22}{60}', \frac{17}{60}', \frac{17}{60}', \frac{2}{60}, \frac{2}{60}\right)$$

$$\beta(v) = \left(\frac{9}{16}', \frac{7}{16}', \frac{7}{16}', \frac{1}{16}, \frac{1}{16}\right)$$
3.2.4 Public Good Index

It was introduced by Holler (1982)
It is similar to the previous index, but it does not account the cardinalities of the minimal winning coalitions

Definition 12 Let $w_i, i \in N$ be the number of minimal winning coalitions including player $i$

The Public Good Index assigns to each player $i \in N$ the power:

$$h_i(v) = \frac{w_i}{\sum_{j \in N} w_j}$$

Also this index in non-monotonic w.r.t. the weights; referring to Example ??:

$$h(v) = \left( \frac{2}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10} \right)$$
3.2.5 Johnston Index

It was introduced by Johnston (1978)
Let $W^q$ be the set of quasi-minimal winning coalitions
The unitary power is equally shared among the quasi-minimal winning coalitions and the quota of each coalition is equally divided among its critical players

**Definition 13** Let $W^q_i, i \in N$ be the set of quasi-minimal winning coalitions for which player $i$ is critical
The Johnston Index assigns to each player $i \in N$ the power:

$$
\gamma_i(v) = \sum_{S_j \in W^q_i} \frac{1}{q} \frac{1}{c_j}
$$

where $q$ is the cardinality of $W^q$ and $c_j$ is the number of critical players in $S_j$

The Johnston Index is efficient and coincides with the Deegan-Packel Index if $W^m = W^q$
Example 7 Referring to Example ??, the quasi-minimal winning coalitions are (critical players are in red):

\[ W^q = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\} \]

The Johnston Index is:

\[ \gamma(v) = \left(\begin{array}{ccccc} 30 & 19 & 19 & 2 & 2 \\ \frac{72}{2} & \frac{72}{2} & \frac{72}{2} & \frac{72}{2} & \frac{72}{2} \end{array}\right) \]
4 The \( FP \) family of power indices for voting games

(see Fragnelli et al., 2009)

Power indices \( \rightarrow \) measure of the influence of each member of a coalition on decisions
In the last decade \( \rightarrow \) increasing attention in political science:
- study the voting power among EU states
- analyze the effects of institutional reforms

- For policy makers:
  Provide a power index with predictive value, designing a family of power indices that may be tailored on different situations with a suitable setting of some parameters
- For game theorists:
  Take into account the role of a party in favoring the formation of a majority, even if its influence on the majority in terms of seats (or percentage of votes) is null
See Moretti and Patrone (2008) and related comments
The Archetype

A suitable analysis of the ideologies allows to represent the parties on a left-right axis → ordering of the parties

\[ \downarrow \]

contiguous winning coalition \( S \): for all \( i, j \in S \) if there exists \( k \in N \) with \( i < k < j \) then \( k \in S \)

- Let \( W^c = \{ S_1, S_2, ..., S_m \} \) be the set of contiguous winning coalitions
- Equal sharing of the unitary power among coalitions \( S_j \in W^c \)
- Equal sharing of the quota assigned to coalition \( S_j \) among its members
- Each player \( i \in N \) sums up all the amounts received in the coalitions he belongs to

\[
FP_i = \sum_{S_j \in W^c, S_j \ni i} \frac{1}{m s_j}, \quad i \in N
\]

where \( s_j \) is the cardinality of \( S_j \)
4.1 The Plausibility Criterion

- Set of winning coalitions
- Probability of a coalition to form
- Sharing rule inside each winning coalition
Set of winning coalitions

- **office-seeking**: size of the coalition
  - minimal winning coalition (MWC) - von Neumann and Morgenstern (1944)
  - minimal number coalition (MNC): minimum of the parties - Leiserson (1966)
  - minimal size coalition (MSC): minimum of the seats - Riker (1962)


- other
Probability of a coalition to form

Equiprobability of coalitions may be not appealing

- ideological distance between the extreme parties - Bilal et al. (2001)
- distance between the preferred point of each party and the position of the coalition - Bilal et al. (2001)
- number of parties (proxy of ideological distance)
- number of parties or number of seats (cost of coordination and agreement): the larger the number, the lower the probability
Sharing rule inside each winning coalition

Equal sharing rule may be not suitable

- number of seats of each party
- critical role of parties (distance from potential opposition)
- median parties within the coalition - Bäck (2001)
4.2 Examples and Comparisons

**Example 8** Let \([9; 6, 2, 6]\) be a weighted majority situation
A unique contiguous winning coalition: \(\{1, 2, 3\}\)

Equally sharing the power among the players:

\[
FP = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\]

\(\varphi = \delta = (\frac{1}{2}, 0, \frac{1}{2})\)
Example 9 Let $[7; 1, 5, 2, 3]$ be a weighted majority situation

Four contiguous winning coalitions: $\{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$

\[ FP = (0.146, 0.354, 0.354, 0.146) \]

Proportional sharing w.r.t. the number of seats

\[ FP = (0.054, 0.573, 0.229, 0.143) \]

Sharing w.r.t. number of parties (probability = (0.353, 0.235, 0.235, 0.176)) and proportional sharing w.r.t. the number of seats

\[ FP = (0.045, 0.597, 0.239, 0.119) \]

\[ \varphi = (0, 0.666, 0.167, 0.167), \delta = (0, 0.50, 0.25, 0.25) \]

A unique minimal contiguous winning coalition: $\{2, 3\}$

\[ FP = (0, 0.50, 0.50, 0) \]
A player \( i \in S \) is \textit{contiguous-critical} for a contiguous winning coalition \( S \) if \( S \setminus \{i\} \) is losing or is no longer contiguous.

\textbf{Example 10} Let \([24; 13, 8, 2, 10, 11]\) be a weighted majority situation.

Three contiguous winning coalitions: \( \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\} \)

Sharing vectors w.r.t. contiguous-critical parties: \( \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right), \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right), (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) \)

\[
FP = \begin{pmatrix}
3 & 10 & 10 & 10 & 3 \\
36' & 36' & 36' & 36' & 36
\end{pmatrix}
\]
Coherence in the choice of the parameters

- minimal size coalition $\leftrightarrow$ number of seats
- minimal number coalition $\leftrightarrow$ number of parties
4.3 Extension

(see Chessa, Fragnelli, 2011)

The hypothesis of contiguity of the parties can be relaxed, using a graph as in the cooperation structure by Myerson (1977)

Consider an undirected graph \( g \) whose vertices correspond to the agents and the edges connect pairs of agents that may communicate

Given a game \( (N, v) \), define the restricted game \( (N, v_g) \) as

\[
v_g(S) = \sum_{T \in S/g} v(T), \quad S \subseteq N
\]

where \( S/g \) is the partition of \( S \) induced by the connected components of \( g \)

A coalition is feasible and its worth is “effective” if the vertices associated to its players are connected, otherwise the worth of the coalition is the sum of the worths of the communicating subcoalitions

The Myerson value of \( (N, v) \) is the Shapley value of \( (N, v_g) \)

The central role played by the contiguous coalitions is now assigned to the connected ones

The model of the left-right axis is a coalition structure represented by a line-graph

This approach allows for a multidimensional space of the parties
5 Evaluation of a Parliament

(see Fragnelli et al, 2005)

The choice of the best electoral system for a Parliament is very hard:

- Too many variables involved $\Rightarrow$ too difficult to balance all of them
- Complex methods are difficult to be understood and managed by voters

Some theorems - Arrow’s (1950) and McKelvey’s (1976) in primis - exclude the possibility of finding out the optimal rule, but no theorem prohibits finding out an empirical criterion of choice among two rules
Compare the performance of the two systems with reference to the same set of real preferences of voters

Votes are affected by the electoral system in use

How would you vote if the electoral system were $X$ in a country where the system is $Y$?
5.1 The Simulative Approach

Simulation is widely used in electoral systems analysis

- examine a specific mixed-member suggestion (Brichta, 1991)
- assess the proportionality of Chilean electoral system via opinion polls (Valenzuela, Siavelis, 1991)
- analyze the contendibility of a two-party system (Bender, Haas, 1996)
- find out equilibria in multiparty spatial models (Lomborg, 1997)
- estimate the effect of the adoption of alternative vote in Canada (Bilodeau, 1999)
- assess the motivations of the electoral reform in Italy (Navarra, Sobbrio, 2001)

Referring to comparison of systems

- analyze the effect in Italy of a change to a number of electoral systems (Gambarelli, Biella, 1992)
- compare six majoritarian systems but without reference to a Parliament (Christensen, 2003)
5.2 The Choice of the Optimal Electoral System

The choice may be affected by a lot of facets of the political process

- **representativeness,** $R$: efficiency in representing electors’ willingness
- **governability,** $G$: effect on the efficiency of the resulting government

In addition:

- incentives for politicians (Riker, 1982; Myerson, 1995)
- corruption (Myerson, 1993, 2001; Persson, Tabellini, Trebbi, 2001)
- information / participation of voters (Mudambi, Navarra, Nicosia, 1995; Mudambi, Navarra, Sobbrio, 1999)
- power of the lobbies (Myerson, 1995)
- strategic choices (Levin, Nalebuff, 1995)
- complexity of the voting system (Levin, Nalebuff, 1995)
- protection of the minorities (Levin, Nalebuff, 1995; Rae, 1995; Sen, 1995)
- risk of extreme choices (Levin, Nalebuff, 1995)
- use of votes as a “voice” device (Sen, 1995; Brennan, Hamlin, 1998)
- overall welfare (Mueller, Stratmann, 2000)
- responsiveness of the government’s choice to the preferences of the voters (Shugart, 2001)
- ...
5.3 The Choice with Two Parameters

Dominance

System "*" is dominant (it is very likely not to exist)
System 4 is dominated (it may safely be excluded)
Systems 1, 2, 3 are alternative systems (Pareto optimality → which system should be chosen?)
Alternative systems may be compared via a social utility function (SUF), e.g. a Cobb-Douglas function

$$U = K g^a r^b$$

where $K$ is a suitable constant

$a$ and $b$ are the partial elasticity of $U$ w.r.t. $g$ and $r$

$$X \succ Y \iff K g_x^a r_x^b > K g_y^a r_y^b$$

Let $p = \frac{a}{b}$

$$X \succ Y \iff \left( \frac{g_x}{g_y} \right)^p > \left( \frac{r_y}{r_x} \right)^b$$

**Remark 1** $p$ may be characterized as the price in terms of a relative variation of $r$ that the community accepts to pay for a given relative opposite variation of $g$

$p = 2$ means that it is worthwhile to accept a $20\%$ reduction of $r$ to gain a $10\%$ increase of $g$
Graphically - Indifference curves

\[ r = \left( \frac{U^*}{K} \right)^{\frac{1}{b}} \]

but ... or ...

the three systems are indifferent
6 Disproportionality Indices

This list is taken from Karpov (2008)
Representativeness and disproportionality are strongly related; for the first the greater the better, for the second the less the better

Notations

\[ N = \{1, \ldots, n\} \] set of parties
\[ S_i, V_i \] number of seats, votes of party \( i \in N \)
\[ s_i, v_i \] percentage of seats, votes of party \( i \in N \)
6.1 Absolute Deviation Indices

**Maximum Deviation Index**

\[ MD = \max_{i \in N} |s_i - v_i| \]

**Rae Index**

\[ I_{Rae} = \frac{1}{n} \sum_{i \in N} |s_i - v_i| \]

**Loosmore-Hanby Index**

\[ I_{LH} = \frac{1}{2} \sum_{i \in N} |s_i - v_i| \]

**Grofman Index**

\[ I_{Gr} = \frac{1}{E} \sum_{i \in N} |s_i - v_i| \]

where \( E = \frac{1}{\sum_{i \in N} v_i^2} \) if the effective number of parties

**Lijphart Index**

\[ I_L = \frac{|s_i - v_i| + |s_j - v_j|}{2} \]

where \( i \) and \( j \) are the two largest parties
6.2 Quadratic Indices

Gallagher Index

\[ Lsq = \sqrt{\frac{1}{2} \sum_{i \in N} (s_i - v_i)^2} \]

Variations

\[ H_k = k \left( \frac{1}{k} \sum_{i \in N} (s_i - v_i)^k \right) \]
\[ \tilde{H}_k = k \left( \frac{1}{2} \sum_{i \in N} (s_i - v_i)^k \right) \]

Monroe Index

\[ I_M = \sqrt{\frac{\sum_{i \in N} (s_i - v_i)^2}{1 + \sum_{i \in N} v_i^2}} \]

Gatev Index

\[ I_{Ga} = \sqrt{\frac{\sum_{i \in N} (s_i - v_i)^2}{\sum_{i \in N} (s_i^2 + v_i^2)}} \]

Ryabtsev Index

\[ I_R = \sqrt{\frac{\sum_{i \in N} (s_i - v_i)^2}{\sum_{i \in N} (s_i + v_i)^2}} \]

Szalai Index

\[ I_S = \sqrt{\frac{\sum_{i \in N} \left( \frac{s_i - v_i}{s_i + v_i} \right)^2}{n}} \]

Variation

\[ \tilde{I}_S = \sqrt{\frac{1}{2} \sum_{i \in N} \left( \frac{s_i - v_i}{s_i + v_i} \right)^2} \]
6.3 Aleskerov-Platonov Index

\[ R = \frac{1}{k} \sum_{i \in O} \frac{s_i}{v_i} \]

where \( O = \{i_1, \ldots, i_k\} \) is the set of overrepresented parties.
6.4 Inequality Indices

**Gini Index**

\[ s_1 \leq s_2 \leq \ldots \leq s_{n-1} \leq s_n \]

\( G \) is the ratio of the areas

**Atkinson Index**

\[
A = 1 - \left[ \sum_{i \in N} v_i \left( \frac{y_i}{\mu} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}
\]

where \( y_i = \frac{s_i}{v_i} \), \( \mu = \frac{S}{V} \) and \( \varepsilon \) is a measure of inequality; equivalently

\[
A = 1 - \left[ \sum_{i \in N} v_i \left( \frac{s_i}{v_i} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}
\]

**Generalized Entropy**

\[
GE = \frac{1}{\alpha^2 - \alpha} \left[ \sum_{i \in N} v_i \left( \frac{y_i}{\mu} \right)^\alpha - 1 \right]
\]

where \( \alpha \) is a parameter; equivalently

\[
GE = \frac{1}{\alpha^2 - \alpha} \left[ \sum_{i \in N} v_i \left( \frac{s_i}{v_i} \right)^\alpha - 1 \right]
\]
6.5 Objective Function Indices

Sainte-Lague Index

\[
SL = \sum_{i \in N} v_i \left( \frac{s_i}{v_i} - 1 \right)^2
\]

D’Hondt Index

\[
H = \max_{i \in N} \frac{s_i}{v_i}
\]
6.6 Three More Indices and the Issue of Power

**Ortona index** (see Fragnelli-Monella-Ortona, 2005)

It is based on the difference between seats assigned by a given electoral system and seats assigned by a perfect proportional system, $PP$, i.e. supposing a unique nation-wide proportional district and assigning the rest to the largest decimals, avoiding to combine votes and seats

$$d = \frac{\sum_{i \in N} |S_i - S_i^{PP}|}{\sum_{i \in N} |S_i^u - S_i^{PP}|}$$

where $N$ is the set of parties, $S_i$ is the number of seats of party $i$ with the system under consideration, $S_i^{PP}$ is the number of seats of party $i$ with the perfect proportional system and $S_i^u$ is the total number of seats for the relative majority party and 0 otherwise.
Fragnelli index (see Fragnelli, 2009)

Starting from the vote share \((v_1, \ldots, v_n)\) and the seat share \((s_1, \ldots, s_n)\), define two simple games \((N, w)\) and \((N, u)\), where \(N = \{1, \ldots, n\}\) is the set of parties and the characteristic functions \(w\) and \(u\) are defined using the vote share and the seat share, respectively.

This disproportionality index measures the distance of the distribution of power on the votes and on the seats, i.e. \(\sum_{i \in N} |\varphi_i(w) - \varphi_i(u)|\)

**Remark 2** The disproportionality is minimal, i.e. equal to 0, when the power of each party is identical in the two distributions.

A particular case happens when the percentages of votes coincide with the percentages of seats.
The index may assume a value larger than 1

**Example 11** Consider a relative majority system with two parties $P_1$ and $P_2$ and three districts $A, B$ and $C$, each one electing a single member

The results of the elections are in the following table:

<table>
<thead>
<tr>
<th>Parties</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>% of votes</th>
<th># of seats</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>16</td>
<td>6</td>
<td>8</td>
<td>60</td>
<td>1</td>
</tr>
<tr>
<td>$P_2$</td>
<td>2</td>
<td>8</td>
<td>10</td>
<td>40</td>
<td>2</td>
</tr>
</tbody>
</table>

In this case $\varphi(w) = (1, 0)$ and $\varphi(u) = (0, 1)$, so the index is $1 + 1 = 2$

Normalize the index in the interval $[0, 1]$, simply dividing by 2, as in the worst case the two distributions of power may assign complementary values:

$$d^\Omega = \frac{\sum_{i \in N} |\varphi_i(w) - \varphi_i(u)|}{2}$$
Gambarelli-Biella index (see Gambarelli-Biella, 1992)

It combines the traditional approach, which considers vote and seat shares, with the idea of measuring the distance of the distributions of power related to the votes and to the seats:

\[
\Delta = \max_{i \in N} \{ |v_i - s_i|, |\varphi_i(w) - \varphi_i(u)| \}
\]

Remark 3 Note that \(d^\Omega\) is based on norm 1, while \(\Delta\) is based on norm \(\infty\), so it is possible to define other indices based on other norms
For computing the Fragnelli and Gambarelli-Biella indices two of the most popular power indices are used:

- the Shapley-Shubik index (1954)

\[
\phi_i(w) = \sum_{S \in W, S \ni i} \frac{(|S| - 1)! (n - |S|)!}{n!} [w(S) - w(S \setminus \{i\})] \quad \forall i \in N
\]

where \( W \) is the set of winning coalitions

- the Public Good Index (Holler, 1982)

\[
PGI_i(w) = \frac{h_i}{\sum_{k \in N} h_k} \quad \forall i \in N
\]

where \( h_i \) is the number of minimal winning coalitions including party \( i \in N \)

The former has the monotonicity property, so that it assigns larger power to parties with greater number of seats, while the latter does not respect monotonicity and the differences of power are often very small w.r.t other non-monotonic indices.
**Case-study 1**

Three parties $A$, $B$, $C$ receive a percentage of votes of 49.5%, 48.5%, 2.0%, respectively and 6 seats have to be assigned. Consider three different seats assignments:

$$
\begin{array}{|c|c|c|}
\hline
& A & B & C \\
\hline
s_{PP} & 3 & 3 & 0 \\
\hline
s & 3 & 2 & 1 \\
\hline
s_{PO} & 2 & 2 & 2 \\
\hline
\end{array}
$$

$s_{PP}$ assigns the seats according to the perfect proportional system; $s$ guarantees one seat to party $C$, so party $A$ receives one seat more than party $B$; finally, $s_{PO}$ ($PO$ after Power Oriented) guarantees a distribution of power on the seat sharing equal to the distribution of power on the vote sharing.

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
& MD & I_{Rae} & I_{LH} & I_{Gr} & I_L & Lsq & I_M & I_{Ga} & I_R & I_S & G & R & SL & H \\
\hline
s_{PP} & 0.020 & 0.013 & 0.020 & 0.019 & 0.010 & 0.018 & 0.021 & 0.026 & 0.018 & 0.577 & 0.020 & 1.021 & 0.021 & 1.031 \\
\hline
s & 0.152 & 0.101 & 0.152 & 0.146 & 0.078 & 0.149 & 0.173 & 0.226 & 0.162 & 0.587 & 0.228 & 1.010 & 1.123 & 8.333 \\
\hline
s_{PO} & 0.313 & 0.209 & 0.313 & 0.301 & 0.157 & 0.271 & 0.315 & 0.425 & 0.315 & 0.598 & 0.313 & 16.667 & 5.009 & 16.667 \\
\hline
\end{array}
$$

$s_{PO}$ is the worst under all the indices.
Shapley-Shubik and Public Good indices:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(v)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\phi(s_{PP})$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\phi(s)$</td>
<td>$\frac{4}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$\phi(s_{PO})$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$PGI(v)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$PGI(s_{PP})$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$PGI(s)$</td>
<td>$\frac{2}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$PGI(s_{PO})$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Added indices:

<table>
<thead>
<tr>
<th></th>
<th>$d$</th>
<th>$d^\Omega$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{PP}$</td>
<td>0</td>
<td>0.333</td>
<td>0.333</td>
</tr>
<tr>
<td>$s$</td>
<td>0.333</td>
<td>0.333</td>
<td>0.167</td>
</tr>
<tr>
<td>$s_{PO}$</td>
<td>0.667</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The index by Ortona that does not account the power assigns the worst value to $s_{PO}$, that on the other hand receives the best score under the index by Fragnelli that is strongly power-oriented; the index by Gambarelli-Biella is influenced by the seat assignment.
Case-study 2

Three parties $A, B, C$ receive a percentage of votes of 49.9%, 25.2%, 24.9%, respectively; suppose two different Parliaments with 9 and 10 seats, respectively. Consider the assignment of the seats under $PP$ ($s_{PP}$) and looking at the power distribution ($s_{PO}$):

\[
\begin{array}{c|ccc}
   & A & B & C \\
9 - s_{PP} & 5 & 2 & 2 \\
9 - s_{PO} & 4 & 3 & 2 \\
10 - s_{PP} & 5 & 3 & 2 \\
10 - s_{PO} & 4 & 3 & 3 \\
\end{array}
\]

$s_{PP}$ gives to party $A$ the absolute majority in the Parliament with 9 seats and the veto power with 10 seats, differently from the distribution of votes and $s_{PO}$

\[
\begin{array}{lcccccccccccc}
   & MD & I_{Rae} & I_{LH} & I_{Gr} & I_L & Lsq & I_M & I_{Ga} & I_R & I_S & G & R & SL & H \\
9 - s_{PP} & 0.057 & 0.038 & 0.057 & 0.042 & 0.043 & 0.049 & 0.059 & 0.078 & 0.056 & 0.579 & 0.057 & 1.113 & 0.013 & 1.113 \\
9 - s_{PO} & 0.081 & 0.054 & 0.081 & 0.061 & 0.068 & 0.072 & 0.087 & 0.119 & 0.084 & 0.584 & 0.082 & 1.323 & 0.035 & 1.323 \\
10 - s_{PP} & 0.049 & 0.033 & 0.049 & 0.037 & 0.025 & 0.049 & 0.059 & 0.079 & 0.056 & 0.580 & 0.073 & 1.096 & 0.019 & 1.190 \\
10 - s_{PO} & 0.099 & 0.066 & 0.099 & 0.074 & 0.074 & 0.086 & 0.103 & 0.143 & 0.102 & 0.583 & 0.100 & 1.198 & 0.039 & 1.205 \\
\end{array}
\]

Again, $s_{PO}$ is the worst under all the indices, with both 9 and 10 seats
### Shapley-Shubik and Public Good indices:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(v)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\phi(9 - s_{PP})$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\phi(9 - s_{PO})$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\phi(10 - s_{PP})$</td>
<td>$\frac{4}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$\phi(10 - s_{PO})$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$PGI(v)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$PGI(9 - s_{PP})$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$PGI(9 - s_{PO})$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$PGI(10 - s_{PP})$</td>
<td>$\frac{2}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$PGI(10 - s_{PO})$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

### Added indices:

<table>
<thead>
<tr>
<th></th>
<th>$d$</th>
<th>$d^\Omega$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>$PGI$</td>
<td>$\phi$</td>
<td>$PGI$</td>
</tr>
<tr>
<td>$9 - s_{PP}$</td>
<td>$0$</td>
<td>$0.667$</td>
<td>$0.667$</td>
</tr>
<tr>
<td>$9 - s_{PO}$</td>
<td>$0.250$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$10 - s_{PP}$</td>
<td>$0$</td>
<td>$0.333$</td>
<td>$0.167$</td>
</tr>
<tr>
<td>$10 - s_{PO}$</td>
<td>$0.200$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Again, the index by Ortona assigns the worst value to $s_{PO}$, that receive the best score under the two power-oriented indices.
Remark 4

- The issue of power allows new perspective for analyzing the disproportionality.
- With a higher number of seats it is possible to reduce the differences among the indices. For instance, referring to case-study 1 with 15 seats, \( s_{PP} \) and \( s \) correspond to \((8, 7, 0)\) and \((7, 7, 1)\), respectively, and the first is easily rejected because Party A would have the absolute majority.

To guarantee the same distribution of power on the votes and on the seats it is sufficient to assign to each party a number of seats greater than or equal to 1 and less than or equal to 7, i.e. \( s_{PO} \) can be selected equal to \( s \).
7 Governability indices

Ortona index (see Fragnelli-Monella-Ortona, 2005)

\[ g_h = \frac{1}{m_h + 1} + \frac{1}{m_h(m_h + 1)} \frac{f_h - \frac{T}{2}}{\frac{T}{2}} \]

where \( m_h \) is the number of parties of the governing coalition under system \( h \) that may destroy the majority if they withdraw, \( f_h \) is the number of seats of the majority under system \( h \) and \( T \) is the total number of seats in the Parliament.
7.1 Power and Governability

(see Fragnelli, 2009)

Why not to use power indices also for governability?

Propensity to disrupt (Gately, 1974) for player $i \in N$:

$$x(N \setminus \{i\}) - v(N \setminus \{i\}) \quad \frac{x_i - v(i)}{x_i - v(i)}$$

Remark 5 $x_i - v(i)$ and $x(S \setminus \{i\}) - v(S \setminus \{i\})$ may be used to evaluate the stability of a parliamentary coalition $S \subseteq N$, while the propensity to disrupt is usually equal to $-1$

When $S$ represents a majority coalition, it is possible to emphasize its power using the Owen value (Owen, 1977), $\Omega$, instead of the Shapley value, because it assigns null power to each player not in the majority

$$\Omega_i(K) = \sum_{H \subseteq K} \sum_{S \subseteq T_j} \frac{h!(k - h - 1)!s!(t_j - s - 1)!}{k!t_j!} [v(H \cup S \cup \{i\}) - v(H \cup S)], \ i \in N$$

where $K = \{T_1, ..., T_k\}$ is the a priori unions structure, $n = |N|, s = |S|, h = |H|, k = |K|$ and $t_j = |T_j|$
Consider \( \Omega_i(S) - \Omega_i(S \setminus \{i\}) \), \( i \in S \), where \( \Omega(S) \) is the Owen value when the a priori unions structure considers the coalition \( S \) and all the other players as singletons, i.e. \( K = \{S, \{i_1\}, ..., \{i_{n-s}\}\} \), where \( s = |S| \)

\( \Omega_i(S) - \Omega_i(S \setminus \{i\}) \) is a measure of the propensity of party \( i \) to stay in the majority

Consequently, the higher is the propensity of the parties to stay in the majority, the higher is the stability of the majority

Summing up on all the players in \( S \), we get the following governability index

\[
\sum_{i \in S} (\Omega_i(S) - \Omega_i(S \setminus \{i\})) = 1 - \sum_{i \in S} \Omega_i(S \setminus \{i\})
\]

(if \( S \) is a majority \( \sum_{i \in S} \Omega_i(S) = 1 \))

Consider \( \Omega_i(S, N \setminus S) - \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}) \), \( i \in S \), where \( \Omega(S, N \setminus S) \) is the Owen value when the a priori unions structure \( K_S \) considers the coalition \( S \) and the complementary coalition \( N \setminus S \), i.e. \( K = \{S, N \setminus S\} \)

\( \Omega_i(S, N \setminus S) - \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}) \) is a measure of the propensity of party \( i \) to stay in the majority

Summing up on all the players in \( S \), we get the following governability index

\[
\sum_{i \in S} (\Omega_i(S, N \setminus S) - \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})) = 1 - \sum_{i \in S} \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})
\]

(if \( S \) is a majority \( \sum_{i \in S} \Omega_i(S) = 1 \))
Remark 6 According to standard literature, the governability is maximal, i.e. equal to 1, when \( S \) is such that each subcoalition \( S \setminus \{i\} \) is winning, i.e. the majority is not affected whichever party leaves it. Note that for each party \( i \in S \), \( \Omega_i(S \setminus \{i\}) = 0 \) and \( \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}) = 0 \). These indices may assume a negative value.

Example 12 Consider a majority game \((N, v)\) in which only the grand coalition is winning. In this case \( \Omega_i(N) = \frac{1}{n}, i \in N \); if party \( k \) leaves the majority its power becomes \( \Omega_k(N \setminus \{k\}) = \Omega_k(N \setminus \{k\}, \{k\}) = \frac{1}{2} \) and the power of the other \( n - 1 \) parties becomes \( \Omega_i(N \setminus \{k\}) = \Omega_i(N \setminus \{k\}, \{k\}) = \frac{1}{2} \frac{1}{n - 1}, i \in N \setminus \{k\} \). So, both indices \( 1 - \sum_{i \in N} \Omega_i(N \setminus \{i\}) = 1 - \sum_{i \in N} \Omega_i(N \setminus \{i\}, \{i\}) = 1 - \sum_{i \in N} \frac{1}{2} \) are negative when \( n \geq 3 \).

Normalize the index in the interval \([0, 1]\), simply dividing by \( n \), as \( 0 \leq \Omega_i(S \setminus \{i\}) \leq 1 \):

\[
g^\Omega = 1 - \frac{\sum_{i \in S} \Omega_i(S \setminus \{i\})}{n}
\]

Normalize the index in the interval \([0, 1]\), simply dividing by \( n \), as \( 0 \leq \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}) \leq 1 \):

\[
g^\Omega^* = 1 - \frac{\sum_{i \in S} \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})}{n}
\]
7.2 A Comparative Example

(see Fragnelli, Ortona, 2006)

<table>
<thead>
<tr>
<th>Voting system</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>$P_6$</th>
<th>$P_7$</th>
<th>$P_8$</th>
<th>$P_9$</th>
<th>$P_{10}$</th>
<th>$P_{11}$</th>
<th>$P_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PP$ (= $v$)</td>
<td>18</td>
<td>22</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>17</td>
<td>28</td>
<td>0</td>
</tr>
<tr>
<td>$P - 4$</td>
<td>19</td>
<td>22</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>17</td>
<td>28</td>
<td>0</td>
</tr>
<tr>
<td>$P(20)$</td>
<td>14</td>
<td>16</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>12</td>
<td>48</td>
<td>0</td>
</tr>
<tr>
<td>$M$</td>
<td>14</td>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$2R$</td>
<td>14</td>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C$</td>
<td>14</td>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>46</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>9</td>
<td>44</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td>$A$</td>
<td>18</td>
<td>37</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>27</td>
<td>0</td>
</tr>
<tr>
<td>$I - 25$</td>
<td>15</td>
<td>33</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>45</td>
<td>0</td>
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<tr>
<td>$I - 75$</td>
<td>17</td>
<td>25</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>13</td>
<td>33</td>
<td>0</td>
</tr>
</tbody>
</table>

$PP$ Pure Proportionality  $M$ Relative Majority  $B$ Borda Count

$P - n$ Threshold Proportionality  $2R$ Two-round Runoff  $A$ Approval Voting

$P(n)$ Prized Proportionality  $C$ Condorcet Method  $I - n$ Mixed-member

A unique 100-seat constituency for the $PP$ and $P - n$, a unique ($100 - n$)-seat constituency for $P(n)$, a $n$-seat plus $100 - n$ one-seat constituencies for $I - n$ and 100 one-seat constituencies for all the other systems.

Bold numbers identify the parties forming the majority in each system.
Indices:

<table>
<thead>
<tr>
<th>Voting system</th>
<th>( r = 1 - d )</th>
<th>( g )</th>
<th>( r^\Omega = 1 - d^\Omega )</th>
<th>( g^\Omega )</th>
<th>( g^\Omega^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP</td>
<td>1.000</td>
<td>0.201</td>
<td>1.000</td>
<td>0.944</td>
<td>0.895</td>
</tr>
<tr>
<td>( P - 4 )</td>
<td>0.986</td>
<td>0.204</td>
<td>0.958</td>
<td>0.932</td>
<td>0.896</td>
</tr>
<tr>
<td>( P(20) )</td>
<td>0.722</td>
<td>0.367</td>
<td>0.621</td>
<td>0.937</td>
<td>0.917</td>
</tr>
<tr>
<td>( M )</td>
<td>0.500</td>
<td>0.453</td>
<td>0.676</td>
<td>0.931</td>
<td>0.917</td>
</tr>
<tr>
<td>( 2R )</td>
<td>0.500</td>
<td>0.453</td>
<td>0.676</td>
<td>0.931</td>
<td>0.917</td>
</tr>
<tr>
<td>( C )</td>
<td>0.556</td>
<td>0.257</td>
<td>0.801</td>
<td>0.931</td>
<td>0.896</td>
</tr>
<tr>
<td>( B )</td>
<td>0.667</td>
<td>0.343</td>
<td>0.651</td>
<td>0.944</td>
<td>0.917</td>
</tr>
<tr>
<td>( A )</td>
<td>0.795</td>
<td>0.350</td>
<td>0.651</td>
<td>0.944</td>
<td>0.917</td>
</tr>
<tr>
<td>( I - 25 )</td>
<td>0.611</td>
<td>0.201</td>
<td>0.879</td>
<td>0.936</td>
<td>0.896</td>
</tr>
<tr>
<td>( I - 75 )</td>
<td>0.889</td>
<td>0.201</td>
<td>0.915</td>
<td>0.944</td>
<td>0.897</td>
</tr>
</tbody>
</table>
Graphical representation (different vertical scale):

In this case \( r \) and \( g \) produce six Pareto optimal systems.

In this case \( r^\Omega \) and \( g^\Omega \) identify PP as a dominant system.

In this case \( r^\Omega \) and \( g^{\Omega^*} \) produce eight Pareto optimal systems.
References

Aleskerov F (2008), Power distribution in the electoral body with an application to the Russian Parliament, ICER applied mathematics working paper series, n.11.


Lomborg B. (1997), Adaptive Parties in a Multiparty, Multidimensional System with Imperfect Information, Working paper, Department of Political Science, University of Aarhus.


