Orders and Indices of Criticality in Simple Games

Vito Fragnelli
University of Eastern Piedmont
vito.fragnelli@uniupo.it

Joint work with: Marco Dall'Aglio - LUISS University
Stefano Moretti - Université Dauphine

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1 The Setting

Consider a parliamentary majority coalition
If it corresponds to a *minimal winning coalition* all the parties result *critical*, i.e. the majority does not subsist anymore if a party leaves, but when only some parties are critical the majority corresponds to a *quasi-minimal winning coalition*
In the Eighties, the Italian governments included five parties, namely *DC, PSI, PSDI, PRI, PLI*; for some years only DC was critical (then also PSI became critical), but all the parties received ministries and/or departments and in 1981 the premiership of the government was given to Giovanni Spadolini, the leader of PRI

This situation may seem unusual, but the critical parties may give a small amount of power to the non-critical parties, i.e. compensate them in order that they do not leave the majority, getting much more for themselves
Presently, the ALA group that is represented only in the Senate, where the majority supporting the Italian government is very unstable, supports the approval of some reforms proposed by the government, with the consequence that the minority of the Democratic Party decided to support the reforms in order to avoid the criticality of the ALA group

Chessa and Fragnelli (2014) accounted for the possibility of the parties of forming a different majority coalition that excludes another party, which in its turn may propose another majority coalition that does not include the party that started the process, satisfying the hypotheses on which the bargaining set relies (see Aumann and Maschler, 1964)

Critical parties have a *first order of criticality*, while the non-critical ones have a higher order of criticality

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The aim of the first part is to provide a formal definition of the higher orders of criticality and to analyze some properties, proposing an allocation of the power

2 Preliminaries

A *cooperative game with transferable utility* (TU-game) is a pair (N, v), where $N = \{1, 2, ..., n\}$ is the *finite set of players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\varnothing) = 0$
$v(S)$ is the worth of coalition $S \subseteq N$, i.e. what players in S may obtain without the cooperation of the other players

A TU-game (N, v) is *simple* when $v : 2^N \to \{0, 1\}$, with $S \subseteq T \Rightarrow v(S) \leq v(T)$ and $v(N) = 1$
If $v(S) = 0$ then S is a *losing* coalition, while if $v(S) = 1$ then S is a *winning* coalition
Given a winning coalition S, if $S \setminus \{i\}$ is losing then $i \in N$ is a *critical player* for S
When a coalition S contains at least one critical player for it, S is a *quasi-minimal winning* coalition; when all the players of S are critical, it is a *minimal winning* coalition

A particular class of simple games are the *weighted majority games*, where a vector of weights $(w_1, w_2, ..., w_n)$ is associated to the players; the characteristic function of the corresponding weighted majority game (N, w) is:

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}, \quad S \subseteq N$$

where q is the *majority quota*
A *weighted majority situation* is often denoted as $[q; w_1, w_2, \ ..., w_n]$
Not every simple game can be represented as a weighted majority situation

A simple game is *proper* or *N-proper* if when S is winning then $N \setminus S$ is losing; for this aim it is sufficient to choose $q > \frac{1}{2} \sum_{i \in N} w_i$

Given (N, v), an *allocation* is a n-dimensional vector $(x_i)_{i \in N} \in \mathbb{R}^N$ assigning to player $i \in N$ the amount x_i
An allocation $(x_i)_{i \in N}$ is *efficient* if $x(N) = \sum_{i \in N} x_i = v(N)$
A *solution* is a function ψ that assigns an allocation $\psi(v)$ to every TU-game (N, v) belonging to a given class of games $\mathcal{G}$ with player set N

For simple games, a solution is often called a *power index*
Among the several power indices, two are mainly used:

- The *Shapley-Shubik index* (Shapley and Shubik, 1954), ϕ, is the natural version for simple games of the Shapley value (Shapley, 1953)

$$\phi_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(s-1)!(n-s)!}{n!} m_i(S), \ i \in N$$

  where n and s denote the cardinalities of the set of players N and of the coalition S, respectively and $m_i(S) = v(S) - v(S \setminus \{i\})$ denotes the marginal contribution of player $i \in N$ to coalition $S \subseteq N, S \ni i$

- The normalized *Banzhaf index* (Banzhaf, 1965), β, is similar to the Shapley-Shubik index, but it considers the marginal contributions of a player to all possible coalitions, independently from the order of the players

$$\beta_i^*(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N, S \ni i} m_i(S), \ i \in N$$

  then, by normalization we get:

$$\beta_i(v) = \frac{\beta_i^*(v)}{\sum_{j \in N} \beta_j^*(v)}, \ i \in N$$

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a vector of n non-negative numbers such that $\sum_{k=1}^{n} p_k \binom{n}{k} = 1$
Interpreting p_s as the probability that a coalition of size s forms, $\pi^{\mathbf{p}}$ is the *semivalue* (see Dubey, Neyman and Weber, 1981) engendered by $\mathbf{p}$:

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subseteq N, S \ni i} p_s m_i(S)$$

A semivalue $\pi^{\mathbf{p}}$ satisfies the *symmetry property* ($\pi_i^{\mathbf{p}}(v) = \pi_j^{\mathbf{p}}(v)$ if players $i, j \in N$ are *symmetric* , i.e. $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$) and the *null player property* ($\pi_i^{\mathbf{p}}(v) = 0$ if player $i \in N$ is *null*, i.e. $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$)

Both the Shapley-Shubik index ϕ and the Banzhaf index β^* can be defined as particular semivalues $\pi^{\mathbf{p}^{\phi}}$ and $\pi^{\mathbf{p}^{\beta}}$, respectively, where $p_s^{\phi} = \frac{(s-1)!(n-s)!}{n!}$ and $p_s^{\beta} = \frac{1}{2^{n-1}}$
In view of monotonicity, for a simple game a semivalue can be simply written as

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \in W_i(v)} p_s m_i(S)$$

where $W_i(v)$ is the set of winning coalitions containing player $i \in N$
$m_i(S) = 1$ only if i is critical for S in v, for each $S \in W_i(v)$, so $\pi_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i to play a critical role in v

3 Second and Higher Orders of Criticality

Fixed a winning coalition $M \subseteq N$, a critical player $i \in M$ may be called *critical of the first order* for coalition M

Definition 1 *Let $M \subseteq N$, with $|M| \geq 3$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. Player i is* Second Order Critical (SOC) for coalition M, via player $j \in M \setminus \{i\}$ *iff $v(M \setminus \{i, j\}) = 0$ with $v(M \setminus \{j\}) = 1$*

Proposition 1 *If player $i \in M$ is critical of the second order for coalition M, via player $j \in M$, then player j is critical of the second order for coalition M, via player i*

Remark 1 *When there are critical players of the second order, they are at least two, but they can be more*

Example 1 *Consider the weighted majority situation [51; 40, 8, 5, 5, 5]; the first party is the unique critical one, while the other four parties are critical of the second order, even if the last three parties are critical only via the second party*

Definition 1 can be extended to further orders as follows

Definition 2 *Let $k \geq 2$ be an integer, let $M \subseteq N$, with $|M| \geq k+2$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. Player i is* critical of the order $k+1$ for coalition M, via coalition $K \subseteq M \setminus \{i\}$, *with $|K| = k$ iff*

$$v(M \setminus K) - v(M \setminus (K \cup \{i\})) = 1 \tag{1}$$

and K is the set of minimal cardinality satisfying (1), i.e. $v(M \setminus (T \cup \{i\})) = 1$ for any $T \subset K$ with $|T| < k$

The notion of minimal cardinality is crucial to unambiguously assign the order of criticality of a player in a coalition

Example 2 *Consider the weighted majority situation [31; 21,5,3,3,3,3,2]; player 7 becomes critical whenever either a coalition involving player 2 and any of the players 3-6 are involved (they are the coalition K in the definition), or three of the players 3-6 are involved. According to Definition 2, player 7 is critical of the third order*

Definition 2 encompasses the definitions for lower orders. First order criticality requires $K = \varnothing$ of 0-cardinality, so $v(M) - v(M \setminus \{i\}) = 1$; for second order criticality consider $K = \{j\}$, then, by (1), $v(M \setminus \{j\}) = 1$ and $v(M \setminus \{i, j\}) = 0$; moreover, since $\{j\}$ is the set of minimal cardinality which makes i critical, then $v(M \setminus \{i\}) = 1$

Remark 2 *A null player is never critical*

Example 3 *Consider the weighted majority situation [51; 44, 3, 3, 3, 3, 3, 3]; in this case the first one is the unique critical party, while the other six parties are critical of the fourth order; note that there are no parties critical of order 2 or 3*

Remark 3 *Let $M \subseteq N$ be a winning coalition, then the players in M may be partitioned into those who are critical of some order and those who are never critical*

Proposition 2 *Let $i \in M$ be a player critical of the order $k+1$ for coalition M, via coalition $K \subset M$; if a player $j \in K$ leaves the coalition, then i is a player critical of the order k for coalition $M \setminus \{j\}$, via coalition $K \setminus \{j\}$*

After defining the various orders of criticality, an index for measuring how much a player may profit from being critical of the second order is introduced

The first step is to measure the power of a player w.r.t. a given coalition, accounting also his order of criticality; then, the power of a player w.r.t. all coalitions he may belong to is computed

The probability for player $i \in N$ to be SOC in v for some coalitions via player $j \in N \setminus \{i\}$ may be computed as follows

First, consider a coalition $S \in 2^{N \setminus \{i,j\}}$ with $v(S \cup \{i,j\}) = 1$ and define $C_{ij}(S) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\} - v(S)$

i is SOC for $S \cup \{i,j\}$ via j is when $v(S \cup \{i\}) = 1, v(S \cup \{j\}) = 1$ and $v(S) = 0$, so $C_{ij}(S) = 1$; in general, $C_{ij}(S) = C_{ji}(S)$

Let $\mathbf{p} = (p_0, \ldots, p_{n-1})$ be a probability vector as defined in Section 2

The probability that i is SOC for some coalitions via j is:

$$\Gamma_{ij}^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} p_{s+1} C_{ij}(S)$$

By Proposition 1, it immediately follows that $\Gamma_{ij}^{\mathbf{p}}(v) = \Gamma_{ji}^{\mathbf{p}}(v)$ for each $i, j \in N$

The total probability that player i is SOC for some coalition via some other player is:

$$C_i^{\mathbf{p}}(v) = \sum_{j \in N \setminus \{i\}} \Gamma_{ij}^{\mathbf{p}}(v)$$

Now, consider the game (N, v^{ij}) such that for each $S \in 2^{N\setminus\{i,j\}}$ then $v^{ij}(S) = v(S)$ and

$$v^{ij}(S \cup \{i,j\}) = v^{ij}(S \cup \{i\}) = v^{ij}(S \cup \{j\}) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\}$$

i.e. the value of each coalition M containing either i or j is lowered to the worst value between $v(M \cup \{j\} \setminus \{i\})$ and $v(M \cup \{i\} \setminus \{j\})$, then

$$\Gamma^{\mathbf{p}}_{ij}(v) = \pi^{\mathbf{p}}_i(v^{ij})$$

Example 4 *Consider the simple game (N, v) with $N = \{1,2,3,4\}$ whose minimal winning coalitions are $\{1,2,3\}$ and $\{1,2,4\}$. Note that 3 (resp. 4) is SOC for $\{1,2,3,4\}$ via 4 (resp. 3), and no other player is SOC via another player for some coalition. Taking the vector $\mathbf{p}^\beta$ as the probability vector yielding the Banzhaf index $\pi^{\mathbf{p}^\beta}$ (see Section 2), then $\Gamma^{\mathbf{p}^\beta}_{34}(v) = \Gamma^{\mathbf{p}^\beta}_{43}(v) = \frac{1}{8}$ and $\Gamma^{\mathbf{p}^\beta}_{ij}(v) = 0$ otherwise*
$v^{34}(S) = v^{43}(S) = v(S)$ *for each $S \in 2^N$, so $\Gamma^{\mathbf{p}^\beta}_{34}(v) = \Gamma^{\mathbf{p}^\beta}_{43}(v) = \pi^{\mathbf{p}^\beta}_3(v^{34}) = \pi^{\mathbf{p}^\beta}_4(v^{43}) = \frac{1}{8}$*
In addition, $\Gamma^{\mathbf{p}^\beta}_{ij}(v) = \pi^{\mathbf{p}^\beta}_i(v^{ij}) = 0$ for all i and j with $\{i,j\} \neq \{3,4\}$
The total probability to be SOC is $C^{\mathbf{p}^\beta}_1(v) = C^{\mathbf{p}^\beta}_2(v) = 0$ and $C^{\mathbf{p}^\beta}_3(v) = C^{\mathbf{p}^\beta}_4(v) = \frac{1}{8}$

It is possible to apply the notions of criticality of first and second order to the situation of the Italian Senate during the Eighties

Example 5 *The distribution of seats among the political parties of the largest alliance in the Italian Senate during the IX Legislature (1979-1983) was as follows:*

Party	*seats*
Democrazia Cristiana (DC)	*145*
Partito Socialista Italiano (PSI)	*32*
Partito Socialdemocratico Italiano (PSDI)	*9*
Partito Repubblicano Italiano (PRI)	*6*
Partito Liberale Italiano (PLI)	*2*

The majority quota was 162*, leading to the weighted majority situation* $[162; 145, 32, 9, 6, 2]$
The minimal winning coalitions were $\{\{DC, PSI\}, \{DC, PSDI, PRI, PLI\}\}$
The symmetric relation of SOC involved PSI vs. PSDI, PSI vs. PRI and PSI vs. PLI

The probability to be critical of the first order (i.e., the Banzhaf index $\pi^{\mathbf{p}^\beta}$) and of the second order (using the index $C^{\mathbf{p}^\beta}$) were

Party	$\pi^{\mathbf{p}^\beta}$	$C^{\mathbf{p}^\beta}$
DC	$\frac{9}{16}$	0
PSI	$\frac{7}{16}$	$\frac{3}{16}$
PSDI	$\frac{1}{16}$	$\frac{1}{16}$
PRI	$\frac{1}{16}$	$\frac{1}{16}$
PLI	$\frac{1}{16}$	$\frac{1}{16}$

In 1983 the PSI threatened to leave the five-parties alliance unless Bettino Craxi, the PSI party's leader, was made Prime Minister. The DC party accepted this compromise in order to avoid a new election
Maybe, the DC had evaluated the threaten of the PSI as likely in view of the high index $C^{\mathbf{p}^\beta}$ for the PSI

4 Criticality Indices

The notion of criticality itself, that considers just a single coalition, is not sufficiently rich to encompass all the possible interactions among players

Example 6 *Consider two simple games* (N, u) *and* (N, w) *with* $N = \{1, 2\}$ *and*

$$u(\{1,2\}) = 1, u(\{1\}) = u(\{2\}) = 0 \qquad w(\{1,2\}) = w(\{1\}) = w(\{2\}) = 1$$

In u *both players are critical of the first order for coalition* $\{1, 2\}$
In w*, each player* $i \in N$ *is critical of the first order for coalition* $\{i\}$ *and both players are critical of the second order for coalition* $\{1, 2\}$ *via the other player (note that* w *is not proper)*

All the most popular power indices assign the same value to the players in both games, even if, without any further assumption about the probability to form coalitions of different size, the two situations are quite different:

- In u, the grand coalition may be threatened by both players
- In w, each player becomes critical only if the other one leaves the grand coalition

The question whether a player is more powerful in one game than in the other one is highly context-specific
Nevertheless, the information contained in the characteristic function of a simple game cannot be adequately represented by a single attribute which exclusively relies on the notion of criticality

Being the order of criticality of a party different in the various possible majorities it may belong to, it may be interesting to associate to each party a unique value that aggregates the different orders of criticality that it has in the different winning coalitions

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The aim of this part is to propose a collective index for aggregating the indices corresponding to different order of criticality into a unique value for each player

Then, the collective index is compared with classical power indices, analysing some properties for weighted majority games, then it is proved that within a given game, the behaviour of a collective index is coherent with the one of classical power indices, while a disaggregate representation of the criticality indices provides a more refined picture of the variation of each player's power across different games

4.1 Indices á la Shapley

Given an ordering π of the players, the power of player $i \in N$ in being critical of order $k = 1, ..., n$ is:

$$\phi_{i,k}(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \sigma_{i,k}(\pi)$$

where

$$\sigma_{i,k}(\pi) = \begin{cases} 1 & \text{if } i \text{ is critical of order } k \text{ in } P_{\pi,i} \cup \{i\} \\ 0 & \text{otherwise} \end{cases}$$

and $P_{\pi,i}$ is the set of players preceding i in ordering π

With simple combinatorial arguments, it is possible to rewrite the indices as:

$$\phi_{i,k}(v) = \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!} dc_k(i, S \cup \{i\})$$

where

$$dc_k(i, M) = \begin{cases} 1 & \text{if } i \text{ is critical of order } k \text{ in the winning coalition } M \\ 0 & \text{otherwise} \end{cases}$$

It is possible to associate a whole distribution of indices of criticality to player $i \in N$:

$$\Phi_i(v) = (\phi_{i,1}(v)\, \phi_{i,2}(v), \ldots, \phi_{i,n}(v))$$

Example 7 *Referring to Example 6, it is immediate to verify that:* $\Phi_1(u) = \Phi_2(u) = \left(\frac{1}{2}, 0\right)$ *and* $\Phi_1(w) = \Phi_2(w) = \left(\frac{1}{2}, \frac{1}{2}\right)$

Example 8 *Consider a simple game with $N = \{1,2,3,4,5,6\}$ and the following minimal winning coalitions:*

$$\{1,2,3\} \qquad \{1,2,4\} \qquad \{1,2,5\} \qquad \{1,3,4\} \qquad \{1,3,5\}$$

In the winning coalition $\{1,2,3,4,5\}$, 1 is critical of the first order, 2 and 3 are critical of the second order, 4 and 5 are critical of the third order; in the grand coalition, the order of criticality for the first 5 players does not change, while player 6 is never critical

The distribution of indices are:

$$\Phi_1(v) = \left(\frac{17}{30}, 0, 0, 0, 0, 0\right) \qquad \Phi_2(v) = \Phi_3(v) = \left(\frac{3}{20}, \frac{3}{10}, 0, 0, 0, 0\right)$$

$$\Phi_4(v) = \Phi_5(v) = \left(\frac{1}{15}, \frac{3}{20}, \frac{1}{5}, 0, 0, 0\right) \qquad \Phi_6(v) = (0, 0, 0, 0, 0, 0)$$

Bounds

Proposition 3 *The following inequalities hold*

$$1 \leq \sum_{k=1}^{n} \sum_{i \in N} \phi_{i,k}(v) \leq n$$

and these are sharp, i.e., they can be attained.

To verify that the lower bound can be attained, consider the unanimity game

$$\underline{v}(S) = \begin{cases} 1 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases}$$

The players are symmetrical and no player can be critical beyond the first order, thus

$$\Phi_i(\underline{v}) = \left(\frac{1}{n}, 0, \ldots, 0\right)$$

To verify that the upper bound can be attained, consider the following game

$$\overline{v}(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{otherwise} \end{cases}$$

For each permutation π, player $\pi(1)$ is critical of the first order, player $\pi(2)$ is critical of the second order, and so on, thus

$$\Phi_i(v) = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$$

Non-critical players

The role of non-critical players leads to the following index:

$$\phi_{i,NC}(v) = \sum_{S \not\ni i} \frac{|S|!(n - |S| - 1)!}{n!} \, dc_{NC}(i, S \cup \{i\})$$

where

$$dc_{NC}(i, M) = \begin{cases} 1 & \text{if } i \text{ is never critical in the winning coalition } M \\ 0 & \text{otherwise} \end{cases}$$

Example 9 *Referring to Example 8:*

$$\phi_{1,NC}(v) = \phi_{2,NC}(v) = \phi_{3,NC}(v) = \phi_{4,NC}(v) = \phi_{5,NC}(v) = 0 \qquad \phi_{6,NC}(v) = \frac{23}{60}$$

4.2 Indices á la Bahnzaf

The Bahnzaf indices of criticality account for the coalitions a player may belong to; the power of player $i \in N$ in being critical of order $k = 1, ..., n$ is

$$\beta_{i,k}(v) = \sum_{S \not\ni i} \frac{dc_k(i, S \cup \{i\})}{2^{n-1}}$$

The distribution of the indices is:

$$B_i(v) = (\beta_{i,1}(v)\, \beta_{i,2}(v), \dots, \beta_{i,n}(v))$$

and the index of non-criticality is

$$\beta_{i,NC}(v) = \sum_{S \not\ni i} \frac{dc_{NC}(i, S \cup \{i\})}{2^{n-1}}$$

Example 10 *Referring to Example 8:*

$$B_1(v) = \left(\frac{5}{8}, 0, 0, 0, 0, 0\right) \qquad B_2(v) = B_3(v) = \left(\frac{1}{4}, \frac{3}{16}, 0, 0, 0, 0\right)$$

$$B_4(v) = B_5(v) = \left(\frac{1}{8}, \frac{3}{16}, \frac{1}{16}, 0, 0, 0\right) \qquad B_6(v) = (0, 0, 0, 0, 0, 0)$$

$$B_{1,NC}(v) = B_{2,NC}(v) = B_{3,NC}(v) = B_{4,NC}(v) = B_{5,NC}(v) = 0 \qquad B_{6,NC}(v) = \frac{5}{16}$$

5 Collective Indices

It is possible to synthesize the whole distribution and compare the power among players

Definition 3 *The Collective Shapley-Shubik (CSS) power index through all orders of criticality for player $i \in N$ is*

$$\bar{\Phi}_i(v) = \frac{\sum_{h=1}^{n} \phi_{i,h}(v) h^{-1}}{\sum_{h=1}^{n} h^{-1}}$$

Similarly, the Collective Banzhaf (CB) power index through all orders of criticality for player $i \in N$ is

$$\bar{B}_i(v) = \frac{\sum_{h=1}^{n} \beta_{i,h}(v) h^{-1}}{\sum_{h=1}^{n} h^{-1}}$$

These indices range between 0 and 1 and allow for comparing the power of different players

Let $\succeq_{CSS}$ and $\succeq_{CB}$ denote the order established among the players by the two Collective indices

Example 11 *Referring to Example 6:*

$$\bar{\Phi}_1(u) = \bar{\Phi}_2(u) = \frac{1}{3} \qquad \bar{\Phi}_1(w) = \bar{\Phi}_2(w) = \frac{1}{2}$$

Example 12 *Referring to Example 8:*

$$\bar{\Phi}_1(v) = \frac{34}{147} \qquad \bar{\Phi}_2(v) = \bar{\Phi}_3(v) = \frac{6}{49} \qquad \bar{\Phi}_4(v) = \bar{\Phi}_5(v) = \frac{25}{294} \qquad \bar{\Phi}_6(v) = 0$$
$$\bar{B}_1(v) = \frac{25}{98} \qquad \bar{B}_2(v) = \bar{B}_3(v) = \frac{55}{392} \qquad \bar{B}_4(v) = \bar{B}_5(v) = \frac{31}{317} \qquad \bar{B}_6(v) = 0$$

$1 \succ_{CSS} 2 \sim_{CSS} 3 \succ_{CSS} 4 \sim_{CSS} 5 \succ_{CSS} 6$ *and the same order applies through the Collective Bahnzaf index*

6 An Example

Referring to the weighted majority situation [51; 44, 3, 3, 3, 3, 3, 3] given in Example 3, party 1 is critical of order 1, while the other six parties are critical of order 4 and there are no parties critical of order 2 or 3

Accounting the ordering $44, 3, 3, 3, 3, 3, 3$ party 4 is critical of order 1, party 5 is critical of order 2 via party 4 (or 2 or 3), party 6 is critical of order 3 via parties 4 and 5 (or any pair of parties from 2 to 5), party 7 is critical of order 4 via parties 4, 5 and 6 (or any triple of parties from 2 to 6); the same reasoning applies to all the permutations of the parties with weight 3
Accounting the ordering $3, 3, 3, 3, 44, 3, 3$ party 5 is critical of order 1, party 6 is critical of order 3, party 7 is critical of order 4; again, the same reasoning applies to all the permutations of the parties with weight 3
All the possible orderings produce the table on the right (Roman number below parties denotes the order of criticality)

44	3	3	3	3	3	3
			I	II	III	IV
3	44	3	3	3	3	3
			I	II	III	IV
3	3	44	3	3	3	3
			I	II	III	IV
3	3	3	44	3	3	3
			I	II	III	IV
3	3	3	3	44	3	3
				I	III	IV
3	3	3	3	3	44	3
					I	IV
3	3	3	3	3	3	44
						I

In terms of distributions, we have

$$\Phi_1 = \left(\frac{24}{42}, 0, 0, 0, 0, 0, 0\right) \qquad \Phi_2 = \cdots = \Phi_7 = \left(\frac{3}{42}, \frac{4}{42}, \frac{5}{42}, \frac{6}{42}, 0, 0, 0\right) \qquad \bar{\Phi} = \left(\frac{80}{363}, \frac{3}{40}, \dots, \frac{3}{40}\right)$$

$$B_1 = \left(\frac{21}{32}, 0, 0, 0, 0, 0, 0\right) \qquad B_2 = \cdots = \Phi_7 = \left(\frac{5}{32}, \frac{5}{32}, \frac{5}{64}, \frac{1}{64}, 0, 0, 0\right) \qquad \bar{B} = \left(\frac{245}{968}, \frac{1}{10}, \dots, \frac{1}{10}\right)$$

7 Possible Developments

- The credibility of a threat of a party, i.e. the possibility of forming an alternative majority joining to some of the parties in the opposition and/or in the current majority, accounting the ideological contiguity of the parties on a left-right axis (see Chessa and Fragnelli, 2011) or the previous majorities
- Different aggregation rules of indices of each player for defining new collective indices
- Extend the concept of criticality to other values
- A better understanding of the nature of power associated to each index of criticality: how such a power may be carried out by means of the bargaining abilities of players? which trade-off exists between the different orders of criticality and between the corresponding indices?
- The indices of criticality may be used to compare the possibility that a coalition forms over different games with equivalent distribution of first order criticality

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Thanks!