Bankruptcy Problems

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Summary

Bankruptcy problems

Bankruptcy Rules and Min Cost Flow Problems

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Bankruptcy problems

Allocation of scarce resources (see O'Neill, 1982, Aumann and Maschler, 1985 and Curiel, Maschler and Tijs,1987)

$$BP = (N, c, E) = (E; c_1, ..., c_n)$$

where	$N = \{1, ..., n\}$	set of claimants
	$c = (c_1, ..., c_n) \in \mathbb{R}^n_>$	vector of claims
	$E \in \mathbb{R}_\geq$	estate, with $E \leq \sum_{i \in N} c_i = C$

A *solution* is an n-dimensional real vector $x = (x_1, ..., x_n)$, where x_i represents the monetary amount assigned to claimant $i \in N$, satisfying the following conditions:

$$0 \leq x_i \leq c_i, \qquad i \in N \qquad \text{(rationality)}$$

and

$$\sum_{i \in N} x_i = E \qquad \text{(efficiency)}$$

A *solution rule* is a function ψ that assigns a solution to each bankruptcy problem

Solutions

There exist four classical solutions (see Herrero and Villar, 2001)

- Proportional

  $PROP$ - The assignments are proportional to the claims:

$$PROP(N, c, E)_i = \frac{E}{C}\, c_i \qquad i \in N$$

- Constrained Equal Awards

  CEA - The assignments are the same for all the agents, but not larger than the claims:

$$CEA(N, c, E)_i = \min\{\alpha, c_i\} \qquad i \in N$$

  where α is such that $\sum_{i \in N} CEA(N, c, E)_i = E$

- Constrained Equal Losses

  CEL - The assignments are equal to the claims reduced of the same amount for all the agents, but non-negative:

$$CEL(N,c,E)_i = \max\{c_i - \beta, 0\} \qquad i \in N$$

  where β is such that $\sum_{i\in N} CEL(N,c,E)_i = E$

- Talmud

$$TAL(N,c,E) = \begin{cases} CEA(N,c/2,E) & \text{if } E \leq C/2 \\ c/2 + CEL(N,c/2,E-C/2) & \text{if } E > C/2 \end{cases}$$

Example 1 (Solutions) *Given the bankruptcy problem* $(72; 6, 9, 24, 33, 36)$

$$C = 108; \frac{E}{C} = \frac{2}{3}$$
$$PROP = (4, 6, 16, 22, 24)$$
$$CEA = (6, 9, 19, 19, 19) \qquad [\alpha = 19]$$
$$CEL = (0, 1.5, 16.5, 25.5, 28.5) \qquad [\beta = 7.5]$$
$$TAL = (3, 4.5, 14.5, 23.5, 26.5)$$

◊

$PROP$ is the most intuitive
CEA favors smaller claims
CEL favors larger claims
TAL is related to the nucleolus of the bankruptcy game

Duality

Given a solution rule ψ, the dual solution rule ψ^* produces the same solution when used for allocating the losses w.r.t. the claims:

$$\psi^*(N, c, E) = c - \psi(N, c, C - E)$$

CEA and CEL are dual rules, while $PROP$ and TAL are *self-dual*:

- $CEA(N, c, E) = c - CEL(N, c, C - E)$
- $CEL(N, c, E) = c - CEA(N, c, C - E)$
- $PROP(N, c, E) = c - PROP(N, c, C - E)$
- $TAL(N, c, E) = c - TAL(N, c, C - E)$

Comments

Very simple model of real-world situations

Most important additional elements in the existing literature:

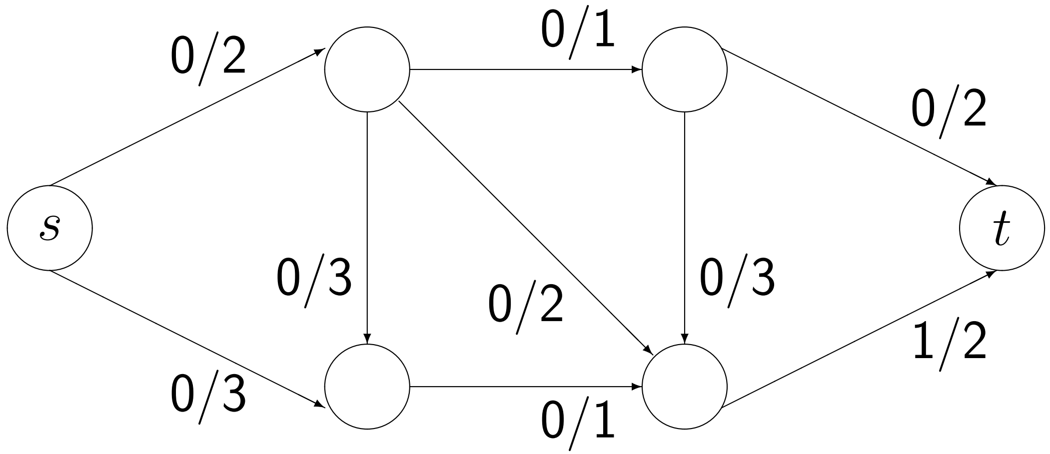
- different priorities of the claimants (Young, 1994, Bebchuck and Fried, 1996, Schwarcz, 1997 and Kaminski, 2000)
- minimal rights of the claimants (Curiel, Maschler and Tijs, 1987 and Pulido, Sanchez-Soriano and Llorca, 2002)
- multiple issues (Calleja, Borm, Hendrickx, 2005 and Moreno-Ternero, 2009)
- negative claims and estate (Herrero, Maschler and Villar, 1999, Branzei, Ferrari, Fragnelli and Tijs, 2008 and 2011)
- non-transferable utility situations (Orshan, Valenciano and Zarzuelo, 2003 and Carpente, Casas-Mendez, Gozalvez, Llorca, Pulido and Sanchez-Soriano, 2013)

Two important surveys are due to Thomson (2003 and 2015)

Bankruptcy Rules and Min Cost Flow Problems

Standard flow problem

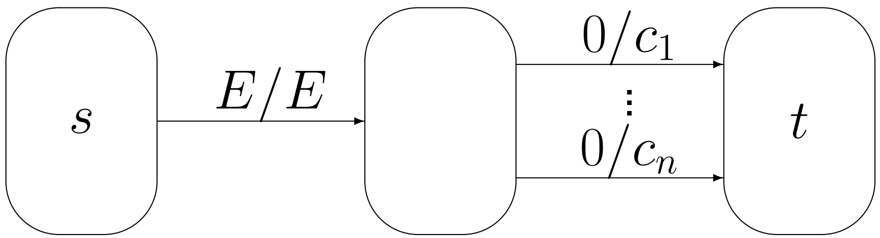
Let $G(N, A)$ be a network with two particular nodes, the source s with no entering arcs and the sink t with no outgoing arcs; arcs have minimal and maximal capacity constraints



A flow is a function $x : A \to \mathbb{R}_+$ that respects the capacity constraints and such that

$$\sum_{j\in N} x_{ij} = \sum_{j\in N} x_{ji}, \forall\, i \in N \setminus \{s, t\}$$

A classical bankruptcy problem can be represented as a standard flow problem



Each feasible flow corresponds to a solution of the bankruptcy problem

Min cost flow approach with suitable cost functions $k_i, i \in N$ can lead to classical rules, or suggest new divisions according to different fairness criteria, tailoring the amount on each agent

- $PROP$: $k_i(x_i) = \dfrac{x_i^2}{c_i},\ i \in N$
- CEA: $k_i(x_i) = x_i^2,\ i \in N$
- CEL: $k_i(x_i) = (c_i - x_i)^2,\ i \in N$
- TAL:
  - if $E \leq \frac{1}{2}\sum_{i \in N} c_i$:
  $$k_i(x_i) = x_i^2,\ i \in N$$
  setting the maximal capacity of the arcs corresponding to the claimants to $\frac{1}{2}c_i$
  - if $E > \frac{1}{2}\sum_{i \in N} c_i$:
  $$k_i(x_i) = (c_i - x_i)^2,\ i \in N$$
  setting the minimal capacity of the arcs corresponding to the claimants to $\frac{1}{2}c_i$

Defining the minimal right of agent $i \in N$ as

$$m_i(N, c, E) = \max\{0, E - \sum_{j \in N \setminus \{i\}} c_j\},\ i \in N$$

it is possible to introduce the *Adjusted Proportional* rule (Curiel, Maschler and Tijs, 1987):

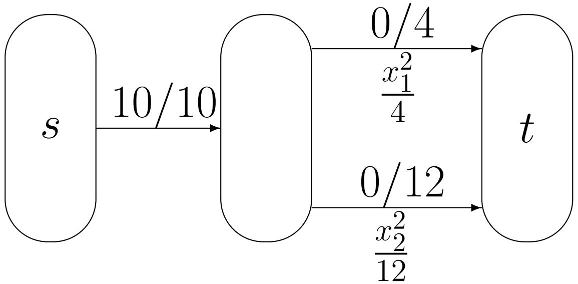
$$APROP_i(N, c, E) = m_i(N, c, E) + PROP_i(Nc', E'),\ i \in N$$

where $E' = E - \sum_{j \in N} m_j$ and $c'_i = min\{E', c_i - m_i(N, c, E)\},\ i \in N$
It coincides with the τ-value (Tijs, 1981) of the bankruptcy game

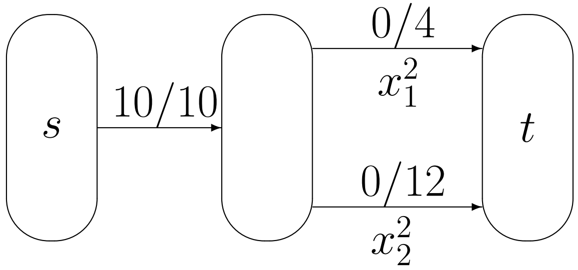
The corresponding cost functions are:

$$k_i(x_i) = \begin{cases} 0 & \text{if } x_i \leq m_i \\ \dfrac{(x_i - m_i)^2}{c'_i - m_i} & \text{if } x_i > m_i \end{cases},\ i \in N$$

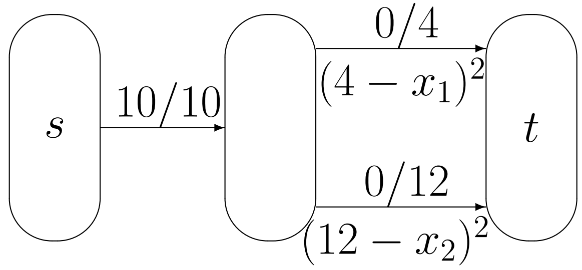
Example 2 *Consider the bankruptcy problem with* $N = \{1, 2\}; c = (4, 12); E = 10$. *The flow problems associated to the above five rules are depicted below (the notations for the arcs are "min capacity/max capacity" above the arc and "cost function" below the arc)*



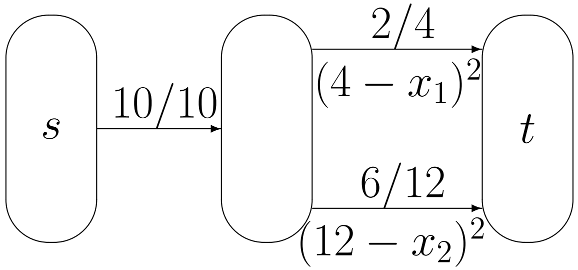
$PROP : x^* = (2.5, 7.5)$



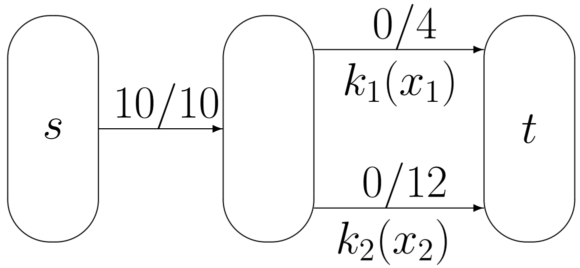
$CEA : x^* = (4, 6)$



$CEL : x^* = (1, 9)$



$TAL : x^* = (2, 8)$



$APROP : x^* = (2, 8)$

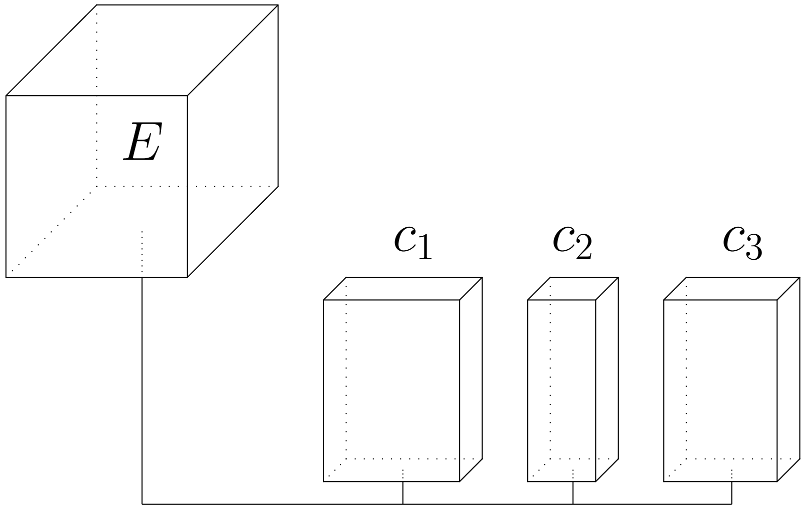
$$k_1(x_1) = \begin{cases} 0 & \textit{if } x_1 \le 0 \\ \dfrac{x_1^2}{4} & \textit{if } x_1 > 0 \end{cases}$$

$$k_2(x_2) = \begin{cases} 0 & \textit{if } x_2 \le 6 \\ \dfrac{(x_2 - 6)^2}{4} & \textit{if } x_2 > 6 \end{cases}$$

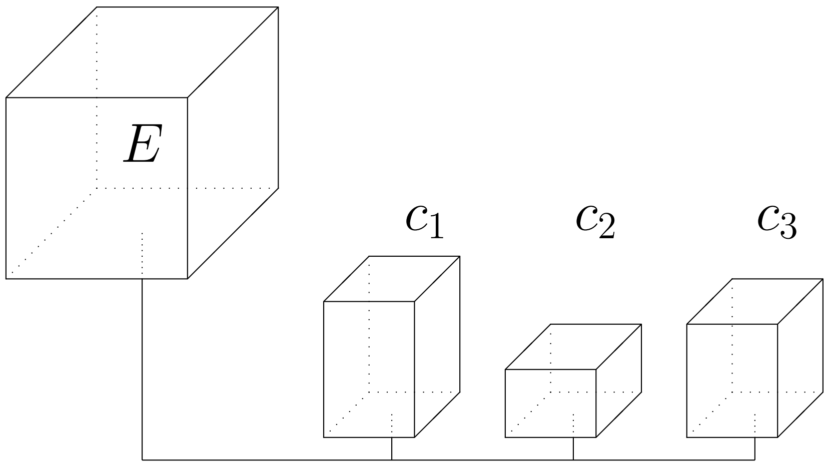
Bankruptcy Rules and Hydraulic Systems

According to Kaminski (2000) "hydraulic" rules can be represented as a system of connected vessels

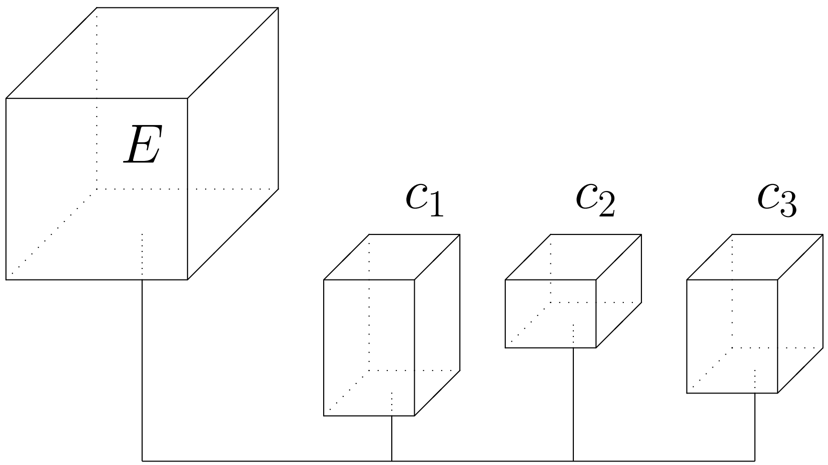
- $PROP$ rule



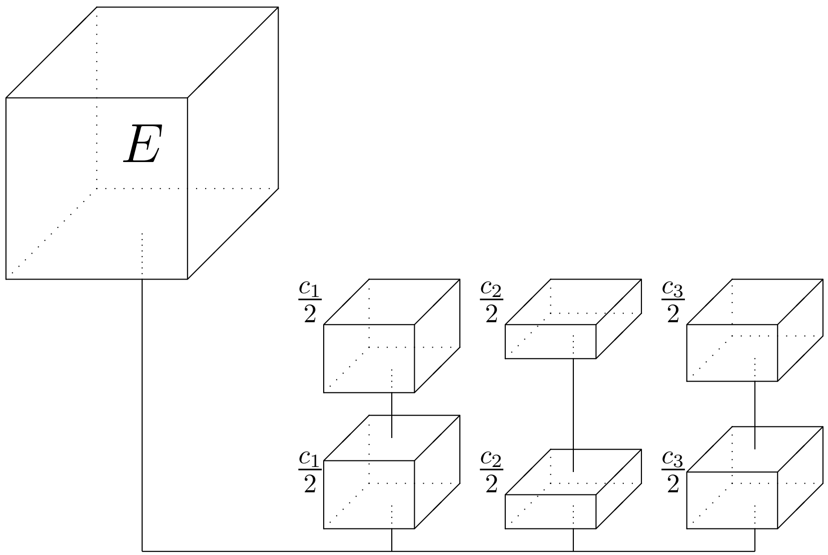
- CEA rule



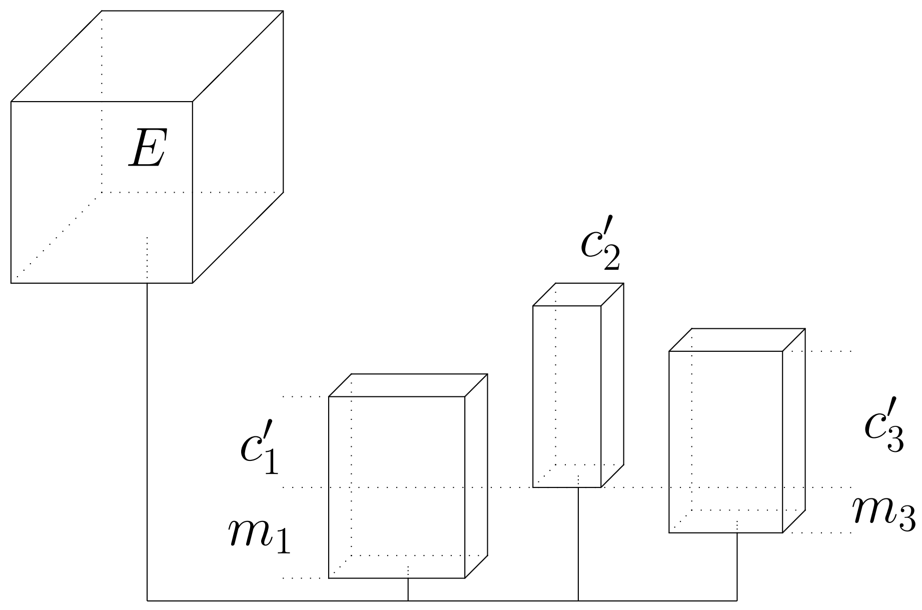
- CEL rule



- TAL rule



- $APROP$ rule



Bankruptcy Games

It is possible to define two TU-games, the pessimistic, (N, v_P), and the optimistic, (N, v_O):

$$v_P(S) = max\left(0, E - \sum_{i \in N \setminus S} c_i\right) \qquad S \subseteq N$$

$$v_O(S) = min\left(E, \sum_{i \in S} c_i\right) \qquad S \subseteq N$$

Example 3 (Inconsistency of the optimistic game) *Given the bankruptcy problem* $(5; 3, 4)$, *the two games are:*

$$v_O(1) = 3; v_O(2) = 4; v_O(12) = 5$$
$$v_P(1) = 1; v_P(2) = 2; v_P(12) = 5$$

The optimistic game assigns to the players as singletons 3 *and* 4, *even if the estate is* 5 ◊

The core of (N, v_P) coincides with the set of rational solutions of the bankruptcy problem:

$$x \in core(v_P) \iff \begin{cases} a) & \sum_{i \in N} x_i = E \\ b) & 0 \leq x_i \leq c_i, \qquad i \in N \end{cases}$$

Game theoretic bankruptcy rules

A bankruptcy rule ψ is a game theoretic rule if it is possible to construct a solution concept F for cooperative games such that

$$\psi(N, c, E) = F(N, v_P)$$

for all bankruptcy problems (N, c, E), where (N, v_P) is the pessimistic TU-game associated to (N, c, E)

Curiel, Maschler and Tijs (1987) proved that a bankruptcy rule ψ is a game theoretic rule if and only if *truncation property* holds, i.e.

$$\psi(N, c, E) = \psi(N, \bar{c}, E)$$

where $\bar{c}_i = min\{c_i, E\}, i \in N$