



imus



Bankruptcy Problems Weighted Solutions

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Casas-Méndez, García-Jurado, Fragnelli (2011) Weighted Bankruptcy Rules and the Museum Pass Problem. EJOR 215, 161-168

Motivation

Bankruptcy problem: several agents claim portions of an estate, the sum of the claims being larger than the estate

Seminal paper by O'Neill (1982)

Recent survey by Thomson (2003)

Weighted bankruptcy problems

Moulin (2000) provides a characterization of a family of rules including the proportional rule, the weighted constrained equal awards rule and the weighted constrained equal losses rule

Hokari and Thomson (2003) characterize the family of weighted Talmud rules

[Here](#)

Four new axiomatic characterizations: the weighted constrained equal awards rule, the weighted constrained equal losses rule, the weighted constrained proportional rule, the weighted constrained adjusted proportional rule

Relevancy of weighted bankruptcy approach applying it to the museum pass problem

Museum pass problem

Museum pass problem (Ginsburgh and Zang, 2003): several museums issue a joint pass and have to share the income

A museum pass problem is a triplet $\mu = (N, \pi, p)$ where N is the set of museums, π is the pass price and $p = (p_i)_{i \in N} \in \mathbb{R}^N$ are the prices of the regular tickets of the museums, with $\pi \leq \sum_{i \in N} p_i$

A realization of μ is a triplet (ν, u, v) where ν is the number of passes sold and $u, v \in \mathbb{R}^N$ are the vector of visitors who have used the pass and the vector of visitors who have not used the pass, respectively

We assume that $\pi > 0$, $\nu > 0$, and $p_i > 0$, $u_i > 0$, $v_i > 0$, for all $i \in N$

In Ginsburgh and Zang (2003) this problem is modelled as a coalitional game and the Shapley value is used for allocating the benefits; in Ginsburgh and Zang (2004), the Shapley value and other rules are numerically compared

Estévez-Fernández et al. (2004 and 2010) propose to build a bankruptcy problem out of every museum pass problem with a realization, and to choose one rule for bankruptcy problem to share the income; given the problem $\mu = (N, \pi, p)$ and the realization (ν, u, v) , they consider the bankruptcy problem (E, c) , where $E = \nu\pi$ and $c_i = u_i p_i, i \in N$

Here

Build a bankruptcy problem (E, c) , where $E = \pi$ and $c = p$; then choose one weighted rule for bankruptcy problem

Every particular realization gives rise to a particular vector of weights and each museum receives ν times its portion of π

u seems to be the most natural vector of weights but sometimes v or $u + v$ may be more reasonable

This approach always ends in a bankruptcy problem (i.e. $E \leq \sum_{i \in N} c_i$), whereas Estévez-Fernández et al.'s approach this is not always true (several buyers visit only few museums)

Weighted rules for bankruptcy problems

Let (E, c, a) be a weighted bankruptcy problem, where $a = (a_i)_{i \in N}$ is the vector of integer positive weights of the claimants

A weighted rule for bankruptcy problem is a map ψ^w that associates to every (E, c, a) a vector $\psi^w(E, c, a) \in \mathbb{R}^N$ s.t. $0 \leq \psi_i^w(E, c, a) \leq c_i, i \in N$ and $\sum_{i \in N} \psi_i^w(E, c, a) = E$

- Weighted constrained proportional rule

$$PROP_i^w(E, c, a) = \min \{ \lambda a_i c_i, c_i \}$$

where $\lambda \in \mathbb{R}$ is s.t. $\sum_{i \in N} \min \{ \lambda a_i c_i, c_i \} = E$

- Weighted constrained equal awards rule

$$CEA_i^w(E, c, a) = \min \{ \lambda a_i, c_i \}$$

where $\lambda \in \mathbb{R}$ is s.t. $\sum_{i \in N} \min \{ \lambda a_i, c_i \} = E$

- Weighted constrained equal losses rule

$$CEL_i^w(E, c, a) = \max \left\{ c_i - \frac{\lambda}{a_i}, 0 \right\}$$

where $\lambda \in \mathbb{R}$ is s.t. $\sum_{i \in N} \max \left\{ c_i - \frac{\lambda}{a_i}, 0 \right\} = E$

- Weighted constrained adjusted proportional rule

$$APROP_i^w(E, c, a) = m_i(E, c, a) + \min \{ \lambda a_i c'_i, c'_i \}$$

where $m_i(E, c, a) = \max \left\{ 0, E - \sum_{j \in N \setminus \{i\}} c_j \right\}$, $E' = E - \sum_{i \in N} m_i(E, c, a)$, $c'_i = \min \{ c_i - m_i(E, c, a), E' \}$, and $\lambda \in \mathbb{R}$ is s.t. $\sum_{i \in N} \min \{ \lambda a_i c'_i, c'_i \} = E'$

Axioms

- *Invariance under ticket prices truncation* This property (cf. Curiel et al., 1987) says that truncating each museum ticket price to the pass price does not influence the outcome
- *Composition of minimal rights* This lower bound requirement (cf. Curiel et al., 1987) implies that each museum receives at least its minimum right and also that the allocation does not change by first giving the minimal rights to all museums
- *Composition* According to this property (cf. Young, 1988) we can divide the pass price among the museums using two different procedures, maintaining the same number of visitors, which result in the same outcome. In the first one we divide the pass price directly using ψ^w . In the other procedure we first divide a part π' of the pass price and then divide remainder $\pi - \pi'$ on the basis of the remaining ticket prices, both times using ψ^w
- *Path independence* If a weighted rule ψ^w satisfies path independence (cf. Moulin, 1987) we can divide the pass price using two procedures, maintaining the same number of visitors, yielding the same result. The first procedure is to divide the pass price directly using ψ^w . In the second one we first divide a larger pass price $\pi' \geq \pi$ and then use the outcome $\psi^w(\pi', p, v)$ as prices to divide the real pass price π , both times using ψ^w
- *Equal treatment* Two museums with the same ticket price and the same number of visitors have to obtain the same share of the pass price. This property has a flavour similar to the basic symmetry requirement of equal treatment of equals (cf. O'Neill, 1982)

- *Restricted full reimbursement w.r.t. ticket prices and number of visitors* This property (cf. Herrero and Villar, 2001) refers to the behaviour of a weighted rule when museums' ticket prices are very unequal. It implies that when the ticket price of a museum is smaller than a proportional division of the price of the pass w.r.t. the ticket price of each museum, the weighted rule should give it the full ticket price
- *Full reimbursement w.r.t. ticket prices and number of visitors* This property has an interpretation similar to the property of *restricted full reimbursement w.r.t. ticket prices and number of visitors*. It implies that when the ticket price of a museum is smaller than a proportional division of the price of the pass w.r.t. the product of the ticket price of each museum times its number of visitors, the weighted rule should give it the full ticket price
- *Full reimbursement w.r.t. number of visitors* This property (cf. Herrero and Villar, 2001) refers to the behaviour of a weighted rule when museums' ticket prices are very unequal. It implies that when the ticket price of a museum is smaller than the proportional division of the price of the pass w.r.t. the number of visitors of each museum, the weighted rule should give it the full ticket price
- *Exemption* This property (cf. Herrero and Villar, 2001) also refers to the behaviour of a weighted rule when museums' ticket prices are very unequal. It implies that when the ticket price of a museum is smaller than the proportional division w.r.t. the number of visitors of each museum of the sum of the ticket prices minus the pass price, the weighted rule should give it nothing

- *Non manipulability of number of visitors when pass price increases* This property says that, when the pass price increases, a museum whose ticket price is larger than the proportional division w.r.t. the number of visitors of each museum of the sum of the ticket prices minus the pass price has no incentive to split up into several ones with the same ticket price and whose number of visitors sum up to the number of visitor of the initial museum
- *Non manipulability of ticket prices* This property (cf. Curiel et al., 1987) says that a museum has no incentive to split up in several ones with the same number of visitors and such that the ticket price of the initial museum is the sum of the new ticket prices
- *Non manipulability of number of visitors* This property (cf. Curiel et al., 1987) says that a museum, which is not fully reimbursable w.r.t. ticket prices and number of visitors, has no incentive to split up in several ones with the same ticket price and such that the number of visitors of the initial museum is the sum of the new numbers

- *Restricted non manipulability of ticket prices* This property (cf. Curiel et al., 1987) says that a museum has no incentive to split up in several ones with the same number of visitors and such that the ticket price of the initial museum is the sum of the new ticket prices, whether the ticket price of the initial museum is less than the pass price and its minimum right is equal to zero
- *Restricted non manipulability of number of visitors* This property (cf. Curiel et al., 1987) says that a museum, which is not fully reimbursable w.r.t. ticket prices and number of visitors, has no incentive to split up in several ones with the same ticket price and such that the number of visitors of the initial museum is the sum of the new numbers, whether the ticket price of the initial museum is less than the pass price and its minimum right is equal to zero
- *Modified non manipulability of number of visitors* This property has an interpretation similar to *restricted non manipulability of number of visitors*. The difference is that the new property refers to museums which are not fully reimbursable w.r.t. number of visitors, whereas the old property refers to museums which are not fully reimbursable w.r.t. ticket prices and number of visitors

Note that if a weighted rule for bankruptcy problems satisfies full reimbursement w.r.t. ticket prices and number of visitors, non manipulability of ticket prices, and non manipulability of number of visitors, then it also satisfies equal treatment

Theorem 1 *A weighted rule ψ^w for bankruptcy problems satisfies full reimbursement w.r.t. ticket prices and number of visitors, non manipulability of ticket prices and non manipulability of number of visitors if and only if $\psi^w = PROP^w$.*

A weighted rule ψ^w for bankruptcy problems satisfies invariance under ticket prices truncation, equal treatment, composition, full reimbursement w.r.t. number of visitors, and modified non manipulability of number of visitors if and only if $\psi^w = CEA^w$.

A weighted rule ψ^w for bankruptcy problems satisfies composition of minimal rights, equal treatment, path independence, exemption, and non manipulability of number of visitors when the pass price increases if and only if $\psi^w = CEL^w$.

A weighted rule ψ^w for bankruptcy problems satisfies invariance under ticket prices truncation, composition of minimal rights, restricted full reimbursement w.r.t. ticket prices and number of visitors, restricted non manipulability of ticket prices and restricted non manipulability of number of visitors if and only if $\psi^w = APROP^w$.

We refer to Casas-Méndez et al. (2011) for the proof of the theorem and for the logical independence of the axioms

	$PROP^w$	CEA^w	CEL^w	$APROP^w$
Invariance under ticket prices truncation	—	×	—	×
Composition of minimal rights	—	—	×	×
Composition	×	×	×	—
Path independence	×	×	×	—
Equal treatment	×	×	×	×
Restricted full reimbursement w.r.t. ticket prices and number of visitors	×	—	—	×
Full reimbursement w.r.t. number of visitors	—	×	—	—
Full reimbursement w.r.t. ticket prices and number of visitors	×	—	—	—
Exemption	—	—	×	—
Non manipulability of number of visitors when pass price increases	—	—	×	—
Non manipulability of ticket prices	×	—	—	—
Non manipulability of number of visitors	×	—	—	—
Restricted non manipulability of ticket prices	×	—	—	×
Restricted non manipulability of number of visitors	×	—	—	×
Modified non manipulability of number of visitors	—	×	—	—

Real-world example: The case of the Municipality of Genova

The “Card Musei” can be used to visit any number of times within a 48 hours period, sixteen museums in Genova, for the price of sixteen euros

The data provided by *Direzione Settore Musei del Comune di Genova* refer to 2007

The weighted adjusted proportional rule coincides with the weighted proportional rule

Our approach requires simple data sets, while the proposal by Ginsburgh and Zang needs a complete history of the use of each pass

Museum	Ticket (€)	Visitors	PROP ^w	CEA ^w	CEL ^w
Musei dei Palazzi Rosso, Bianco, Tursi	8	122,747	6.09	5.04	6.81
Museo d'Arte Orientale "E. Chiossone"	4	8,175	0.20	0.34	0
Museo di S. Agostino	4	17,912	0.44	0.74	0
Museo del Risorgimento	4	13,289	0.33	0.55	0
Museo di Archeologia Ligure	4	13,594	0.34	0.56	0
Museo Navale di Pegli	4	4,713	0.12	0.19	0
Museo di Storia e Cultura Contadina	2.8	2,030	0.04	0.08	0
Raccolte Frugone	4	10,243	0.25	0.42	0
Museo "G. Luxoro"	4	1,705	0.04	0.07	0
Galleria d'Arte Moderna	6	9,690	0.36	0.40	0
Museo di Storia Naturale "G. Doria"	4	31,671	0.79	1.30	0
Museo di Arte Contemporanea	4	49,494	1.23	2.03	1.06
Museo del Castello D'Albertis	6	18,531	0.69	0.76	0
Collezione Wolfson	5	3,372	0.10	0.14	0
Museo del Tesoro di S. Lorenzo	5.5	4,426	0.15	0.18	0
Galata Museo del Mare	10	77,927	4.83	3.20	8.13

Weights: Total number of visitors

Museum	Ticket (€)	Visitors	PROP ^w	CEA ^w	CEL ^w
Musei dei Palazzi Rosso, Bianco, Tursi	8	1,701	6.29	5.29	6.66
Museo d'Arte Orientale "E. Chiossone"	4	161	0.30	0.50	0
Museo di S. Agostino	4	279	0.52	0.87	0
Museo del Risorgimento	4	902	1.67	2.81	1.46
Museo di Archeologia Ligure	4	32	0.06	0.10	0
Museo Navale di Pegli	4	33	0.06	0.10	0
Museo di Storia e Cultura Contadina	2.8	30	0.04	0.09	0
Raccolte Frugone	4	179	0.33	0.56	0
Museo "G. Luxoro"	4	42	0.08	0.13	0
Galleria d'Arte Moderna	6	115	0.32	0.36	0
Museo di Storia Naturale "G. Doria"	4	80	0.15	0.25	0
Museo di Arte Contemporanea	4	92	0.17	0.29	0
Museo del Castello D'Albertis	6	388	1.08	1.21	0.10
Collezione Wolfson	5	20	0.05	0.06	0
Museo del Tesoro di S. Lorenzo	5.5	58	0.15	0.18	0
Galata Museo del Mare	10	1,028	4.75	3.20	7.77

Weights: Visitors with "Card Musei"

The most important museums receive larger quotas of the pass price, as they have larger numbers of visitors and larger prices

CEL^w emphasizes the differences between the museums

All the museums receive a quota when CEA^w and $PROP^w$ are adopted, but CEA^w is less influenced by the price of the tickets than $PROP^w$

When the income due for the cards is a small part of the museums' budgets, CEL^w referred to the total number of visitors can be seen as a kind of bonus for the most important museums

When the income due for the cards represents the main financial support for the museums, CEA^w referred to the visitors with pass is a good solution for its flattening effect on the allocation

Comments and concluding remarks

The purpose of our study is not to select the “best” rule, but to help the managers in understanding their differences and the type of situations in which each of these rules seems to be more appropriate

PROP^w allocates the pass price proportionally to the ticket prices of the museums and to the number of visitors, remaining in the mid-way

This rule prevents the formation of groups among museums with the same ticket prices or with the same number of visitor

Due to the property of full reimbursement w.r.t. ticket prices and number of visitors, this rule also protects small ticket prices but the effect is corrected with respect to the weighted constrained equal awards rule because it also takes into account the number of visitors when defining fully reimbursable museums

CEA^w , like in the non weighted case, follows a protective criterion in which ticket prices are interpreted as maximum aspirations and the museums with small prices are benefit

The excess of a ticket price above the card price is irrelevant and according to the property of full reimbursement w.r.t. number of visitors, those museums with smaller prices are given priority in the distribution and can actually be fully reimbursable

Equal treatment seems difficult to object when we consider problems in which museums have no relevant differences

Composition ensures coherence with respect to subdivisions of the pass price

The modified non manipulability of number of visitors prevents that two museums with the same ticket price present themselves as a “single museum” accumulating their numbers of visitors

CEL^w benefits the museums with high ticket prices

Exemption conveys the opposite message that full reimbursement w.r.t. number of visitors: those museums with very low ticket prices are to be disregarded

Equal treatment ensures coherence as for CEA^w

Path independence and non manipulability of number of visitors when the pass price increases guarantee against subdivisions of the pass price and alterations due to the formation of groups or divisions of museums with the same ticket prices

$APROP^w$ gives to each museum its minimal right and then it allocates the remaining pass price proportionally to the corrected ticket prices; it also satisfies invariance under ticket price truncation
It coincides with $PROP^w$ when all ticket prices are smaller than the pass price and minimal rights of all museums are null

The weighted model allows to consider additional information with respect to the elements involved in the problem, other than the ticket prices

For the Museum Pass Problem, it seems difficult to see how CEL^w could be accepted by most of the (small) museums

$PROP^w$ adjusts the allocations taking into account the number of visitors and the differences in ticket prices, so most probably museum managers would favour this rule over CEA^w

References

- Curiel, I., Maschler, M., Tijs, S., 1987. Bankruptcy games. *Zeitschrift für Operations Research (Now: Mathematical Methods of Operations Research)* 31, 143-159.
- Dagan, N., 1996. New characterizations of old bankruptcy rules. *Social Choice and Welfare* 13, 51-59.
- Estévez-Fernández, A., Borm, P., Hamers, H., 2004. "The museum pass game and its value" revisited. CentER Report 2004-07. Tilburg University. The Netherlands.
- Estévez-Fernández, A., Borm, P., Hamers, H., 2010. A Note on Passepartout Problems. Discussion Paper TI 2010-031/1. Tinbergen Institute. The Netherlands.
- Ginsburgh, V., Zang, I., 2003. The museum pass game and its value. *Games and Economic Behavior* 43, 322-325.
- Ginsburgh, V., Zang, I., 2004. Sharing the income of a museum pass program. *Museum Management and Curatorship* 19, 371-383.
- Herrero, C., Villar, A., 2001. The three musketeers: Four classical solutions to bankruptcy problems. *Mathematical Social Sciences* 42, 307-328.
- Hokari, T., Thomson, W., 2003. Claims problems and weighted generalizations of the Talmud rule. *Economic Theory* 21, 241-261.
- Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory* 16, 161-186.
- Moulin, H., 2000. Priority rules and other asymmetric rationing methods. *Econometrica* 68, 643-684.
- O'Neill, B., 1982. A problem of rights arbitration from the Talmud. *Mathematical Social Sciences* 2, 345-371.
- Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences* 45, 249-297.
- Young, H.P., 1988. Distributive justice in taxation. *Journal of Economic Theory* 44, 321-335.