The Adjusted Winner procedure by S.Brams and A.Taylor

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The framework

The Players $N = \{I, II\}$

The goods $M = \{1, 2, \dots, m\}$

The utility a_{ij} evaluation of good j by player i

Main assumption. Utilities are:

normalized $\sum_{j \in M} a_{ij} = 1$ for every $i = 1, 2$,

linear if player i gets share $t_j \in [0, 1]$ of item j and share $t_k \in [0, 1]$ of item k, she gets a total utility of $t_j a_{ij} + t_k a_{ik}$.

Preferences are described by a **matrix**

	item 1	item 2	$\cdots$	item m
pl.1	a_{11}	a_{12}	$\cdots$	a_{1m}
pl.2	a_{21}	a_{22}	$\cdots$	a_{2m}

How to obtain a Pareta optimal allocation

Definition

The pl.1 to pl.2 **valuation ratio** for item j is defined as $r_j = \frac{a_{1j}}{a_{2j}}$
with the assumpion that
If $a_{1j} > 0$ and $a_{2j} = 0$ then $r_j = +\infty$
(If $a_{1j} = a_{2j} = 0$ item j is of no interest in the division)

Example

	item 1	item 2	*item*3	item 4
pl.1	0.2	0.6	0.1	0.1
pl.2	0.1	0.2	0.05	0.65
r	2	3	2	0.1538

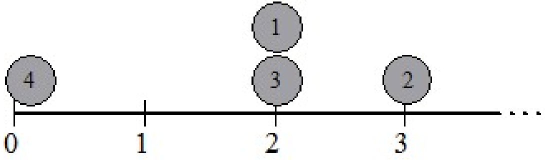
PO allocations

1st idea: allocation ratios

Plot all the allocation ratios on the positive half line of real numbers

Example 1

	item 1	item 2	*item*3	item 4
pl.1	0.2	0.6	0.1	0.1
pl.2	0.1	0.2	0.05	0.65
r	2	3	2	0.1538



PO allocations

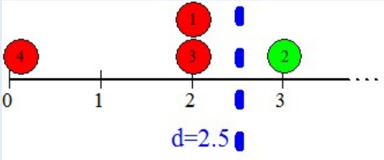
2nd idea: Threshold division

Create a division of the items by drawing a vertical mark:

- items on the right of the mark are given to Player 1
- items on the left of the mark are given to Player 2
- items on the mark can be assigned to any of the players, or **can be split between them**

Example: Threshold division when $d = 2$

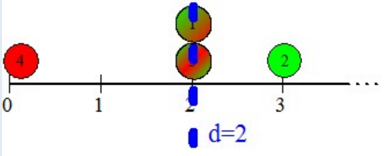
- pl.1 gets item 2
- pl.2 gets items 1,3 & 4



A Theorem

Example: Threshold division when $d = 2$

- pl.1 gets item 2
- pl.2 gets items 4
- items 1 and 3 can be assigned to both players



The following allocations of items 1 and 3 are all compatible :

- pl.1 gets 1, pl.2 gets 3
- pl.1 3/4 of 1, 1/5 of 3, pl.2 gets 1/4 of 1 and 4/5 of 3
- pl.1 gets 1 and 1/10 of 3, pl2 gets 9/10 of 3
- ...

A Theorem

The threshold divisions are **precisely** the PO divisions

PO and fairness

- Pareto optimality alone is not enough to guarantee fairness. In the previous example:

  If $d = 0$ All items go to pl.1

  If $d = 1000$ All items go to pl.2

  which are efficient but totally **unfair** allocations.
- There is one way of placing the vertical line that leads to an **equitable division**

$$\mu_1(\text{items to pl.1}) = \mu_2(\text{items to pl.2})$$

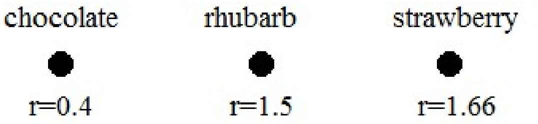
- The **Adjusted Winner (AW)** method finds this equitable division
- It was proposed in 1994 by Steven Brams and Alan Taylor and it was patented in 1999

Example 2

Players 1 and 2 want to divide a cake that consists of a strawberry component, a rhubarb component and a chocolate component, with valuations

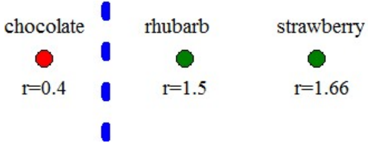
	chocolate	rhubarb	strawberry
pl.1	0.2	0.3	0.5
pl.2	0.5	0.2	0.3
r	0.4	1.5	1.66

We consider a simpler plot for the valuation ratios



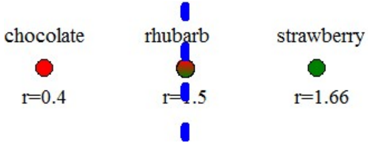
Example 2 (continued)

B & T suggest to begin with a vertical line at $d = 1$



$\mu_1 = 0.8$
$\mu_2 = 0.5$

The partition is not equitable: The smallest meaningful move to increase μ_2 (and decrease μ_1) is to move the vertical line rightward



rhubarb can be assigned to any of the two players

If rhubarb to pl.1 $\Rightarrow \mu_1 = 0.8 \quad \mu_2 = 0.5 \Rightarrow$ **pl.1 "wins"**
If rhubarb to pl.2 $\Rightarrow \mu_1 = 0.5 \quad \mu_2 = 0.7 \Rightarrow$ **pl.2 "wins"**
$\Rightarrow$ An equitable allocation is obtained by properly **splitting rhubarb**

Example 2 (continued)

How should we split rhubarb?
$p \in (0,1)$ = share of rhubarb assigned to pl.1
The equitable allocation must satsify

$$\mu_1 = 0.5 + 0.3p = 0.5 + 0.2(1-p) = \mu_2 \quad \Rightarrow p = 0.4$$

Therefore the equitable allocation is:

pl.1 gets strawberry and a 0.4 share of rhubarb

pl.2 gets chocolate and a 0.6 share of rhubarb

and $\mu_1 = \mu_2 = 0.62$

Theorem

The Adjusted Winner procedure returns an allocation which is **Pareto Optimal** and **Equitable**.

Example 1 (again)

	item 1	item 2	*item*3	item 4
pl.1	0.2	0.6	0.1	0.1
pl.2	0.1	0.2	0.05	0.65
r	2	3	2	0.1538

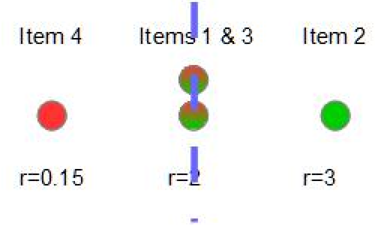
The vertical line is set at $d = 1$

$\mu_1 = 0.9$
$\mu_2 = 0.65$

The partition is not equitable:

Example 1 (continued)

The smallest meaningful move is to move the vertical line rightward



items 1 & 3 can be assigned to any of the two players

both to pl.1 $\Rightarrow \mu_1 = 0.9 \quad \mu_2 = 0.65 \Rightarrow$ **pl.1 "wins"**
both to pl.2 $\Rightarrow \mu_1 = 0.6 \quad \mu_2 = 0.8 \Rightarrow$ **pl.2 "wins"**
$\Rightarrow$ An equitable allocation is obtained by properly **splitting items 1 & 3**
$p \in (0,1) =$ share of items 1 & 3 assigned to pl.1
The equitable allocation must satsify

$$0.6 + 0.3p = 0.65 + 0.15(1-p) = \mu_2 \quad \Rightarrow p = 4/9$$

Example 1 (continued)

A solution would be to split both items 1 & 3

$$\mu_1(\text{item } 2 + 4/9 \text{ of item } 1 \ + 4/9 \text{ of item } 3\ ) = 0.7333$$
$$\mu_2(\text{item } 4 + 5/9 \text{ of item } 1 \ + 5/9 \text{ of item } 3\ ) = 0.7333$$

Actually we obtain the same result by splitting item 1 only

$$\mu_1(\text{items } 2 \ \& \ 3 + 1/6 \text{ of item } 1\ ) = 0.7333$$
$$\mu_2(\text{item } 4 + 5/6 \text{ of item } 1\ ) = 0.7333$$

Example 1 (continued)

A solution would be to split both items 1 & 3

$$\mu_1(\text{item } 2 + 4/9 \text{ of item } 1 \; + 4/9 \text{ of item } 3\;) = 0.7333$$
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Result

At most one item need to be split in the AW procedure

What about envy-freeness?

A Pareto optimal and equitable partition is always proportional.
Since the solution of AW is Pareto optimal and equitable $\Rightarrow$ the solution is also proportional
Since there are only two players $\Rightarrow$ the solution is also envy-free

Theorem

The AW solution is **Pareto optimal, Equitable** and **Envy-free**

What happens when $n \geq 3$

Proposition (D. and Hill, 2003)

for each $n \geq 3$, there exist mutually absolutely continuous atomless measures $\mu_1, \mu_2, \ldots, \mu_n$ such that no maximin-optimal partition [which is Pareto optimal and equitable] is envy-free.

Consider the following situation for $n = 3$

	item 1	item 2	item 3
pl.1	0.4	0.5	0.1
pl.2	0.3	0.4	0.3
pl.3	0.3	0.3	0.4

The allocation where pl.i gets item i ($i = 1, 2, 3$) is Pareto optimal and equitable, but not envy-free