On 1-convexity and nucleolus of co-insurance games

Theo Driessen[†], Vito Fragnelli[‡], Ilya Katsev[§], Anna Khmelnitskaya[§]

[†] Twente University, The Netherlands

[‡] Universitá del Piemonte Orientale, Italy

[§] St. Petersburg Institute for Economics and Mathematics,
Russian Academy of Sciences, Russia

Universitá del Piemonte Orientale, Alessandria

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Outline

A *cooperative TU game* is a pair $\langle N, v\rangle$ where
$N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players,
$v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is a *characteristic function*.

A subset $S \subseteq N$ (or $S \in 2^N$) of s players is a *coalition*,
$v(S)$ presents the *worth* of the coalition S.

$\mathcal{G}_N$ is the class of TU games with a fixed player set N.
($\mathcal{G}_N = \mathbb{R}^{2^n-1}$ of vectors $\{v(S)\}_{\substack{S\subseteq N\\ S\neq\emptyset}}$)

Any vector $x \in \mathbb{R}^n$ can be considered as a *payoff vector* in a game $v \in \mathcal{G}_N$,
the real number x_i is the *payoff* to player i.

A payoff vector $x \in \mathbb{R}^n$ is said to be *efficient* in a game v, if $x(N) = v(N)$.

A *subgame* of a game v is a game $v|_T$ with a player set $T \subset N$, $T \neq \emptyset$, and
$v|_T(S) = v(S)$ for all $S \subseteq T$.

A game v is *nonnegative* if $v(S) \geq 0$ for all $S \subseteq N$.

A game v is *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.

The *imputation set* of a game v is a set of efficient and individually rational payoff vectors

$$I(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \geq v(i),\ \text{for all } i \in N\}.$$

The *core* of a game v (Gillies, 1953) is a set of efficient payoff vectors that are not dominated by any coalition

$$C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S),\ \text{for all } S \subseteq N\}.$$

A game v is *balanced* if $C(v) \neq \emptyset$.

For a game v, the *excess* of a coalition $S \subseteq N$ with respect to a payoff vector $x \in \mathbb{R}^n$ is

$$e^v(S, x) = v(S) - x(S).$$

The *nucleolus* of a game v (Schmeidler, 1969) is a minimizer of the lexicographic ordering of components of the excess vector of a given game v arranged in decreasing order of their magnitude over the imputation set $I(v)$:

$$\nu(v) = x \in I(v): \quad \theta(x) \preceq_{lex} \theta(y),\ \forall y \in I(v),$$

where $\theta(x) = (e(S_1, x), e(S_2, x), ..., e(S_{2^n-1}, x))$,
while $e(S_1, x) \geq e(S_2, x) \geq ... \geq e(S_{2^n-1}, x)$.

In a balanced game v the nucleolus $\nu(v) \in C(v)$.

The *imputation set* of a game v is a set of efficient and individually rational payoff vectors

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For a game v we consider a *marginal worth vector* $m^v \in \mathbb{R}^n$ equal to the vector of marginal contributions to the grand coalition,

$$m_i^v = v(N) - v(N \backslash \{i\}), \qquad \text{for all } \ i \in N,$$

and the *gap* vector $g^v \in \mathbb{R}^{2^N}$,

$$g^v(S) = \begin{cases} \sum_{i \in S} m_i^v - v(S), & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases}$$

that for every coalition $S \subseteq N$ measures the total coalitional surplus of marginal contributions to the grand coalition over its worth.

In fact, $g^v(S) = -e^v(S, m^v)$.

It is easy to check that in any game v, the vector m^v provides upper bounds of the core: for any $x \in C(v)$,

$$x_i \leq m_i^v, \qquad \text{for all } \ i \in N.$$

In particular, for an arbitrary game v, the condition

$$v(N) \leq \sum_{i \in N} m_i^v$$

is a necessary (but not sufficient) condition for non-emptiness of the core,

i.e., a strictly negative gap of the grand coalition $g^v(N) < 0$ implies $C(v) = \emptyset$.

A game v is *convex* if for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T),$$

or equivalently, if for all $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

Any convex game has a nonempty core (Shapley, 1971).

Proposition

In any convex game $v \in \mathcal{G}_N$,

$$g^v(N) \geq 0,$$

$$g^v(N) \geq g^v(S), \qquad \text{for all } S \subseteq N.$$

A game $v \in \mathcal{G}_N$ is *1-convex* if

$$0 \le g^v(N) \le g^v(S), \qquad \text{for all } \; S \subseteq N,\, S \neq \emptyset.$$

As it is shown in Driessen and Tijs (1983) and Driessen (1985), in a 1-convex game v,

- every 1-convex game has a nonempty core $C(v)$;
- for every efficient vector $x \in \mathbb{R}^n$,

  $$x_i \le m_i^v, \;\text{ for all } i \in N \implies x \in C(v);$$

  in particular, the characterizing property of a 1-convex game is:

  $$\bar{m}^v(i) = \{\bar{m}_j^v(i)\}_{j \in N} \in C(V),$$

  $$\bar{m}_j^v(i) = \begin{cases} v(N) - m^v(N \backslash i) = m_i^v - g^v(N), & j = i, \\ m_j^v, & j \neq i, \end{cases} \qquad \text{for all } \; j \in N;$$

  moreover, $\{\bar{m}^v(i)\}_{i \in N}$ is a set of extreme points of $C(v)$, and $C(v) = co(\{\bar{m}^v(i)\}_{i \in N})$;
- the nucleolus coincides with the barycenter of the core vertices, and is given by

  $$\nu_i(v) = m_i^v - \frac{g^v(N)}{n}, \qquad \text{for all } \; i \in N,$$

  i.e., the nucleolus defined as a solution to some optimization problem that, in general, is difficult to compute, appears to be linear and thus simple to determine.

Proposition

In any convex game $v \in \mathcal{G}_N$,

$$g^v(N) \geq 0,$$

$$g^v(N) \geq g^v(S), \qquad \textit{for all } S \subseteq N.$$

Corollary

A convex game $v \in \mathcal{G}_N$ is 1-convex, if and only if

$$g^v(N) = g^v(S), \qquad \textit{for all } S \subseteq N,\ S \neq \emptyset.$$

Consider the problem in which a risk is evaluated too much heavy for a single insurance company, but it can be insured by the finite set N of companies that share the risk and the premium.

By hypothesis it is assumed that

- every company $i \in N$ evaluates an insurable risk X through a real-valued nonnegative function $H_i(X)$ such that $H_i(0) = 0$;
- for any nonempty set $S \subseteq N$ of companies an optimal decomposition of the given risk exists, and therefore,

$$\min_{X \in \mathcal{A}(S)} \sum_{i \in S} H_i(X_i) := \mathcal{P}(S) \text{ is well-defined;}$$

  here $\mathcal{A}(S) = \{X \in \mathbb{R}^S \mid \sum_{i \in S} X_i = \mathcal{R}\}$ represents the (non-empty) set of feasible decompositions of the risk $\mathcal{R}$ over companies in S.

The real-valued set function $\mathcal{P}(S)$ can be seen as the evaluation of the optimal decomposition of the risk $\mathcal{R}$ by the companies in coalition S as a whole.

$\mathcal{P}$ is nonnegative and non-increasing, i.e., for all $S \subseteq T \subseteq N$, $S \neq \emptyset$, $0 \leq \mathcal{P}(T) \leq \mathcal{P}(S)$.

In case of *constant quotas* specified by a priori given quotas $q_i > 0$, $i \in N$, $\sum_{i \in N} q_i = 1$, it is assumed that for each insurable risk $\mathcal{R}$,

- for every insurance company $i \in N$,

$$H_i(\mathcal{R}) = q_i H\Big(\frac{\mathcal{R}}{q_i}\Big),$$

  where H is a certain a priori fixed convex function,

- for any $S \subseteq N$, $S \neq \emptyset$, there exists the optimal decomposition $\left(\frac{q_i}{q(S)}\mathcal{R}\right)_{i \in S} \in \mathbb{R}^S$.

In this case the evaluation function $\mathcal{P}$ is simply given by

$$\mathcal{P}(S) = \sum_{i \in S} H_i\Big(\frac{q_i}{q(S)}\mathcal{R}\Big) = q(S) H\Big(\frac{\mathcal{R}}{q(S)}\Big), \quad \text{for all } S \subseteq N, S \neq \emptyset.$$

If insurance companies evaluate the risk $\mathcal{R}$ according to the *variance principle*

$$H_i(\mathcal{R}) = E(\mathcal{R}) + a_i Var(\mathcal{R}), \quad a_i > 0, \qquad \text{for all } i \in N,$$

($E(\mathcal{R})$ and $Var(\mathcal{R})$ denote the expectation and variance of a random variable $\mathcal{R}$) then we are in case of a priori given quotas

$$q_i = \frac{a(N)}{a_i}, \qquad a(N) = \Big(\sum_{i \in N} \frac{1}{a_i}\Big)^{-1}$$

(cf. Deprez and Gerber (1985), Fragnelli and Marina (2004)).

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$\mathcal{P}$ is nonnegative and non-increasing, i.e., for all $S \subseteq T \subseteq N$, $S \neq \emptyset$, $0 \leq \mathcal{P}(T) \leq \mathcal{P}(S)$.

For a given premium Π and an evaluation function $\mathcal{P}\colon 2^N \setminus \emptyset \to \mathbb{R}$, the associated *co-insurance game* $v_{\Pi,\mathcal{P}} \in \mathcal{G}_N$ is defined in Fragnelli and Marina (2004) by

$$v_{\Pi,\mathcal{P}}(S) = \begin{cases} \max\{0, \Pi - \mathcal{P}(S)\}, & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

The co-insurance game $v_{\Pi,\mathcal{P}}$ is nonnegative and monotonic, i.e.,

$$0 \leq v_{\Pi,\mathcal{P}}(S) \leq v_{\Pi,\mathcal{P}}(T), \qquad \text{for all } S \subseteq T \subseteq N.$$

Let the evaluation function $\mathcal{P}$ be fixed and consider the co-insurance game $v_{\Pi,\mathcal{P}}$ with respect to the variable premium Π.

To avoid trivial situations we suppose $\Pi > \mathcal{P}(N)$.

The following results are already shown in Fragnelli and Marina (2004):

- If Π is small enough, $\Pi \leq \max_{i \in N} \mathcal{P}(N \backslash \{i\})$, then $v_{\Pi,\mathcal{P}}$ is balanced:

  $C(v_{\Pi,\mathcal{P}})$ contains the efficient allocation $\xi = \{\xi_i\}_{i \in N}$, where $\xi_{i^*} = v_{\Pi,\mathcal{P}}(N)$, $i^* = \arg\max_{i \in N} \mathcal{P}(N \backslash \{i\})$, and $\xi_i = 0$, for all $i \neq i^*$.
- If $\Pi > \bar{\alpha}_{\mathcal{P}} = \sum_{i \in N} [\mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N)] + \mathcal{P}(N)$, then $C(v_{\Pi,\mathcal{P}}) = \emptyset$.
- For all $\Pi \leq \bar{\alpha}_{\mathcal{P}}$, under the hypothesis of *reduced concavity* of function $\mathcal{P}$:

  $$\mathcal{P}(S) - \mathcal{P}(S \cup \{i\}) \geq \mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N), \quad \text{for all } i \in N \backslash S,\ S \subsetneq N,\ S \neq \emptyset,$$

  $C(v_{\Pi,\mathcal{P}}) \neq \emptyset$.

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- For all $\Pi \leq \bar{\alpha}_{\mathcal{P}}$, under the hypothesis of *reduced concavity* of function $\mathcal{P}$:

  $$\mathcal{P}(S) - \mathcal{P}(S \cup \{i\}) \geq \mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N), \quad \text{for all } i \in N \backslash S,\ S \subsetneq N,\ S \neq \emptyset,$$

  $C(v_{\Pi,\mathcal{P}}) \neq \emptyset$.

To ensure strictly positive worth $v_{\Pi,\mathcal{P}}(S) > 0$ for every $S \subseteq N$, $S \neq \emptyset$, we assume that $\Pi \geq \underline{\alpha}_{\mathcal{P}} = \max_{i \in N} \mathcal{P}(\{i\})$.

$$m_i^{v_{\Pi,\mathcal{P}}} = v_{\Pi,\mathcal{P}}(N) - v_{\Pi,\mathcal{P}}(N \backslash \{i\}) = \mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N), \qquad \text{for all } \ i \in N,$$

$$g^{v_{\Pi,\mathcal{P}}}(S) = \sum_{i \in S} m_i^{v_{\Pi,\mathcal{P}}} - v_{\Pi,\mathcal{P}}(S) = \sum_{i \in S} \big[\mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N)\big] + \mathcal{P}(S) - \Pi, \ \ \forall S \subseteq N, \ S \neq \emptyset.$$

We distinguish the two cases $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$ and $\bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}}$ respectively.

Theorem

If $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$, then the following statements are equivalent:

(i) *the evaluation function $\mathcal{P}$ meets 1-concavity condition*

$$\mathcal{P}(S) - \mathcal{P}(N) \geq \sum_{i \in N \backslash S} \big[\mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N)\big], \qquad \textit{for all } \ S \subseteq N, \ S \neq \emptyset;$$

(ii) *$v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is balanced;*

(iii) *$C(v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$ is a singleton and coincides with the marginal worth vector $m^{v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}}$;*

(iv) *the co-insurance game $v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is 1-convex.*

Remark

The 1-concavity condition is weaker then the condition of reduced concavity used in Fragnelli and Marina (2004).

Theorem

If for some fixed premium $\Pi^* \geq \underline{\alpha}_{\mathcal{P}}$, the co-insurance game $v_{\Pi^*,\mathcal{P}}$ is 1-convex, then for all variable premium $\underline{\alpha}_{\mathcal{P}} \leq \Pi \leq \Pi^*$, the corresponding co-insurance games $v_{\Pi,\mathcal{P}}$ are 1-convex as well.

As a corollary to both theorems above, we obtain

Theorem

If $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$ and the evaluation function $\mathcal{P}$ is 1-concave, then for any premium $\underline{\alpha}_{\mathcal{P}} \leq \Pi \leq \bar{\alpha}_{\mathcal{P}}$,

- (i) *the corresponding co-insurance game $v_{\Pi,\mathcal{P}}$ is 1-convex;*
- (ii) *the core $C(v_{\Pi,\mathcal{P}}) \neq \emptyset$;*
- (iii) *the nucleolus $\nu(v_{\Pi,\mathcal{P}})$ is the barycenter of the core $C(v_{\Pi,\mathcal{P}})$ and is given by*

$$\nu_i((v_{\Pi,\mathcal{P}})) = \mathcal{P}(N\backslash\{i\}) - \mathcal{P}(N) + \frac{\Pi - \bar{\alpha}_{\mathcal{P}}}{n}, \qquad \text{for all } i \in N.$$

Remark

The statement of the last theorem remains in force if the 1-concavity condition for the evaluation function $\mathcal{P}$ is replaced by any one of the equivalent conditions given by the first theorem, in particular if $C(v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}) \neq \emptyset$ or if the co-insurance game $v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is 1-convex.

If $\bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}}$, then even if the co-insurance game $v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is 1-convex, for the co-insurance game $v_{\Pi,\mathcal{P}}$ with $\Pi < \bar{\alpha}_{\mathcal{P}}$ the 1-convexity may be lost because the co-insurance worth of at least one coalition turns out to be at zero level.

Example

Let the evaluation function $\mathcal{P}$ for 3 insurance companies be $\mathcal{P}(\{1\})=5, \mathcal{P}(\{2\})=4, \mathcal{P}(\{3\})=3, \mathcal{P}(\{12\})=\mathcal{P}(\{13\})=\mathcal{P}(\{23\})=2, \mathcal{P}(\{123\})=1$.
In this case, $4 = \bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}} = 5$.

- If the premium $\Pi = 4$, then the co-insurance game $v_{4,\mathcal{P}}$:
  $v_{4,\mathcal{P}}(\{1\})=v_{4,\mathcal{P}}(\{2\})=0,\ v_{4,\mathcal{P}}(\{3\})=1,$
  $v_{4,\mathcal{P}}(\{12\})=v_{4,\mathcal{P}}(\{13\})=v_{4,\mathcal{P}}(\{23\})=2,\ v_{4,\mathcal{P}}(\{123\})=3,$
  is a 1-convex game with the minimal for a 1-convex game gap $g^{v_{4,\mathcal{P}}}(\{123\})=0$
  $\Longrightarrow$ the unique core allocation $m^{v_{4,\mathcal{P}}}=(1,1,1)$.
- If the premium $\Pi = 3$, then the co-insurance game $v_{3,\mathcal{P}}$:
  $v_{3,\mathcal{P}}(\{1\})=v_{3,\mathcal{P}}(\{2\})=v_{3,\mathcal{P}}(\{3\})=0,$
  $v_{3,\mathcal{P}}(\{12\})=v_{3,\mathcal{P}}(\{13\})=v_{3,\mathcal{P}}(\{23\})=1,\ v_{3,\mathcal{P}}(\{123\})=2,$
  is a symmetric
  1-convex and convex, since the gap $g^{v_{3,\mathcal{P}}}(S)=1$ is constant for all $S \subseteq N$, $S \neq \emptyset$,
  $C(v_{3,\mathcal{P}})$ is the triangle with three extreme points $(1,1,0)$, $(1,0,1)$, $(0,1,1)$.
- However, for any $2 \le \Pi < 3$, in the zero-normalized and symmetric $v_{\Pi,\mathcal{P}}$:
  $v_{\Pi,\mathcal{P}}(i)=0,\ \ v_{\Pi,\mathcal{P}}(ij)=\Pi-2,\ \ v_{\Pi,\mathcal{P}}(123)=\Pi-1,$
  1-convexity fails because the gap of singletons is strictly less than the gap of N:
  $$g^{v_{\Pi,\mathcal{P}}}(i)=1 < 4-\Pi = g^{v_{\Pi,\mathcal{P}}}(123).$$

If $\bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}}$, then even if the co-insurance game $v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is 1-convex, for the co-insurance game $v_{\Pi,\mathcal{P}}$ with $\Pi < \bar{\alpha}_{\mathcal{P}}$ the 1-convexity may be lost because the co-insurance worth of at least one coalition turns out to be at zero level.

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- If the premium $\Pi = 3$, then the co-insurance game $v_{3,\mathcal{P}}$:
  $v_{3,\mathcal{P}}(\{1\})=v_{3,\mathcal{P}}(\{2\})=v_{3,\mathcal{P}}(\{3\})=0$,
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  1-convex and convex, since the gap $g^{v_{3,\mathcal{P}}}(S)=1$ is constant for all $S \subseteq N$, $S \neq \emptyset$,
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- However, for any $2 \leq \Pi < 3$, in the zero-normalized and symmetric $v_{\Pi,\mathcal{P}}$:
  $v_{\Pi,\mathcal{P}}(i)=0$, $v_{\Pi,\mathcal{P}}(ij)=\Pi-2$, $v_{\Pi,\mathcal{P}}(123)=\Pi-1$,
  1-convexity fails because the gap of singletons is strictly less than the gap of N:
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If $\bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}}$, then even if the co-insurance game $v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is 1-convex, for the co-insurance game $v_{\Pi,\mathcal{P}}$ with $\Pi < \bar{\alpha}_{\mathcal{P}}$ the 1-convexity may be lost because the co-insurance worth of at least one coalition turns out to be at zero level.

Example

Let the evaluation function $\mathcal{P}$ for 3 insurance companies be $\mathcal{P}(\{1\})=5, \mathcal{P}(\{2\})=4, \mathcal{P}(\{3\})=3, \mathcal{P}(\{12\})=\mathcal{P}(\{13\})=\mathcal{P}(\{23\})=2, \mathcal{P}(\{123\})=1$. In this case, $4 = \bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}} = 5$.

- If the premium $\Pi = 4$, then the co-insurance game $v_{4,\mathcal{P}}$:
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Relation between co-insurance games and bankruptcy games

If for each insurance company $i \in N$ there exists a fixed "claim" $d_i \geq 0$ such that $\mathcal{P}(S) = \sum_{i \in N \setminus S} d_i$, for all $S \subseteq N$, $S \neq \emptyset$, then the co-insurance game

$$v_{\Pi,\mathcal{P}}(S) = \begin{cases} \max\{0, \Pi - \mathcal{P}(S)\}, & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

reduces to the *bankruptcy game* (Aumann and Maschler, 1985)

$$v_{E;d}(S) = \begin{cases} \max\{0,\ E - d(N \setminus S)\}, & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases}$$

with estate $E = \Pi$ and vector of claims $d = \{d_i\}_{i \in N}$.

The "bankruptcy" evaluation function $\mathcal{P}$ is nonnegative and non-increasing, $\mathcal{P}(N) = 0$. Moreover, $\bar{\alpha}_{\mathcal{P}} = \sum_{i \in N} d_i$, $\underline{\alpha}_{\mathcal{P}} = \sum_{i \in N} d_i - \min_{i \in N} d_i$, and so, always $\underline{\alpha}_{\mathcal{P}} \leq \bar{\alpha}_{\mathcal{P}}$.

For the bankruptcy situation with the estate (premium) varying between $d(N) - \min_{i \in N} d_i$ and $d(N)$, the last theorem expresses the fact that the nucleolus provides equal losses to all creditors (insurance companies) with respect to their individual claims, which agrees fully with the Talmud rule for bankruptcy situations studied exhaustively in Aumann and Maschler (1985).

Now we introduce an algorithm providing the comparatively easy computation of the nucleolus of a co-insurance game not only in cases when it is a linear function of a given premium as it is stated by the latter theorem. To do that,

1. we uncover the ***relation*** between the classes of ***co-insurance*** games, in particular bankruptcy games, and of the so-called ***veto-removed*** games that are the Davis-Maschler reduced games of monotonic veto-rich games obtained by deleting a veto-player with respect to the nucleolus;
2. we provide an algorithm for computing the nucleolus of a veto-removed game.

A player i is a *veto-player* in the game $v \in \mathcal{G}_N$ if $v(S) = 0$, for every $S \subseteq N \setminus i$. A game $v \in \mathcal{G}_N$ is a *veto-rich* game if it has at least one veto-player.

For a game $v \in \mathcal{G}_N$, a coalition $S \subseteq N$, $S \neq \emptyset$, and an efficient payoff vector $x \in \mathbb{R}^n$, the *Davis-Maschler reduced game* with respect to S and x is the game $v_{S,x} \in \mathcal{G}_S$ defined in Devis and Maschler (1965) by

$$v_{S,x}(T) = \begin{cases} 0, & T = \emptyset, \\ v(N) - x(N \backslash S), & T = S, \\ \max_{Q \subseteq N \backslash S}(v(T \cup Q) - x(Q)), & \text{otherwise}, \end{cases} \qquad \text{for all } T \subseteq S.$$

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Denote a veto-player by 0 and let $N_0 := N \cup \{0\}$.

$\mathcal{G}^m_{N_0}$ is the class of monotonic veto-rich games with a player set N_0

With any monotonic game $v \in \mathcal{G}_N$ we associate the monotonic veto-rich game $v^0 \in \mathcal{G}^m_{N_0}$,

$$v^0(S) = \begin{cases} 0, & S \not\ni 0, \\ v(S\backslash\{0\}), & S \ni 0, \end{cases} \qquad \text{for all } S \subseteq N_0.$$

$\mathcal{R}_N$ is the class of *veto-removed* games $v \in \mathcal{G}_N$ that are the Davis-Maschler reduced games of games $v^0 \in \mathcal{G}^m_{N_0}$ obtained by deleting the veto-player 0 in accordance to the nucleolus payoff.

As it follows from Arin and Feltkamp (1997)

- All veto-removed games are balanced since every monotonic veto-rich game is balanced and the Davis-Maschler reduced game inherits the core property.
- the nucleolus payoff to a veto-player $\nu_0(v^0) > 0$ in every $v^0 \in \mathcal{G}^m_{N_0}$, since the nucleolus gives maximal payoff to a veto-player and because the worth of the grand coalition in any nontrivial monotonic game is strictly positive.

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Theorem

(*i*) *Every game* $v \in \mathcal{R}_N$ *can be presented as a co-insurance game* $v_{\Pi,\mathcal{P}} \in \mathcal{G}_N$*;*

(*ii*) *if* $v_{\Pi^*,\mathcal{P}} \in \mathcal{R}_N$*, then for all premium* $\Pi \leq \Pi^*$*,* $v_{\Pi,\mathcal{P}} \in \mathcal{R}_N$ *as well;*

(*iii*) *for every evaluation function* $\mathcal{P}\colon 2^N \backslash \{\emptyset\} \to \mathbb{R}$*, for every premium* Π*,*

$$\Pi \leq \Pi^*_{\mathcal{P}} = \mathcal{P}(N) + \frac{n^2}{n+1} \min_{\substack{S \subsetneq N \\ S \neq \emptyset}} \frac{\mathcal{P}(S) - \mathcal{P}(N)}{n-s+1},$$

the co-insurance game $v_{\Pi,\mathcal{P}} \in \mathcal{R}_N$*.*

The above estimation of $\Pi^*_{\mathcal{P}}$ is rather rough.

In the particular bankruptcy case it guarantees $v_{E;d} \in \mathcal{R}_N$ only if $E \leq \dfrac{n^2}{2(n+1)} \min_{i \in N} d_i$.

We may impose weaker conditions on the parameters of $v_{E;d}$ to guarantee $v_{E;d} \in \mathcal{R}_N$.

Theorem

For any estate $E \in \mathbb{R}_+$ *and any vector of claims* $d \in \mathbb{R}^n_+$ *such that*

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For any estate $E \in \mathbb{R}_+$ and any vector of claims $d \in \mathbb{R}^n_+$ such that

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the corresponding bankruptcy game $v_{E;d} \in \mathcal{R}_N$.

For any game $w \in \mathcal{G}_M$, for every $S \subsetneq M$ we define a number

$$\kappa_w(S) = \begin{cases} \frac{w(M)-w(S)}{m-s+1}, & S \neq \emptyset, \\ \frac{w(M)}{m}, & S = \emptyset. \end{cases}$$

Algorithm 1 of constructing a payoff vector, say $x \in \mathbb{R}^N$, in a veto-removed game $v \in \mathcal{R}_N$:

0. Set $M = N$ and $w = v$.
1. Find a coalition $S \subsetneq M$ with minimal size such that $\kappa_w(S) = \min_{T \subsetneq M} \kappa_w(T)$.
2. For $i \in M \backslash S$, set $x_i = \kappa_w(S)$. If $S = \emptyset$, then stop, otherwise go to Step 3.
3. Construct the Davis-Maschler reduced game $w_{S,x} \in \mathcal{G}_S$. Set $M = S$ and $w = w_{S,x}$ and return to Step 1.

Theorem

For any veto-removed game $v \in \mathcal{R}_N$, Algorithm 1 yields the nucleolus payoff $x = \nu(v)$.

The proof is by the comparison of outputs of two algorithms yielding nucleoli, Algorithm 1 applied to a veto-removed game $v \in \mathcal{R}_N$ and another Algorithm 2 applied to the associated monotonic veto-rich game $v^0 \in \mathcal{G}^m_{N_0}$. It is based on the Davis-Maschler consistency of the prenucleolus (Sobolev, 1975) and the coincidence of the nucleolus and the prenucleolus because of the balancedness of all games in $\mathcal{G}^m_{N_0}$.

Remark

For application of Algorithm 1 to $v \in \mathcal{R}_N$ there is no need in construction of $v^0 \in \mathcal{G}^m_{N_0}$.

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Remark

For application of Algorithm 1 to $v \in \mathcal{R}_N$ there is no need in construction of $v^0 \in \mathcal{G}^m_{N_0}$.

$\mathcal{G}^+_{N_0}$ is the class of nonnegative veto-rich games with a player set N_0, a veto-player 0, that satisfies the property $v^0(N_0) \geq v^0(S)$, for all $S \subseteq N_0$.

Algorithm 2 of constructing a payoff vector, say $y \in \mathbb{R}^{N_0}$, of a game $v^0 \in \mathcal{G}^+_{N_0}$:

0. Set $M = N_0$ and $w = v^0$.
1. Find a coalition $S_0 \subsetneq M$ with minimal size such that $\kappa_w(S_0) = \min_{T_0 \subsetneq M} \kappa_w(T_0)$.
2. For $i \in M \backslash S_0$, set $y_i = \kappa_w(S_0)$. If $S_0 = \{0\}$, set $y_0 = v^0(N_0) - \sum_{i \in N} y_i$ and stop, otherwise go to Step 3.
3. Construct the Davis-Maschler reduced game $w_{S_0,y} \in \mathcal{G}_{S_0}$. Set $M = S_0$ and $w = w_{S_0,y}$ and return to Step 1.

Theorem

For any veto-rich game $v^0 \in \mathcal{G}^+_{N_0}$, Algorithm 2 yields the nucleolus payoff $y = \nu(v^0)$.

Algorithm 2 is closed conceptually to the algorithm for computing the nucleolus for veto-rich games suggested in Arin and Feltkamp (1997).

Since $\mathcal{G}^m_{N_0} \subset \mathcal{G}^+_{N_0}$, Algorithm 2 is applicable to any game $v^0 \in \mathcal{G}^m_{N_0}$ as well.

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The main idea of Algorithm 2 is based on the corollary to the Kohlberg's (1971) characterization of the prenucleolus stating that the collection of coalitions with maximal excess values with respect to the nucleolus is balanced[1]. Whence it follows

- among coalitions with the maximal excess there exists $S_0 \subsetneqq N_0$.
- every singleton $\{i\}$, $i \notin S_0$, also has the maximal excess,

and we show that this maximal excess is equal to $-\kappa_{v^0}(S_0) = -\min\limits_{T_0 \subsetneqq M} \kappa_{v^0}(T_0)$.

[1] A set of coalitions $\mathcal{B} \subset 2^N \backslash \{N\}$ is a *set of balanced coalitions*, if positive numbers λ_S, $S \in \mathcal{B}$, exist such that

$$\sum_{S \in \mathcal{B}:\ S \ni i} \lambda_S = 1, \qquad \text{for all } i \in N.$$

Concluding remarks

- A co-insurance game appears to be a very natural extension of the well-known bankruptcy game.
- The study of 1-convex/1-concave TU games possessing a nonempty core and for which the nucleolus is linear was initiated by Driessen and Tijs (1983) and Driessen (1985), but until recently appealing abstract and practical examples of these classes of games were missing. The first practical example of a 1-concave game, the so-called library cost game, and the 1-concave complementary unanimity basis for the entire space of TU games were introduced in Driessen, Khmelnitskaya, and Sales (2005). A co-insurance game under some conditions provides a new practical example of a 1-convex game. Moreover, in this paper we also show that a bankruptcy game is not only convex but 1-convex as well when the estate is sufficiently large comparatively to the given claims.
- The interest to the class of co-insurance games is not only because they reflect the well defined actual economic situations but also it is determined by the fact that every monotonic TU game may be represented in the form of a co-insurance game:

$$\mathcal{P}(S) = v(N) - v(S), \ \forall S \subseteq N \ \& \ \Pi = v(N).$$

This allows to glance into the nature of a monotonic game from another angle. In particular, the results of this paper are applicable to any monotonic game.

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- A co-insurance game appears to be a very natural extension of the well-known bankruptcy game.
- The study of 1-convex/1-concave TU games possessing a nonempty core and for which the nucleolus is linear was initiated by Driessen and Tijs (1983) and Driessen (1985), but until recently appealing abstract and practical examples of these classes of games were missing. The first practical example of a 1-concave game, the so-called library cost game, and the 1-concave complementary unanimity basis for the entire space of TU games were introduced in Driessen, Khmelnitskaya, and Sales (2005). A co-insurance game under some conditions provides a new practical example of a 1-convex game. Moreover, in this paper we also show that a bankruptcy game is not only convex but 1-convex as well when the estate is sufficiently large comparatively to the given claims.
- The interest to the class of co-insurance games is not only because they reflect the well defined actual economic situations but also it is determined by the fact that every monotonic TU game may be represented in the form of a co-insurance game:

$$\mathcal{P}(S) = v(N) - v(S), \ \forall S \subseteq N \ \& \ \Pi = v(N).$$

This allows to glance into the nature of a monotonic game from another angle. In particular, the results of this paper are applicable to any monotonic game.

Concluding remarks

- A co-insurance game appears to be a very natural extension of the well-known bankruptcy game.
- The study of 1-convex/1-concave TU games possessing a nonempty core and for which the nucleolus is linear was initiated by Driessen and Tijs (1983) and Driessen (1985), but until recently appealing abstract and practical examples of these classes of games were missing. The first practical example of a 1-concave game, the so-called library cost game, and the 1-concave complementary unanimity basis for the entire space of TU games were introduced in Driessen, Khmelnitskaya, and Sales (2005). A co-insurance game under some conditions provides a new practical example of a 1-convex game. Moreover, in this paper we also show that a bankruptcy game is not only convex but 1-convex as well when the estate is sufficiently large comparatively to the given claims.
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Thank You!

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