Free-riding and Cost Allocation in Common Facilities Sharing

Vito Fragnelli
Università del Piemonte Orientale
vito.fragnelli@uniupo.it

Joint work with:
Federica Briata
federica.briata@libero.it

SING 13 – Paris – July 5-7 2017

Thanks to Stefano

Thanks to my co-authors

Alessandro	Rodica	Chiara	Fabio	Ana	Monserrat	Izabella
Encarnacion	Federica	Theo	Nicola	Luciano	Flavio	Roberto
Zeynep	Giuliana	Giulio	Anna	Giovanni	Giovanni	Elena
Daniela	Carlo	Giuliana	Enrico	Stefano	Joaquin	Angela
Roberto	Balbina	Josep	Ilya	Henk	Simona	Stef
Marco	Michela	Stefano	Anna	Guido	Lluis	Anna
Cesarino	Marco	Gianfranco	Nati	Stefania	Roberto	Silvia
Stefano	Rita	Ignacio	Maria	Fioravante	Enrico	Laura

The size represents the number of joint papers (cf. MathSciNet)
Alphabetical order according to family name

Contents

1 Introduction

- Some potential users have to share the maintenance cost C of a facility (sophisticated medical instruments, supercomputers, etc.)
- Equal sharing of C is a good possibility without further information; it is unfair, e.g. non-users are charged like users
- A check, whose cost is χ, may assess who the users are

There are several contributions on sharing costs, among them:

- the most popular is the airport problem (Littlechild and Thompson, 1977)
- telecommunication network (van den Nouweland et al., 1996)
- transport of the students in the area of Alicante (Sánchez-Soriano et al., 2002)
- railway networks (Norde et al., 2002)
- collecting and disposing urban solid wastes (Fragnelli and Iandolino, 2004)
- highways (Kuipers et al., 2013 and Sudhölter and Zarzuelo, 2017)

A nice survey is Fiestras-Janeiro el al. (2011)

Here

In a non-cooperative setting:

- Avoid free-riding situations
  - non-users pay for the cost of users (the check may eliminate them)
  - non-users may not to ask for the check, with the hope that the other non-users ask and pay for it (the check may cause them)
- Modify the rules
  - for obtaining the check
  - for sharing the maintenance cost and the checking cost

In a cooperative setting:

- improve the fairness of the cost allocation problem

2 Basic definitions and notations

Cost game in strategic form: $(X_1, X_2, \ldots, X_n, c_1, c_2, \ldots, c_n)$ where $N = \{1, 2, ..., n\}$ is the finite set of players, X_i is the non-empty set of strategies of player $i \in N$ and $c_i : \prod_{k \in N} X_k \longrightarrow \mathbb{R}$ is the cost function of player $i \in N$ $c_i(x_1, x_2, \ldots, x_n)$ is the amount that player $i \in N$ has to pay when the strategy profile $(x_1, x_2, \ldots, x_n)$ is chosen Let $X_{-i} = \prod_{k \neq i} X_k$

Definition 1. *Given a cost game in strategic form* $(X_1, X_2, \ldots, X_n, c_1, c_2, \ldots, c_n)$*, the strategy* $x_i \in X_i$ strongly dominates *the strategy* $y_i \in X_i$ *for player* $i \in N$*, if for each* $x_{-i} \in X_{-i}$

$$c_i(x_i, x_{-i}) < c_i(y_i, x_{-i})$$

The strategy $x_i \in X_i$ weakly dominates *the strategy* $y_i \in X_i$ *for player* i*, if for each* $x_{-i} \in X_{-i}$

$$c_i(x_i, x_{-i}) \leq c_i(y_i, x_{-i})$$

and there exists a strategy $\bar{x}_{-i} \in X_{-i}$ *such that*

$$c_i(x_i, \bar{x}_{-i}) < c_i(y_i, \bar{x}_{-i})$$

The strategy $x_i \in X_i$ *is* strongly (weakly) dominant *for player* i*, if* x_i *strongly (weakly) dominates every strategy* $y_i \in X_i$ *with* $x_i \neq y_i$*, while the strategy* $x_i \in X_i$ *is* strongly (weakly) dominated *if there exists a strategy* y_i *which strongly (weakly) dominates it*

Definition 2. *Given a cost game in strategic form* $(X_1, X_2, \ldots, X_n, c_1, c_2, \ldots, c_n)$*, the strategy profile* (x_i^*, x_{-i}^*) *is a* Nash equilibrium (Nash, 1950) *if for each* $i \in N$ *and for each* $x_i \in X_i$

$$c_i(x_i^*, x_{-i}^*) \leq c_i(x_i, x_{-i}^*)$$

A *TU-cost game* in characteristic function form is a pair (N, c)
where $N = \{1, 2, ..., n\}$ is again the set of players
$c : 2^N \longrightarrow \mathbb{R}$, $c(\varnothing) = 0$ is the *characteristic function*

A subset $S \subset N$ is called *coalition* and N is called *grand coalition*; $c(S), S \subseteq N$ may be viewed as the total cost that the players in S have to pay independently from the choices of the other players

A TU-cost game (N, c) is called *additive* when $c(S \cup T) = c(S) + c(T), S, T \subseteq N$ whenever $S \cap T = \varnothing$

Given a TU-cost game (N, c):

- an *allocation* is a vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where x_i represents the amount assigned to player $i \in N$
- an allocation $(x_1, x_2, \ldots, x_n)$ is *individually rational* if $x_i \leq c(\{i\})$ for each $i \in N$ and it is *efficient* if $\sum_{i \in N} x_i = c(N)$; the set of all individually rational and efficient allocations is called *imputation set*
- an imputation is *coalitionally rational* if $\sum_{i \in S} x_i \leq c(S)$ for each $S \subset N$; the set of all the coalitionally rational imputations is called *core* and is denoted by $core(c)$

3 Recalls

The non-cooperative models in Briata (2011)

Sharing the maintenance cost $C > 0$ of a printer among the potential users $N = \{1, 2, ..., n\}$, with a proper subset $M = \{1, 2, ..., m\} \subset N$ of non-users
A check of cost $\chi > 0$ is necessary in order to identify who the actual users are; this may be viewed as an information cost (see TUIC games by Moretti and Patrone, 2004)
Each player may $name$ the facility ($naming$ procedure)
At least t agents have to name the facility: e.g. one agent for the $Naming\ Game$, the majority of the agents for the $Decision\ Majority\ Game$

The intensity of use of the facility is disregarded
C and χ are fixed and known; moreover, they are comparable
If the check is made, C is equally divided among the users, while χ is equally shared among the agents that asked for it; otherwise, C is equally shared among all the agents
Free-riding behavior of the users, if the check is not made, or of the non-users, if the check is made

Each agent has only two strategies, $\mathcal{A}$ = *asking for check* and $\mathcal{N}$ = *not asking for check*
$\mathcal{A}$ is strongly dominated by $\mathcal{N}$ for a user
Removing strongly dominated strategies, the players in the set M are symmetric and the resulting game is binary, so it has at least one pure Nash equilibrium (see Briata, 2011)

Example 1. *Let* $N = \{1,2,3,4,5\}$, $M = \{1,2,3\}$ *and* $t = 2$. *As the users choose the dominant strategy* $\mathcal{N}$, *and the non-users are symmetric, there are four meaningful situations:*

choice					*individual cost*					
non-users			*users*		*non-users*			*users*		
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\frac{C}{5}$	$\frac{C}{5}$	$\frac{C}{5}$	$\frac{C}{5}$	$\frac{C}{5}$	*Nash equilibrium*
$\mathcal{A}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\frac{C}{5}$	$\frac{C}{5}$	$\frac{C}{5}$	$\frac{C}{5}$	$\frac{C}{5}$	*Nash equilibrium if* $\frac{\chi}{2} \geq \frac{C}{5}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\frac{\chi}{2}$	$\frac{\chi}{2}$	0	$\frac{C}{2}$	$\frac{C}{2}$	*Nash equilibrium if* $\frac{\chi}{2} \leq \frac{C}{5}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{N}$	$\mathcal{N}$	$\frac{\chi}{3}$	$\frac{\chi}{3}$	$\frac{\chi}{3}$	$\frac{C}{2}$	$\frac{C}{2}$	*no Nash equilibrium*

1. $\frac{C}{5} \geq \frac{\chi}{2}$
   *The non-users may obtain an advantage if the check is made; if only one non-user does not ask for it, s/he pays nothing (*free-riding*)*

2. $\frac{\chi}{2} > \frac{C}{5} > \frac{\chi}{3}$
   *The non-users obtain an advantage when the check is made only when all of them ask for the check; if only one non-user does not ask for it, s/he pays nothing (*free-riding*) and the other two pay more (*inefficiency*)*

3. $\frac{\chi}{3} \geq \frac{C}{5}$
   Not ask for the check is a weakly dominant strategy for the non-users

The previous results may be generalized for any value of n, m, t, C, χ

If $t = 1$, the number of Nash equilibria decreases, since the indifferent Nash equilibria disappear

4 The non-cooperative approach

See Briata and Fragnelli (2017)

Modify the model in Briata (2011) with a twofold objective:

- reducing the free-riding behavior via two mechanisms
- increasing the profitability of the check, adding the condition that the check provides also the intensity of usage

4.1 Quorum

Strategy $\mathcal{A}$ may avoid the free-riding behavior of the users, but checking cost can be too expensive if the number of users who ask for it is too scant
An adequate threshold, the *quorum* q, may guarantee that the check is made only if it is non-disadvantageous

$$q = \min_{p \in \mathbb{N}_>} \left\{ p : \frac{\chi}{p} \leq \frac{C}{n} \right\}$$

Mechanism 1. *Fix the threshold equal to the quorum*

Under Mechanism 1, if the quorum is reached, each agent who asked for it pays no more than the equal share of C
But the non-users may still have a free-riding behavior

Example 2. *Let* $N = \{1,2,3,4\}, M = \{1,2\}, C = 12, \chi = 2$*; in this case* $q = 1$*. Let* $Q \subseteq N$ *be the set of agents that ask for the check and* x_Q *the corresponding allocation*

a. no agent asks for the check: $x_{\varnothing} = (3,3,3,3)$

b. agent 1 asks for the check and obtains it: $x_{\{1\}} = (2,0,6,6)$

c. agent 2 asks for the check and obtains it: $x_{\{2\}} = (0,2,6,6)$

d. agents 1 and 2 ask for the check and obtain it: $x_{\{1,2\}} = (1,1,6,6)$

Comparing cases b, c and d, the non-users may have an advantage from not asking the check, with the risk that the exit is a, with an unfair solution

Mechanism 2. *When the check is made, the maintenance cost is equally shared among all the agents that did not ask for the check*

Under Mechanism 2, the non-users that do not ask for the check pay a quota of C and Mechanism 1 guarantees that non-users that ask for the check do not pay more than the equal share of C

Remark 1. *When the check is made the non-users that do not ask for it pay* $\frac{C}{|N \setminus Q|}$ *that is strictly larger than* $\frac{C}{n}$*; this reinforce the incentive for asking for the check which Mechanisms 1 and 2 were designed for*

Example 3 (Example 2 continued)**.** *Let* $N = \{1, 2, 3, 4\}, M = \{1, 2\}, C = 12, \chi = 2$*; under Mechanism 2, the new allocations in cases b and c are* $x_{\{1\}} = (2, 4, 4, 4)$ *and* $x_{\{2\}} = (4, 2, 4, 4)$*, respectively*

Remark 2. *Mechanisms 1 and 2 guarantee that asking for the check is a weakly dominant strategy for the non-user even if they ignore the number* m *of non-users*

When $q > m$ it is still non-disadvantageous to ask for the check, as in this case the check is never obtained

Example 4. *Let* $N = \{1, 2, 3, 4\}, M = \{1, 2\}, C = 24, \chi = 13$*; in this case* $q = 3$*, i.e. the non-users cannot obtain the check and the solution is anyhow* $x = (6, 6, 6, 6)$

4.2 Intensity of usage

Mechanisms 1 and 2 reduce free-riding problems, but there are situations with few non-users versus an expensive checking cost and a cheap maintenance cost (see Example 4)

Suppose that when the check is made, the cost C is shared according to a division rule $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n_{\geq}$, $\gamma(C) = (\gamma_1(C), \gamma_2(C), \ldots, \gamma_n(C))$, where $\gamma_i(C)$ represents the non-negative quota that agent $i \in N$ has to pay
A fair division rule γ satisfies the following properties:

- *efficiency*: $\sum_{i \in N} \gamma_i = C$
- *weak monotonicity w.r.t. the level of usage*: if agent i uses the facility at a strictly higher level than agent j then $\gamma_i \geq \gamma_j$ for each C
- *equal treatment of equal users*: if users i and j use the facility at the same level then $\gamma_i = \gamma_j$ for each C
- *weak monotonicity w.r.t. the cost C*: if $C > C'$, then then $\gamma_i(C) \geq \gamma_i(C')$ for every $i \in N$

Remark 3.

- *The previous rules for dividing the cost C satisfy the properties, so the following approach generalizes the previous models*
- *No agent receives money for using the facility, whatever the level of usage*
- *γ could assign a positive amount to an agent that does not use the facility (accessibility fee)*
- *Each agent knows how much s/he used the facility and may calculate her/his quota of C according to γ*

Possible rules γ

- Different slots of intensity of usage, e.g. low usage (up to a first threshold), medium usage (up to a larger threshold), high usage (over the larger threshold)
- Proportional to the actual usage, e.g. proportional to the number of pages printed

More sophisticated rules can be used, possibly accounting also the right of usage

γ makes the request for the check non-disadvantageous for agent $i \in N \setminus M$ if

$$\frac{\chi}{|M|+1} + \gamma_i \leq \frac{C}{|N \setminus M|} \text{ and } q \leq |M| \tag{1}$$

Under Mechanisms 1 and 2 all the agents in M ask for the check, so if agent $i \in N \setminus M$ asks for the check then the cost χ is divided among at least $|M|+1$ agents, so if adding the individual quota γ_i the total is not larger than the equal sharing of C among the users, it is convenient to ask for the check

Example 5. *Let* $N = \{1, 2, 3, 4, 5\}, M = \{1, 2\}, C = 30, \chi = 8$, *so that* $t = q = 2 = |M|$, *i.e. the non-users obtain the check and the solution is* $x_{\{1,2\}} = (4, 4, 10, 10, 10)$; *suppose that agent 3 uses the facility at a low level, agent 4 uses the facility at a medium level, and agent 5 uses the facility at a high level; let the division rule of the cost* C *be such that* $\gamma(C) = (0, 0, 1, 9, 20)$, *so that agent 3 is the unique user that satisfies condition* (1)*. If the first three agents ask for the check the solution is* $x_{\{1,2,3\}} = (2.666, 2.666, 3.666, 14.5, 14.5)$

Unfortunately, the situation could be more complicated

Example 6 (Example 5 revised)**.** *Reconsider Example 5 with* $\chi = 16$*. In this case condition* (1) *is never satisfied as* $q = 3 > |M|$, *so the check cannot be obtained. On the other hand, it is easy to notice that the first three agents pay in total* $17 = \chi + \gamma_3 = 16 + 1$ *if the check is made and* $18 = \frac{C}{n} \cdot 3 = 6 \cdot 3$ *otherwise, but* $x_{\{1,2,3\}} = (5.333, 5.333, 6.333, 14.5, 14.5)$, *i.e. agent 3 pays more asking for the check*

Remark 4.

- *In Example 5, Agent 4 does not satisfies condition* (1), *as* $\frac{\chi}{|M \cup \{4\}|} + \gamma_4 = 11.666 > 10 = \frac{C}{|3|}$; *on the other hand, it is easy to notice that* $x_{\{1,2,3,4\}} = (2, 2, 3, 11, 20)$, *i.e. agent 4 pays less asking for the check, jointly with agents 1, 2 and 3, but we cannot account this in condition* (1), *as agents 3 and 4 should exchange information on their willingness of asking for the check*

- *In Example 6, agents 1, 2 and 3 may profit asking for the check but a division of the cost* χ *different from the simple equal share among the agents that ask for the check is necessary*

These situations of coalition formation and cost allocation may be faced via a TU-cost game

5 The cooperative approach

This is new!(hopefully)

A group of agents may decide of asking for the check if their total costs of using the facility and of the check is smaller then the total costs they have to pay when the check is not made

Before subscribe binding agreements each agent is able to compute only her/his quota of C according to γ knowing her/his level of usage

The grand coalition never profits from the check

The way of dividing the cost C among the agents plays an important role:

in Example 6, if the division were $\gamma = (0, 0, 3, 10, 17)$, i.e. more favorable to large users, the first three agents should pay in total $19 = \chi + \gamma_3 = 16 + 3$ if the check is made, that is larger than 18

Fixing the division rule γ, it is possible to define the cost game (N, c) where N is the set of agents and the characteristic function c is defined as:

$$c(S) = min\left\{ s\,\frac{C}{n}, \sum_{i\in S}\gamma_i + \chi \right\}, S \subseteq N$$

where $S \subseteq N$ is the set of agents that aim to ask for the check

Remark 5. *Mechanisms 1 and 2 are no longer required*

A basic role is played by the hypothesis that the agents, being aware of the cost C and of their level of usage, are able to compute the amount assigned to them by the rule γ

The most trivial case is $c(S) = s\,\frac{C}{n}$ for each $S \subseteq N$, i.e. the game is additive, and the unique imputation coincides with the equal sharing of the cost C
Assume that there exists at least one coalition $S \subset N$ such that:

$$c(S) = \sum_{i \in S} \gamma_i + \chi \leq s\,\frac{C}{n} \tag{2}$$

and

$$\gamma_i \leq \frac{C}{n}, i \in S \tag{3}$$

Condition (2) implies that the agents in S have no disadvantage from asking for the check
Condition (3) states that all the agents in S may profit from cooperation with the other agents in S; note that condition (3) implies no-subsidization

The cost of the coalition S is allocated among its members, referring to the restricted game (S, c_S), where $c_S(T) = c(T), T \subseteq S$

In view of condition (3), an agent $i \in N$ whose quota γ_i is not larger than the equal share $\frac{C}{n}$ may enter in a non-disadvantageous coalition, so the largest coalition includes at most all and only those agents that satisfy condition (3)

Imputations require $\sum_{i\in S} x_i = \sum_{i\in S} \gamma_i + \chi$ (efficiency) and $x_i \leq c(\{i\}), i \in S$ (individual rationality)
A fairness condition is

$$x_i \geq \gamma_i, i \in S \tag{4}$$

So,

$$x_i = \gamma_i + \alpha_i,\ i \in S$$

with $0 \leq \alpha_i \leq \frac{C}{n} - \gamma_i$ and $\sum_{i\in S} \alpha_i = \chi$

Remark 6. *Apparently, the condition $\alpha_i \leq \frac{C}{n} - \gamma_i$ may seem not enough to guarantee the individual rationality; on the other hand, if $c(\{i\}) = \gamma_i + \chi < \frac{C}{n}$ then $\sum_{j\in N} \alpha_j = \chi$ and $0 \leq \alpha_j \leq \chi, j \in N$ are sufficient for guaranteeing $x_i = \gamma_i + \alpha_i \leq c(\{i\})$*

The *solidarity solution* SOL is defined as follows:

$$SOL_i = \gamma_i + \frac{\left(\frac{C}{n} - \gamma_i\right)}{\sum_{j\in S}\left(\frac{C}{n} - \gamma_j\right)}\,\chi,\ i \in S$$

i.e. agents pay their quota according to γ plus the sharing of χ proportional to the savings
It is a solidarity solution because the more an agent saves with the check, the larger the quota of χ s/he has to pay

Proposition 1. *The solidarity solution is weakly monotonic w.r.t. the cost $\gamma_i, i \in S$*

Remark 7. *The cost χ may be allocated according to other criteria, obtaining different solutions.*
A general approach is a bankruptcy problem *(see Aumann and Maschler, 1985) in which the set of agents is S, the claims are $\frac{C}{n} - \gamma_i, i \in S$ and the estate is χ*
The solidarity solution coincides with the proportional solution *(see Herrero and Villar, 2001), enforcing the meaning of the solution*

Remark 8. *When the usage quota of agent $i \in S$ is $\gamma_i = \frac{C}{n}$, then $SOL_i = \gamma_i$, i.e. the agent does not contribute to χ*
When the usage quota of agent $i \in S$ is $\gamma_i = 0$, if $\gamma_j = 0$ for each $j \in S \setminus \{i\}$, then $SOL_j = \frac{1}{s}\,\chi$ for each $j \in S$; if $\gamma_j = \frac{C}{n}$ for each $j \in S \setminus \{i\}$ then $SOL_i = \chi$; condition (2) implies $\chi \leq \frac{C}{n}$

Proposition 2. *The restricted game (S, c_S) has non-empty core, if and only if conditions (2) and (3) hold, i.e.*

$$core(c_S) \neq \varnothing \Leftrightarrow \sum_{i \in S} \gamma_i + \chi \leq s\,\frac{C}{n} \text{ and } \gamma_i \leq \frac{C}{n}, i \in S$$

Remark 9. *Condition (4), i.e. $x_i \geq \gamma_i, i \in S$, is necessary for core-membership*

Example 7. *Let* $N = \{1, 2, 3, 4\}, M = \{1, 2\}, C = 28, \chi = 10$*; let* $\gamma(C) = (0, 0, 8, 19)$*. In this case* $\frac{C}{n} = 7$ *and the characteristic function is as follows:*

S	1	2	3	4	**12**	13	14	23	24	34	**123**	124	134	234	1234
$c(S)$	7	7	7	7	10	14	14	14	14	14	18	21	21	21	28

The two coalitions in **bold** *satisfy condition* (2)*, but only the first one satisfies condition* (3)*; in fact,* $core(c_{\{1,2\}}) = \{(\alpha, 10 - \alpha) \in \mathbb{R}^2, 3 \le \alpha \le 7\}$*, while* $core(c_{\{1,2,3\}})$ *is empty, as* $c_{\{1,2,3\}}(12) + c_{\{1,2,3\}}(3) = 17 < 18 = c_{\{1,2,3\}}(123)$

Note that the solidarity solution is $SOL = (5, 5)$ *for coalition* $\{1, 2\}$*, i.e. they equally share the cost* χ*, according to Remark 8*

Remark 10. *Referring to the game* $c_{\{1,2,3\}}$*, the allocation* $(5, 5, 8)$ *is profitable for agent* 3 *that pays 14 when the check is asked only by agents* 1 *and* 2

Remark 11. *The solidarity solution may be viewed as the compromise efficient solution (see Tijs and Otten, 1993) among the allocations* $\underline{m} = (\gamma_1, \gamma_2, ..., \gamma_s)$ *and* $\overline{m} = \left(\frac{C}{n}, \frac{C}{n}, ..., \frac{C}{n}\right)$

One of the most important compromise solutions is the τ-value (Tijs, 1981) defined as follows

Definition 3 (Tijs, 1981)**.** *Given a cost game* (N, c), *for each* $i \in N$ *let* $\underline{m}_i = c(N) - c(N \setminus \{i\})$ *and* $\overline{m}_i = \min_{S \ni i} \left\{ c(S) - \sum_{j \in S \setminus \{i\}} \underline{m}_j \right\}$; *if*

1. $\underline{m}_i \leq \overline{m}_i$, *for each* $i \in N$
2. $\sum_{i \in N} \underline{m}_i \leq c(N) \leq \sum_{i \in N} \overline{m}_i$

then the τ*-value,* $\tau(c)$*, is the compromise efficient solution among* $\underline{m}$ *and* $\overline{m}$*, i.e.*

$$\tau(c) = \alpha \underline{m} + (1 - \alpha) \overline{m}$$

with α *s.t.* $\sum_{i \in N} \tau_i(c) = c(N)$, *i.e.* $\alpha = \dfrac{\sum_{i \in N} \overline{m}_i - c(N)}{\sum_{i \in N} \overline{m}_i - \sum_{i \in N} \underline{m}_i}$

Proposition 3. *Given the restricted game* (S, c_S)*, the solidarity solution coincides with the* τ*-value* $\tau(c_S)$ *when:*

a. $c_S(S \setminus \{i\}) = \sum_{j \in S \setminus \{i\}} \gamma_i + \chi$ *for each* $i \in S$

b. $\gamma_i + \chi \geq \frac{C}{n}$ *for each* $i \in S$

Unfortunately, condition $a.$ is quite strong

A different way for obtaining a solution is to share among the agents in S the savings of coalition S

$$\mathcal{R}(S) = s\,\frac{C}{n} - \left(\sum_{i \in S} \gamma_i + \chi\right)$$

In these cases, the solutions x are

$$x_i = \frac{C}{n} - \beta_i,\ i \in S$$

with $0 \leq \beta_i \leq \frac{C}{n} - \gamma_i$ and $\sum_{i \in S} \beta_i = \mathcal{R}(S)$

Remark 12. *Again, a general approach for allocating the saving $\mathcal{R}(S)$ may be to define a bankruptcy problem in which the set of agents is S, the claims are $\frac{C}{n} - \gamma_i, i \in S$ and the estate is $\mathcal{R}(S)$*
The set of solutions of this bankruptcy problem coincides with the set of solutions of the bankruptcy problem in Remark 7, where two solutions coincides if and only if they are obtained via dual rules

6 Influence of the agents in asking for the check

How much the agents may influence the decision of making the check?
The answer makes use of a suitable simple game associated to (N, c)

Definition 4. *A game* (N, w) *is* simple *if* $w : 2^N \longrightarrow \{0, 1\}$*; if* $w(S) = 1$ *then* S *is called* winning*, if* $w(S) = 0$ *then* S *is called* losing*; usually,* $w(N) = 1$ *and the game is monotonic, i.e. whenever* S *is winning also* $T \supset S$ *is winning*
Given a winning coalition $S \subseteq N$*, a player* $i \in S$ *is* critical for S*, if* $S \setminus \{i\}$ *is losing*
A winning coalition S *is* minimal *if all its agents are critical for it and it is* quasi-minimal *if at least one agent in* S *is critical for it*

Given the game (N, c), it is possible to associate it the simple game (N, w^c):

$$w^c(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} \gamma_i + \chi \leq s\dfrac{C}{n} \\ 0 & \text{otherwise} \end{cases}, \quad S \subseteq N$$

Note that the game (N, w^c) is non-standard as the grand coalition is losing; moreover, also the condition of monotonicity does not hold

Among the classical power indices, the Shapley-Shubik index (1954) and the Banzhaf index (1965) have a lot of well-known good properties but cannot be used in this case because they require that the characteristic function is monotonic

The Public Good Index (Holler, 1982) is suitable because it was defined for public goods and in our setting, the check may be considered a public good; moreover, it takes into accounts the minimal winning coalitions, differently from the Johnston index (1978) that accounts the quasi-minimal ones, without considering their cardinality, differently from the Deegan-Packel index (1978)

The Public Good Index is defined as follows:

$$PGI_i(w) = \frac{h_i}{\sum_{k \in N} h_k} \quad i \in N$$

where h_i is the number of minimal winning coalitions including agent $i \in N$

The Public Good Index allows defining the *influential solution* ($INFL$) for the restricted game (S, c_S):

$$INFL_i = \frac{C}{n} - \frac{PGI_i(w^c)}{\sum_{j \in S} PGI_j(w^c)} \mathcal{R}(S), i \in S$$

In this case, the more an agent has the power of influencing the request of the check, the lower the quota of χ s/he has to pay

Example 8. *Let* $N = \{1,2,3,4\}, M = \{1\}, C = 20, \chi = 7$ *and* $\gamma = (0,2,2,16)$

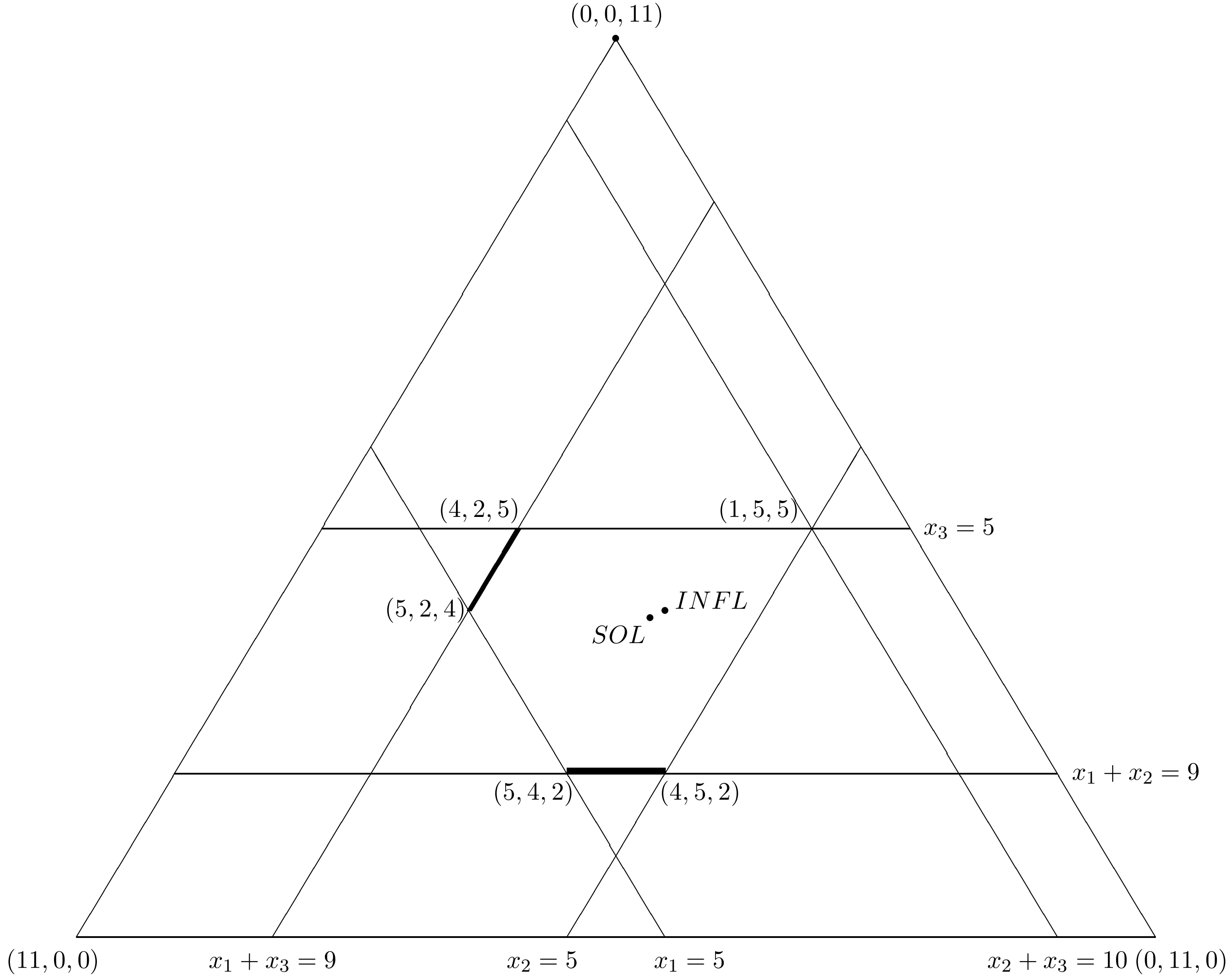
S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
$c(S)$	5	5	5	5	9	9	10	10	10	10	11	15	15	15	20
$w^c(S)$	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0

The picture represents the core of $c_{\{1,2,3\}}$

The thick segments joining points $(5,4,2),(4,5,2)$ *and* $(5,2,4),(4,2,5)$ *represent the divisions of the cost in which agents 1 and 2, or agents 1 and 3, respectively pay as if they asked for the check, and agent 3 or agent 2, respectively, pays only the usage quota* γ

$SOL = (3.182, 3.909, 3.909)$

The minimal winning coalitions of $w^{c_{\{1,2,3\}}}$ *are* $\{1,2\}$ *and* $\{1,3\}$ *so* $PGI(w^{c_{\{1,2,3\}}}) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, *and* $INFL = (3,4,4)$



7 Further developments

- Characterize the solidarity solution
- Analyze other properties of the cooperative game
- Since the total cost is $C+\chi$ in case of check and C otherwise, the users can propose the not-users that if they do not ask for check, they commit to pay them an amount in such a way that each agent may be better off
- A different representation of the cost game, e.g. *partition function form* (Thrall and Lucas, 1963), may allow accounting for Remark 10, i.e. the situation of the players when other agents ask for the check
- Referring to Example 5 and Remark 4, agent 4 does not satisfy condition (3), so it cannot enter in a coalition for asking for the check. On the other hand, if the check is asked by agents in $\{1,2,3\}$ then the solution is $x_{\{1,2,3\}} = (2.666, 2.666, 3.666, 14.5, 14.5)$, while $c(\{1,2,4\}) = 17 < 2.666 + 2.666 + 14.5$ and $c(\{1,2,3,4\}) = 18 < 2.666 + 2.666 + 3.666 + 14.5$, i.e. both coalitions satisfy condition (2). A suitable coalition formation process may allow all the agents involved being better off

References

[1] Aumann, R.J. and M. Maschler (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory* 36, 195-213.

[2] Banzhaf, J.F. (1965) Weighted Voting doesn't Work: A Mathematical Analysis, *Rutgers Law Review* 19, 317-343.

[3] Briata, F. (2011) Noncooperative Games from TU Games with Information Cost, *International Game Theory Review* 13, 301-323.

[4] Briata, F. and V. Fragnelli (2017) Free-riding in Common Facility Sharing, *Transactions on Computational Collective Intelligence*, to appear.

[5] Deegan, J. and Packel, E.W. (1978) A New Index of Power for Simple n-person Games, *International Journal of Game Theory* 7, 113-123.

[6] Fiestras-Janeiro, M.G., I. García-Jurado and M.A. Mosquera (2011) Cooperative games and cost allocation problems, *TOP* 19, 1-22.

[7] Fragnelli, V. and A. Iandolino (2004) A Cost Allocation Problem in Urban Solid Wastes Collection and Disposal, *Mathematical Methods of Operations Research* 59, 447-464.

[8] Herrero C. and Villar A. (2001) The Three Musketeers: Four Classical Solutions to Bankruptcy Problems. *Mathematical Social Sciences* 42, 307-328.

[9] Holler, M.J. (1982) Forming Coalitions and Measuring Voting Power, *Political Studies* 30, 262-271.

[10] Johnston, R.J. (1978) On the Measurement of Power: Some Reactions to Laver, *Environment and Planning A* 10, 907-914.

[11] Kuipers, J., M.A. Mosquera and J.M. Zarzuelo (2013) Sharing costs in highways: A game theoretic approach, *European Journal of Operational Research*, 228: 158-168.

[12] Littlechild S.C. and G.F. Thompson (1977) Aircraft Landing Fees: A Game Theory Approach, *Bell Journal of Economics* 8, 186-204.

[13] Moretti, S. and F. Patrone (2004) Cost Allocation Games with Information Costs, *Mathematical Methods of Operations Research* 59, 419-434.

[14] Nash, J.F. (1950) Equilibrium Points in n-Person Games, *Proceedings of the National Academy of Sciences of the United States of America* 36, 48-49.

[15] Norde, H., V. Fragnelli, I. García-Jurado, F. Patrone and S. Tijs (2002) Balancedness of Infrastructure Cost Games, *European Journal of Operational Research* 136, 635-654.

[16] van den Nouweland, A., P. Borm, W. van Golstein Brouwers, R. Groot Bruinderink and S. Tijs (1996) A Game Theoretic Approach to Problems in Telecommunication, *Management Science* 42: 294-303.

[17] Sánchez-Soriano, J., N. Llorca, A. Meca, E. Molina and M. Pulido (2002) An integrated transport system for Alacant's students. UNIVERCITY, *Annals of Operations Research* 109: 41-60.

[18] Shapley, L.S. and M. Shubik (1954) A Method for Evaluating the Distribution of Power in a Committee System, *American Political Science Review* 48, 787-792.

[19] Sudhölter, P. and J.M. Zarzuelo (2017) Characterizations of highway toll pricing methods, *European Journal of Operational Research*, 260: 161-170.

[20] Thrall, R.M. and W.F. Lucas (1963) n-person games in partition function form, *Naval Research Logistics Quarterly* 10 : 281-298.

[21] Tijs, S.H. (1981) Bounds for the Core and the τ-Value. In O. Moeschlin and D. Pallaschke (eds.) *Game Theory and Mathematical Economics*, North-Holland Publishing Company, Amsterdam, 123-132.

[22] Tijs, S. and G.J. Otten (1993) Compromise values in cooperative game theory, *TOP* 1, 1-36.

Thanks!