An Overview on Power Indices

Vito Fragnelli
Università del Piemonte Orientale
vito.fragnelli@uniupo.it

Elche - 2 NOVEMBER 2015

Summary

The Setting

The Basic Tools

The Survey

The Issue of Infeasibility

The Setting

0-1 decision problems: the result is simply against or in favor of a proposal, with no intermediate position
Question: how to evaluate the influence of each member on the final decision, mainly when the members are not equivalent? *Parties in a Parliament, stackholders with different quotas, etc.*
This question may be answered, inter alia, by using power indices

The Basic Tools

Cooperative game in characteristic form is a pair (N, v)
where $N = \{1, 2, ..., n\}$ is the set of players
$v : 2^N \longrightarrow \mathbb{R}, v(\emptyset) = 0$ is the characteristic function
$v(S), S \subseteq N$ is the worth of the players in S
The game (N, v) is *simple* if $v : 2^N \longrightarrow \{0, 1\}$
In a simple game (N, v) a coalition $S \subseteq N$ is called *winning* if $v(S) = 1$ and *losing* if $v(S) = 0$
A simple game (N, v) is *proper* if $v(S) = 1 \Rightarrow v(N \setminus S) = 0, S \subseteq N$
Usually, for simple games $S \subset T \Rightarrow v(S) \leq v(T)$ (*monotonicity*) and $v(N) = 1$

Weighted majority situation $[q; w_1, w_2, ..., w_n]$
where $N = \{1, 2, ..., n\}$ set of decision-makers
$w_1, w_2, ..., w_n$ weights of decision-makers
q majority quota

Weighted majority game (N, w):

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases} , S \subseteq N$$

The Survey

Each existing power index emphasizes different features of the problem, making it particularly suitable for *specific* situations

- *First indices*

Penrose (1946), Shapley and Shubik (1954), Banzhaf (1965), Coleman (1971)

Ability of a decision-maker to switch the result of the voting session by leaving a set of decision-makers that pass the proposal
The indices of Penrose, Banzhaf and Coleman tally the switches w.r.t. the possible coalitions, while in the Shapley-Shubik index also the order agents form a coalition plays a role

Formally

Swing: A winning coalition $S \subseteq N$ becomes losing when player $i \in S$ leaves it

Player i is said *critical for* S

Using the concept of swing we have:

Penrose-Banzhaf-Coleman index

$$\beta_i = \frac{1}{2^{n-1}} \sum_{S \ni i} SW(i, S), i \in N$$

where $SW(i, S) = 1$ if i critical for S and $SW(i, S) = 0$ otherwise

Normalized Penrose-Banzhaf-Coleman index

$$\bar{\beta}_i = \frac{\beta_i}{\sum_{j \in N} \beta_j}, i \in N$$

Shapley-Shubik index

$$\phi_i = \frac{1}{n!} \sum_{\pi \in \Pi} SW(i, P(i, \pi)), i \in N$$

where Π is the set of permutations of N and $P(i, \pi)$ is the set of predecessors of i in π, including i

• *Relations among agents*
Myerson (1977), Owen (1977)

Myerson proposes to use an undirected graph, G, called *communication structure*, whose vertices are associated to the players and the arcs represent compatible pairs of players; then a *restricted game* (N, v_G) is considered

$$v_G(S) = \sum_{T \in S/G} v(T), S \subseteq N$$

where S/G is the set of coalitions induced by the connected components of the vertices of S in G

Owen introduces the *a priori unions*, or *coalition structure*, i.e. a partition of the set of players, that accounts for existing agreements, not necessarily binding, among some decision-makers

Owen (1986) studies the relationship among the power indices, mainly Shapley-Shubik and Banzhaf, in the original game and in the restricted game á la Myerson
Winter (1989) requires that the different unions may join only according to a predefined scheme, called *levels structure*
Khmelnitskaya (2007) combines communication structures and a priori unions

- *Power sharing*

Deegan and Packel (1978), Johnston (1978), Holler (1982)

Deegan and Packel account only the coalitions in which each agent is critical, while Johnston includes the coalitions in which at least one agent is critical
Both indices divide the unitary power among the coalitions considered; then the power assigned to each coalition is equally shared among its critical agents

Holler introduces the Public Good index, supposing that the worth of a coalition is a *public good*, so the members of the *winning decisive sets*, i.e. those in which all the agents are critical, have to enjoy the same relevance; the power of an agent is proportional to the number of winning decisive sets s/he belongs to

Formally

A coalition $S \subseteq N$ is a *minimal winning coalition* if all the players in S are critical for it

A coalition $S \subseteq N$ is a *quasi-minimal winning coalition* if at least one player in S is critical for it

Deegan-Packel index

$$\delta_i = \sum_{S_j \ni i; S_j \in \mathcal{W}^m} \frac{1}{m} \frac{1}{s_j}, i \in N$$

where $\mathcal{W}^m = \{S_1, ..., S_m\}$ is the set of minimal winning coalitions and $s_j = |S_j|$

Johnston index

$$\gamma_i = \sum_{S_j \in \mathcal{W}_i^q} \frac{1}{\ell} \frac{1}{c_{S_j}}, i \in N$$

where $\mathcal{W}^q = \{S_1, ..., S_\ell\}$ is the set of quasi-minimal winning coalitions, $\mathcal{W}_i^q$ is the set of quasi-minimal winning coalitions which player i is critical for and c_{S_j} is the number of critical players in S_j; Johnston index coincides with Deegan-Packel index if $\mathcal{W}^m = \mathcal{W}^q$

Public Good index

$$h_i(v) = \frac{w_i^m}{\sum_{j \in N} w_j^m}, i \in N$$

where $w_i^m, i \in N$ is the number of minimal winning coalitions including player i

- *Weights*

Kalai and Samet (1987), Haeringer (1999), Chessa and Khmelnitskaya (2015)

Kalai and Samet add a *weight* to the elements characterizing each agent, modifying the Shapley-Shubik index

Haeringer combines weights and communication structure (weighted Myerson index)

Chessa and Khmelnitskaya add a weight, redefining the Deegan-Packel index

• *Restricted cooperation - Permission structures*
Gillies et al. (1992), Van den Brink and Gillies (1996) and Van den Brink (1997)

The papers introduce the conjunctive and the disjunctive permission indices for games with a permission structure

• *Restricted cooperation - Feasible coalitions*
Bilbao et al. (1998), Bilbao and Edelmann (2000), Algaba et al. (2003, 2004)

The first two papers consider the Penrose-Banzhaf-Coleman index and the Shapley-Shubik index on convex geometries, respectively
The last two papers study the Shapley-Shubik index and the Penrose-Banzhaf-Coleman index on antimatroids, respectively

Katsev (2010) surveys indices for games with restricted cooperation

- *Contiguity and connection*

Fragnelli, Ottone and Sattanino (2009), Chessa and Fragnelli (2011)

Fragnelli, Ottone and Sattanino introduce a new family of power indices, called *FP*, accounting the issue of *contiguity* in a monodimensional voting space

Generalizing the scheme of Deegan-Packel they consider the set $\mathcal{W}^c = \{S_1, S_2, ..., S_m\}$ of winning coalitions with contiguous players, i.e. given two players $i, j \in S$ if there exists $k \in N$ with $i < k < j$ then $k \in S$

$$FP_i = \sum_{S_j \in \mathcal{W}^c; S_j \ni i} \frac{1}{m} \frac{1}{s_j}, \ i \in N$$

They allow for different sharing rules of the power among the coalitions and among the players inside each coalition

Chessa and Fragnelli extend the *FP* accounting the issue of *connection* instead of contiguity in a possibly multidimensional voting space

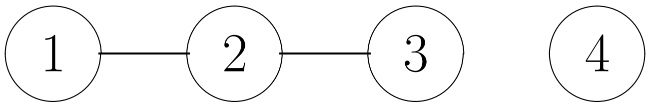
In both cases, non-contiguous and non-connected coalitions are ignored

The idea of monodimensionality is already considered in Amer and Carreras (2001)

The Issue of Infeasibility

In Myerson (1977) compatibility is represented by an undirected graph

Example 1 *Consider the weighted majority situation* $[51; 35, 30, 25, 10]$
The winning coalitions are $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$
Suppose that the communication structure is represented by the graph G*:*



In the restricted game (N, v_G)*, coalitions* $\{1,3\}$ *and* $\{1,3,4\}$ *are no longer winning*

Comments

i. According to the graph G, coalitions $\{1,2\}$, $\{2,3\}$, $\{1,2,3\}$ are feasible while coalition $\{1,3\}$ is infeasible
Suppose that parties 1 and 3 never want to stay in the same coalition, so that coalition $\{1,2,3\}$ is infeasible; introducing the idea of *complete subgraph* for representing feasible coalitions, the feasibility of coalition $\{1,2,3\}$ implies that also coalition $\{1,3\}$ is feasible
Look at the graph, account feasible coalitions and assign them a probability (FP indices)

ii. $v_G(\{1,2,4\}) = v(\{1,2\}) + v(\{4\}) = 1$, even if it is not feasible
Revise the concept of swing involving infeasible coalitions

References

Algaba, E., Bilbao, J.M., van den Brink, R. and Jiménez-Losada, A. (2003), Axiomatizations of the restricted Shapley value for cooperative games on antimatroids, *Mathematical Methods of Operations Research* 57 : 49-65.

Algaba, E., Bilbao, J.M., van den Brink, R. and Jiménez-Losada, A. (2004), An axiomatization of the Banzhaf value for cooperative games on antimatroids, *Mathematical Methods of Operations Research* 59 : 147-166.

Amer, R. and Carreras, F. (2001), Power, cooperation indices and coalition structures, in M.J. Holler and G. Owen (eds.), *Power Indices and Coalition Formation*, Dordretch, Kluwer Academic Publishers : 153-173.

Banzhaf, J.F. (1965), Weighted Voting doesn't Work: A Mathematical Analysis, *Rutgers Law Review* 19 : 317-343.

Bilbao, J.M., Jiménez, A. and Lopez, J.J. (1998), The Banzhaf power index on convex geometries, *Mathematical Social Sciences* 36 : 157-173.

Bilbao, J.M. and Edelman, P.H. (2000), The Shapley value on convex geometries, *Discrete Applied Mathematics* 103 : 33-40.

Chessa, M. and Fragnelli, V. (2011), Embedding classical indices in the FP family, *AUCO Czech Economic Review* 5 : 289-305.

Chessa, M. and Khmelnitskaya, A. (2015), Weighted and restricted Deegan-Packel power indices, *Proceedings of SING 11 Meeting*.

Coleman, J.S. (1971), Control of Collectivities and the Power of a Collectivity to Act, in B. Lieberman (ed.), *Social Choice*, London, Gordon and Breach : 269-300.

Deegan, J. and Packel, E.W. (1978), A New Index of Power for Simple n-person Games, *International Journal of Game Theory* 7 : 113-123.

Fragnelli, V., Ottone, S. and Sattanino R. (2009), A new family of power indices for voting games, *Homo Oeconomicus* 26 : 381-394.

Gilles, R.P., Owen, G. and Van den Brink, R. (1992), Games with permission structures: the conjunctive approach, *International Journal of Game Theory* 20 : 277-293.

Haeringer, G. (1999), Weighted Myerson Value, *International Game Theory Review* 1 : 187-192.

Holler, M.J. (1982), Forming Coalitions and Measuring Voting Power, *Political Studies* 30 : 262-271.

Johnston, R.J. (1978), On the Measurement of Power: Some Reactions to Laver, *Environment and Planning A* 10 : 907-914.

Kalai, E. and Samet, D. (1987), On weighted Shapley values, *International Journal of Game Theory* 16 : 205-222.

Katsev, I.V. (2010), The Shapley Value for Games with Restricted Cooperation. Available at SSRN: http://ssrn.com/abstract=1709008

Khmelnitskaya, A.B. (2007), Values for Graph-restricted Games with Coalition Structure, Memorandum 1848, University of Twente.

Myerson, R. (1977), Graphs and Cooperation in Games, *Mathematics of Operations Research* 2 : 225-229.

Owen, G. (1977), Values of Games with a Priori Unions, *Lecture Notes in Economic and Mathematical Systems* 141 : 76-88.

Owen, G. (1986), Values of Graph-Restricted Games, *SIAM Journal of Algebraic Discrete Methods* 7 : 210-220.

Penrose, L.S. (1946), The Elementary Statistics of Majority Voting, *Journal of the Royal Statistical Society* 109 : 53-57.

Shapley, L.S. and M. Shubik (1954), A Method for Evaluating the Distribution of Power in a Committee System, *American Political Science Review* 48 : 787-792.

Van den Brink, R. (1997), An axiomatization of the disjunctive permission value for games with a permission structure, *International Journal of Game Theory* 26 : 27-43.

Van den Brink, R. and Gilles, R.P. (1996), Axiomatizations of the conjunctive permission value for games with permission structures, *Games and Economic Behavior* 12 : 113-126.

Winter, E. (1989), A Value for Cooperative Games with Levels Structure of Cooperation, *International Journal of Game Theory* 18 : 227-240.

Thanks!