A Simulative Approach for Evaluating Electoral Systems

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- The Choice of the Optimal Electoral System
- The Choice with Two Parameters
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- Index of Governability, \( g \)
- The Choice with One Parameter
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1 The Setting

The choice of the best electoral system for a Parliament is very hard:

- Too many variables involved $\Rightarrow$ too difficult to balance all of them
- Complex methods are difficult to be understood and managed by voters

Some theorems - Arrow’s and McKelvey’s *in primis* - exclude the possibility of finding out the optimal rule, but no theorem prohibits finding out an *empirical* criterion of choice among two rules.
Compare the performance of the two systems
with reference to the same set of real preferences of voters

Votes are affected by the electoral system in use

↓

How would you vote were the electoral system \(X\) in a country where the system is \(Y\)?

A simulative approach requires a set of preferences
2 The Simulative Approach

Simulation is widely used in electoral systems analysis

- examine a specific mixed-member suggestion (Brichta, 1991)
- assess the proportionality of Chilean electoral system via opinion polls (Valenzuela and Siavelis, 1991)
- analyze the contendibility of a two-party system (Bender and Haas, 1996)
- find out equilibria in multiparty spatial models (Lomborg, 1997)
- estimate the effect of the adoption of alternative vote in Canada (Bilodeau, 1999)
- assess the motivations of the electoral reform in Italy (Navarra and Sobbrio, 2001)

Referring to comparison of systems

- analyze the effect in Italy of a change to a number of electoral systems (Gambarelli and Biella, 1992)
- compare six majoritarian systems but without reference to a Parliament (Christensen, 2003)
3 The Choice of the Optimal Electoral System

The choice may be affected by a lot of facets of the political process

Here

- **representativeness, \( R \):** efficiency in representing electors’ will
- **governability, \( G \):** effect on the efficiency of the resulting government

In addition:

- incentives for politicians (Riker, 1982; Myerson, 1995)
- corruption (Myerson, 1993 and 2001; Persson, Tabellini and Trebbi, 2001)
- information / participation of voters (Mudambi, Navarra and Nicosia, 1995; Mudambi, Navarra and Sobbrio, 1999)
- power of the lobbies (Myerson, 1995)
- strategic choices (Levin and Nalebuff, 1995)
- complexity of the voting system (Levin and Nalebuff, 1995)
- protection of the minorities (Levin and Nalebuff, 1995; Rae, 1995; Sen, 1995)
- risk of extreme choices (Levin and Nalebuff, 1995)
- use of votes as a voice device (Sen, 1995; Brennan and Hamlin, 1998)
- public spending (Persson and Tabellini, 1998 and 2001; Milesi-Ferretti, Perotti and Rostagno, 2000)
- overall welfare (Mueller and Stratmann, 2000)
- responsiveness of the government’s choice to the preferences of the voters (Shugart, 2001)
- ...
4 The Choice with Two Parameters

Dominance

System ? is dominant (is very likely not to exist)
System 4 is dominated (may safely be excluded)
Systems 1, 2, 3 are alternative systems (Pareto optimality → which system should be chosen?)
Alternative systems may be compared via a social utility function \((SUF)\), e.g. a Cobb-Douglas function

\[
U = K g^a r^b
\]

where \(K\) is a suitable constant  

\(a\) and \(b\) are the partial elasticity of \(U\) w.r.t. \(g\) and \(r\)

\[
X \succ Y \iff Kg_X^a r_X^b > Kg_Y^a r_Y^b
\]

Let \(p = \frac{a}{b}\)

\[
X \succ Y \iff \left(\frac{g_X}{g_Y}\right)^p > \left(\frac{r_Y}{r_X}\right)^b
\]

**Remark 1** \(p\) may be characterized as the price in terms of a relative variation of \(r\) that the community accepts to pay for a given relative opposite variation of \(g\)

\(p = 2\) means that it is worthwhile to accept a 20\% reduction of \(r\) to gain a 10\% increase of \(g\)
Graphically - Indifference curves

\[
r = \frac{(U^*)^{\frac{1}{b}}}{K^{\frac{a}{b}}} \cdot g^{\frac{b}{b}}
\]
5 Our simulative Program

Concrete voting situations using different electoral systems in an hypothetical country

- **constituencies** are characterized by a real number $c$ in the electoral space (left-right) $[-1, 1]$.
  The location of the constituencies can be selected by the user or by the program, with a sequence of random numbers.

- **parties** are characterized by their position $\ell$ on the same space $[-1, 1]$.

- **voters** are characterized by their profiles of preferences (sequence of hexadecimal numbers $\Rightarrow$ up to 16 parties).
  Each profile identifies a position $e$ of the voter on the space $[-1, 1]$, as a weighted sum of the positions of the parties, truncated at $-1$ or $1$.
  
  $567B9A832014$ expresses the preference of the fourth party ($B = 12$ points), then the sixth party ($A = 11$ points), then the fifth party ($9 = 10$ points) and so on till the tenth party ($0 = 1$ point) and identifies the position $e = 1 \times \ell_4 + \frac{1}{2} \times \ell_6 + \frac{1}{3} \times \ell_5 + \ldots + \frac{1}{12} \times \ell_{10}$.

Voters are obtained from a representative, nation-wide survey of the complete preferences for the existing parties of Italian citizens in 1997 (Data collected by ISPO - Institute for the Study of the Public Opinion, Milano).
The relative incidence $y$ of a profile is transformed in the effective percentage $w$ increasing or decreasing its weight according to its coherence with the constituency:

- $w = \frac{y}{1 - ce}$ if $ce > 0$
- $w = y(1 + ce)$ if $ce < 0$

The aim is to represent geographical opinion clusters

$c = 0$ neutralizes the procedure

Output:

A parliament with the parties and their seats and the index of *representativeness*

In addition, it is possible to determine a majority and the index of *governability*

The majority is a minimal winning coalition of adjacent parties
6 Index of Representativeness, $r$

The representativeness index cannot be based on the difference between the share of votes and of seats, as a proportionality index.

Our index is based on the difference between votes cast in a nation-wide proportional district and seats assigned by a given electoral system:

$$r_h = 1 - \frac{\sum_{i \in N} |S_i^h - S_i^{PP}|}{\sum_{i \in N} |S_i^u - S_i^{PP}|}$$

where $N$ is the set of parties,

- $S_i^h$ is the number of seats of party $i$ with system $h$,
- $S_i^{PP}$ is the number of seats of party $i$ with the perfect proportional system,
- $S_i^u$ is the total number of seats for the relative majority party under system $h$ and it is 0 otherwise.

**Example 1** Suppose three parties, L, C and R, in a parliament of 100 seats. Under PP they obtain 49, 31 and 20 seats respectively, under majority ($M$) 90, 10 and 0, and under some other system ($S$) 30, 55 and 15. So $r_M = 1 - \frac{41+21+20}{51+31+20} = 0.196$ and $r_S = 1 - \frac{19+24+5}{49+45+20} = 0.579$ (obviously $r_{PP} = 1 - \frac{0}{51+31+20} = 1$).
7 Index of Governability, $g$

It depends on the number of parties of the governing coalition that may destroy the majority if they withdraw, $m$ (more important), and on the share of seats of the majority, $f$ (less important). $m$ is used to define a lower bound $\frac{1}{m+1}$ and an upper bound $\frac{1}{m}$ and $f$ specifies the value of the index depending on the number of seats of the majority coalition:

$$\frac{gf}{m - \frac{1}{m+1}} = \frac{f - \frac{T}{2}}{T - \frac{T}{2}}$$

from which:

$$gf = \frac{1}{m(m+1)} \frac{f - \frac{T}{2}}{\frac{T}{2}}$$

where $T$ is the total number of seats in the Parliament.

So

$$g = \frac{1}{m+1} + \frac{1}{m(m+1)} \frac{f - \frac{T}{2}}{\frac{T}{2}}$$

If there are 100 seats and the governing majority is made up of one party with 59 members, we have $gf = \frac{1.9}{250} = 0.09$; this value is added to 0.5, to give $g = 0.59$

$g = 1$ if a party has all the seats
$g \rightarrow 0$ if the number of parties increases
### Example 2: Seats assignments:

<table>
<thead>
<tr>
<th>Voting system</th>
<th>( P )</th>
<th>( P(=v) )</th>
<th>( P-4 )</th>
<th>( P(20) )</th>
<th>( M )</th>
<th>( 2R )</th>
<th>( C )</th>
<th>( B )</th>
<th>( A )</th>
<th>( I-25 )</th>
<th>( I-75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( P )</td>
<td>18 22 5 0 5 0 1 0 17 28 0</td>
<td>19 22 5 0 5 0 0 0 17 28 0</td>
<td>14 16 3 3 0 3 0 1 12 48 0</td>
<td>14 36 0 0 0 0 0 0 0 50 0</td>
<td>14 36 0 0 0 0 0 0 0 50 0</td>
<td>14 36 0 0 0 0 0 0 0 4 46 0</td>
<td>9 44 0 0 0 0 0 0 0 15 32 0</td>
<td>18 37 0 0 0 0 0 0 0 18 27 0</td>
<td>15 33 1 1 0 1 0 0 0 4 45 0</td>
<td>17 25 4 3 0 4 0 1 13 33 0</td>
</tr>
<tr>
<td></td>
<td>Pure Proportionality</td>
<td>( M )</td>
<td>Relative Majority</td>
<td>( B )</td>
<td>Borda Count</td>
<td>( P-n )</td>
<td>Threshold Proportionality</td>
<td>( 2R )</td>
<td>Two-round Runoff</td>
<td>( A )</td>
<td>Approval Voting</td>
</tr>
</tbody>
</table>

We consider a unique 100-seat (or 80-seat) constituency for the proportional systems, a \( n \)-seat plus 100 - \( n \) one-seat constituencies for \( I-n \) and 100 one-seat constituencies for all the others.
## Systems locations:

<table>
<thead>
<tr>
<th>Voting system</th>
<th>r</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>1.000</td>
<td>0.201</td>
</tr>
<tr>
<td>P – 4</td>
<td>0.986</td>
<td>0.204</td>
</tr>
<tr>
<td>P(20)</td>
<td>0.722</td>
<td>0.367</td>
</tr>
<tr>
<td>M</td>
<td>0.500</td>
<td>0.453</td>
</tr>
<tr>
<td>2R</td>
<td>0.500</td>
<td>0.453</td>
</tr>
<tr>
<td>C</td>
<td>0.556</td>
<td>0.257</td>
</tr>
<tr>
<td>B</td>
<td>0.667</td>
<td>0.343</td>
</tr>
<tr>
<td>A</td>
<td>0.795</td>
<td>0.350</td>
</tr>
<tr>
<td>I – 25</td>
<td>0.611</td>
<td>0.201</td>
</tr>
<tr>
<td>I – 75</td>
<td>0.889</td>
<td>0.201</td>
</tr>
</tbody>
</table>

### Six alternative systems

P, P – 4, P(20), M, 2R, A

M and 2R weakly dominate each other.
8 The Choice with One Parameter

Two problems

• determine the power of a party on the basis of the distribution of voters and of seats in the Parliament

• measure the distance of the two distributions
Game Theoretic Approach

Define a TU-game \((N, v)\) where:

- \(N\) is the set of parties (players)
- \(v\) is the characteristic function that assumes value 1 for the majority coalitions and value 0 for the minority coalitions
Power Indices

*Shapley-Shubik index* (Shapley and Shubik, 1954)

\[
\phi_i = \frac{1}{|N|!} \sum_{\pi \in \Pi} [v(P(i, \pi) \cup \{i\}) - v(P(i, \pi))] \quad \forall \ i \in N
\]

*Normalized Banzhaf-Coleman index* (Banzhaf, 1965, and Coleman, 1971)

\[
\beta_i^* = \frac{1}{2^{|N|}-1} \sum_{S \subseteq N, S \ni i} [v(S) - v(S \setminus \{i\})] \quad \forall \ i \in N
\]

and normalizing:

\[
\beta_i = \frac{\beta_i^*}{\sum_{j \in N} \beta_j^*} \quad \forall \ i \in N
\]

*Deegan-Packel index* (Deegan and Packel, 1978)

\[
\delta_i = \sum_{S_k \in W, S_k \ni i} \frac{1}{|W|} \frac{1}{|S_k|} \quad \forall \ i \in N
\]

*Holler index* (Holler, 1982, and Holler and Packel, 1983)

\[
H_i = \frac{c_i}{\sum_{j \in N} c_j} \quad \forall \ i \in N
\]
Measures for the distances

- $d_1^h = \sum_{i \in N} |v_i - s_i^h|

- $d_2^h = \sqrt{\sum_{i \in N} (v_i - s_i^h)^2}$

- $d_\infty^h = \max_{i \in N} |v_i - s_i^h|$
Example 3 Four parties receive 40, 25, 20 and 15 per cent of the votes; the majority quota is 50 per cent. The parliament consists of 4 seats and two voting systems generate the partitions \((2, 1, 1, 0)\) and \((1, 1, 1, 1)\). Distances of the two partitions from the distribution of voters:

<table>
<thead>
<tr>
<th></th>
<th>((2, 1, 1, 0))</th>
<th>((1, 1, 1, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_1)</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>(d_2)</td>
<td>0.01√350</td>
<td>0.01√350</td>
</tr>
<tr>
<td>(d_\infty)</td>
<td>0.15</td>
<td>0.15</td>
</tr>
</tbody>
</table>

The two voting systems seem to be equivalent.

Indices of the majority games on voters \(w(v)\), on the first parliament \(w(s^1)\) and on the second parliament \(w(s^2)\):

<table>
<thead>
<tr>
<th>game</th>
<th>(\phi)</th>
<th>(\beta)</th>
<th>(\delta)</th>
<th>(H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w(v))</td>
<td>((\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}))</td>
<td>((\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}))</td>
<td>((\frac{9}{24}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24}))</td>
<td>((\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}))</td>
</tr>
<tr>
<td>(w(s^1))</td>
<td>((\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0))</td>
<td>((\frac{2}{3}, \frac{1}{5}, \frac{1}{5}, 0))</td>
<td>((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0))</td>
<td>((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0))</td>
</tr>
<tr>
<td>(w(s^2))</td>
<td>((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}))</td>
<td>((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}))</td>
<td>((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}))</td>
<td>((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}))</td>
</tr>
</tbody>
</table>

Distances between the power w.r.t. the voters and to each parliament:

<table>
<thead>
<tr>
<th></th>
<th>((2, 1, 1, 0))</th>
<th>((1, 1, 1, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_1)</td>
<td>0.333</td>
<td>0.500</td>
</tr>
<tr>
<td>(d_2)</td>
<td>0.236</td>
<td>0.289</td>
</tr>
<tr>
<td>(d_\infty)</td>
<td>0.167</td>
<td>0.250</td>
</tr>
</tbody>
</table>

The distances on the power indices distinguish the two systems.
Another measure of the distance (Gambarelli and Biella, 1992):

\[
\Delta = \max_{i \in N} \left| v_i - s^h_i \right| - \left| \varphi_i - \varphi^h_i \right|
\]

where \( v \) are the percentages of distribution of voters
\( s^h \) are the percentages of seats according to an electoral system \( h \)
\( \varphi \) and \( \varphi^h \) are the power of the parties related to the votes and to the seats

**Example 4** Referring to the data of Example 2 the distances are:

<table>
<thead>
<tr>
<th>Voting system</th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \varphi = \beta )</td>
</tr>
<tr>
<td>( P - 4 )</td>
<td>0.013</td>
</tr>
<tr>
<td>( P(20) )</td>
<td>0.286</td>
</tr>
<tr>
<td>( M )</td>
<td>0.136</td>
</tr>
<tr>
<td>( 2R )</td>
<td>0.136</td>
</tr>
<tr>
<td>( C )</td>
<td>0.097</td>
</tr>
<tr>
<td>( B )</td>
<td>0.146</td>
</tr>
<tr>
<td>( A )</td>
<td>0.176</td>
</tr>
<tr>
<td>( I - 25 )</td>
<td>0.113</td>
</tr>
<tr>
<td>( I - 75 )</td>
<td>0.029</td>
</tr>
</tbody>
</table>
9 Conclusions

If votes are those actually cast in a plurality election, they are useless to compare the distribution of power with that of preferences.

The distribution of votes may be assumed as a proxy to that of preferences only in proportional systems with large districts.

Real data cannot provide useful information, the simulation does.

Accumulate experimental (i.e. simulative) evidence would probably provide relevant suggestions for real world analysis and policing.
10 Further researches

• measures of the electoral systems:
  – representativeness → dispersion index (Gini)
  – governability → propension to disrupt index (Gately)

• robustness:
  – high governability when random elements are considered (e.g. absence of some members in a voting)
  – capacity to limit the possibility to manipulate the elections (e.g. merging and splitting)