Integer Solutions to Bankruptcy Problems with Non-integer Claims

Vito Fragnelli
Università del Piemonte Orientale
vito.fragnelli@mfn.unipmn.it

Joint work with:
Stefano Gagliardo - stefano.gagliardo@ge.imati.cnr.it
Fabio Gastaldi - fabio.gastaldi@mfn.unipmn.it

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1 Introduction

Standard bankruptcy problem (BP)
A scarce resource (*estate*) has to be shared among a set of agents (*claimants*) that claim different amounts of it, with the same rights and with the condition that the estate is not sufficient for fully satisfying all the requests

Solution of a BP
Divide the whole estate (*efficiency*), assigning to each claimant a non-negative amount not greater than her/his claim (*rationality*)

Very simple model of real-world situations

Most important additional elements in the existing literature:

- different priorities of the claimants (Young, 1994, Bebchuck and Fried, 1996, Schwarcz, 1997 and Kaminski, 2000)
- minimal rights of the claimants (Curiel et al., 1987 and Pulido et al., 2002)
- multiple issues (Calleja et al., 2005 and Moreno-Ternero, 2009)
- negative claims and estate (Herrero et al., 1999, Branzei et al., 2008 and 2011)
- non-transferable utility situations (Orshan et al., 2003 and Carpente et al., 2012)

Two important surveys are due to Thomson (2003 and 2015)

Here

Integer BP *(*IBP*)*

The estate is not continuously divisible and is represented by a positive integer, but the claimants may have non-integer claims

Examples:

- the most popular is the electoral situation where the estate are the seats, but the claims are the percentages of votes
- items whose conversion to money is largely disadvantageous, e.g. works of art, luxury furniture, real estate and so on
- allocation of quotas of hedge funds, minimal stock or bond shares
- assignment of radio-frequencies
- deployment of emergency intervention units (ambulances, fire vehicles and so on)

Literature on BP with integer estate and integer claims:

- the notion of type (Young, 1994)
- the issue of symmetry (Moulin, 2000)
- the quantitatively and qualitatively relevant contributions by Herrero and Martinez (2004), (2008a), (2008b), (2011)
- the application to radio-frequencies assignment (Gozalves et al., 2007 and Lucas-Estañ et al., 2012)
- the systematic favourability (Chen, 2014)

Herrero and Martinez (2004) suppose the existence of an *a priori* ordering for assigning each unity of the estate to the claimants, guaranteeing the uniqueness of the solution in the case of integer claims; moreover, a suitable ordering for assigning the unities enables the agents to reach any possible solution

The situation of non-integer claims with integer estate is essentially new

2 Preliminaries and recalls

Formally, a BP is a triplet (N, c, E), where $N = \{1, ..., n\}$ is the set of claimants; $c = (c_1, ..., c_n)$ is a positive n-dimensional real vector, with c_j representing the monetary amount corresponding to the *claim* of agent $j \in N$; E is a positive real number representing the monetary amount corresponding to the estate, with the condition that $E \leq \sum_{j \in N} c_j = C$

Given a BP, a *solution* is an n-dimensional real vector $x = (x_1, ..., x_n)$, where x_j represents the monetary amount assigned to claimant $j \in N$, satisfying the following conditions:

$$0 \leq x_j \leq c_j, \qquad j \in N \qquad \text{(rationality)}$$

and

$$\sum_{j \in N} x_j = E \qquad \text{(efficiency)}$$

Two classical solutions for a BP (N, c, E) are the Constrained Equal Losses (CEL):

$$CEL_j(N, c, E) = \max\{c_j - \beta, 0\}, \qquad j \in N$$

where β is a positive real number satisfying $\sum_{k \in N} CEL_k(N, c, E) = E$ and the Constrained Equal Awards (CEA):

$$CEA_j(N, c, E) = \min\{c_j, \alpha\}, \qquad j \in N$$

where α is a positive real number satisfying $\sum_{k \in N} CEA_k(N, c, E) = E$

In an IBP each agent can only receive integer unities of E, so a solution x must satisfy a further condition

$$x_j \in \mathbb{N}, \qquad j \in N \qquad \text{(integrity)}$$

Notation

$$[[C]] = \sum_{j \in N} [c_j] \qquad \text{(total integer claim)}$$

where $[x]$ represents the *integer part* of x

Note that $[[C]] \leq [C]$; moreover, under our assumptions $[[C]]$ may vanish when $c_j < 1, j \in N$

Construction of an integer solution:

- computing an overall non-integer solution and then rounding it (Herrero and Martinez, 2004)
- assigning the unities of the estate to the claimants one by one (Herrero and Martinez, 2008a)

Whatever the method is, in our setting of non-integer claims, integer solutions might not exist: in the bankruptcy situation $(\{1, 2, 3\}, (0.6, 0.4, 0.4), 1)$, the three vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are not rational, while the vector $(0, 0, 0)$ is not efficient; in addition, $(0, 1, 0)$ and $(0, 0, 1)$ do not satisfy the *minimal right principle* and the *equal treatment of equals* (Thomson, 2003)

Necessary and sufficient condition for the existence of integer solutions to IBP

Theorem 1. *An integer solution to IBP exists if and only if $E \leq [[C]]$; if $E = [[C]]$, then the integer solution is unique (and coincides with $[c]$); if $E < [[C]]$, then the integer solution is unique if and only if all claims but one are strictly smaller than 1.*

3 What can be done when $E > [[C]]$

Since we cannot expect that an integer solution exists, satisfying both the rationality and the efficiency conditions, we may relax our requirements on integer solutions, from either point of view; however, we still require that the relaxed solutions have non-negative integer components

This will lead to the concept of inefficiency (resp. irrationality) residual and the existence of an integer vector that satisfies the rationality (resp. efficiency) condition and minimizes the residual is proved

Uniqueness of this minimizer in the inefficiency case holds without any further assumption, while in the irrationality case it is obtained when a ranking among the agents is supposed to exist

Keeping the rationality condition and relaxing efficiency, consider the finite set of rational integer vectors

$$U = \{u = (u_1, \ldots, u_n) : \; 0 \leq u_j \leq [c_j] \text{ and } u_j \in \mathbb{N}, \; \forall j \in N\}$$

and for every $u \in U$ introduce the *inefficiency residual* $\delta_e(u) = E - \sum_{j \in N} u_j$: note that $\delta_e(u) > 0$ for every $u \in U$. Clearly, the vector $u^* = ([c_1], \ldots, [c_n])$ belongs to U and $\delta_e(u^*) < \delta_e(u)$ for every $u \in U \backslash \{u^*\}$. Thus, u^* is *the only minimizer of the inefficiency residual* in the set U and we call it *the less inefficient solution*

Relaxing the rationality assumption and maintaining efficiency, consider the finite set of efficient integer non-negative vectors

$$W = \left\{ w = (w_1, \ldots, w_n) : \; w_j \in \mathbb{N} \; \forall j \in N \text{ and } \sum_{j \in N} w_j = E \right\}$$

and for every $w \in W$ introduce the *irrationality residual*

$$\delta_r(w) = \sum_{j \in N} (w_j - c_j)^+ \tag{1}$$

A mere minimization of the irrationality residual may not give a satisfactory result, as the following example shows: consider the bankruptcy problem $(\{1,2,3\}, (0.5, 0.5, 1.4), 2)$, where

$$\begin{aligned} &W = \{(1,1,0),\; (1,0,1),\; (0,1,1),\; (2,0,0),\; (0,2,0),\; (0,0,2)\} \\ &\delta_r(1,1,0) = 1, \quad \delta_r(1,0,1) = \delta_r(0,1,1) = 0.5 \\ &\delta_r(0,0,2) = 0.6, \quad \delta_r(2,0,0) = \delta_r(0,2,0) = 1.5 \end{aligned}$$

In particular, the minimum value of δ_r is 0.5 and is reached by two elements of W, $(1,0,1)$ and $(0,1,1)$; moreover, the solutions do not even satisfy the *equal treatment of equals*; therefore, some special criterion of choice has to be enforced whenever several agents have equal claims but the estate is not enough to treat them equally, in an efficient way and minimizing the irrationality residual

Definition 2. *Let $u, v \in W$ be such that $u \neq v$ and $\delta_r(u) = \delta_r(v)$. We say that u is* preferred to v *if, for every pair of agents $j,\ k \in N$*

$$\text{if} \qquad \begin{cases} u_k > v_k \\ u_j < v_j \end{cases} \qquad \text{then} \quad k < j$$

Assumption 3. N is endowed with a strict total order $\succ$.

The order $\succ$ enables us to define a further strict total order $\underset{dec}{\succ}$ in N:

$$j \underset{dec}{\succ} i \quad \text{if and only if} \quad \begin{cases} \gamma_j > \gamma_i & \text{if } \gamma_j \neq \gamma_i \\ j \succ i & \text{if } \gamma_j = \gamma_i \end{cases} \qquad j,\ i \in N$$

where $\gamma_i = c_i - [c_i],\ i \in N$ is the *decimal part* of c_i

Theorem 4. *Suppose that the estate E satisfies $E > [[C]]$ and that Assumption 3 hold and $j < i \iff j \underset{dec}{\succ} i$. Let $D = E - [[C]]$ and let w^* be such that, for every $j \in N$*

$$w_j^* = \begin{cases} [c_j] + 1 & \text{if } j \leq D \\ [c_j] & \text{if } D < j \leq n \end{cases} \tag{2}$$

Then, w^* minimizes δ_r in W and

1. *if $\gamma_D > \gamma_{D+1}$, then w^* is the only minimizer of δ_r in W;*
2. *if $\gamma_D = \gamma_{D+1}$, then w^* is preferred to any other minimizer $v \in W$, in the sense of Definition 2.*

The vector w^* defined in (2) is called *the less irrational integer solution*: it minimizes the irrationality residual under the constraint given by the efficiency and, in case more than one minimizer exists, it is preferred to any other in the sense of Definition 2

Other irrationality residuals are possible

4 Solving IBP via the box method

When the necessary condition is satisfied but the integer solution is not unique, we introduce the *minimal loss* procedure $ML(x)$, depending on a non-integer solution x and consisting in a truncation of it and subsequent distribution of the remainders, in such a way that the loss w.r.t. the claims is minimized
Possible ambiguities still persisting in the choice of the integer solution may be solved by introducing a priority ordering among the agents, according to Assumption 3

A constructive method, named *box method*, yields an integer solution, named *box solution* b_x, fulfilling an optimality criterion which may be expressed in terms of a lexicographical comparison among the losses

5 The minimal loss procedure

Let x be *any* (rational and efficient) solution to BP; the following *minimal loss* $ML(x)$ procedure provides an integer solution:

$ML_1(x)$. Let $[x] = ([x_1], [x_2], \ldots, [x_n])$, $[[x]] = \sum_{j\in N}[x_j]$ and $\Delta(x) = E - [[x]]$; note that $\Delta(x) = \sum_{j\in N}(x_j - [x_j]) < n$ and $\Delta(x) \in \mathbb{N}$

$ML_2(x)$. Find n non-negative integers $d_1, d_2, \ldots, d_n$ such that

$$\begin{cases} (a) & d_j \leq [c_j] - [x_j], \text{ for every } j \in N, \\ (b) & \displaystyle\sum_{j\in N} d_j = \Delta(x), \\ (c) & d = (d_1, \ldots, d_n) \text{ minimizes the quantity } \; m(d) = \max\{c_j - [x_j] - d_j,\; j \in N\} \\ & \text{under the constraints } (a) \text{ and } (b). \end{cases}$$

$ML_3(x)$. If step $ML_2(x)$ can be performed, let $u = [x] + d$: we claim that u is an integer solution to IBP

Step $ML_2(x)$ is meaningful if and only if there exists a solution to the problem

$$\text{find} \quad \delta \in \Gamma(x) \quad \text{such that} \quad m(\delta) \leq m(d), \quad \text{for every} \quad d \in \Gamma(x) \tag{3}$$

where $\Gamma(x) = \{d = (d_1, \ldots, d_n) \in \mathbb{N}^n : \; (a) \text{ and } (b) \text{ hold}\}$

If $\Gamma(x) \neq \emptyset$ then *for every* $d \in \Gamma(x)$ the vector $u = [x] + d$ introduced in step $ML_3(x)$ is an integer solution to problem IBP; the $ML(x)$ procedure provides some integer solutions by solving problem (3): these are the elements of the set

$$S(x) = \{[x] + \delta : \ \delta \text{ solves } (3)\}$$

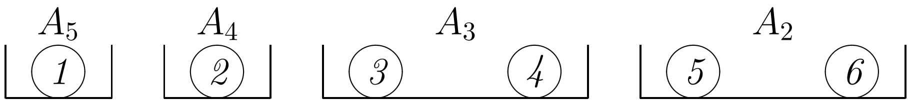
Note that $S(x) \neq \emptyset$ if and only if $\Gamma(x) \neq \emptyset$

A solutions to problem (3) exists if and only if $\Gamma(x) \neq \emptyset$
Theorem 1 gives the following necessary condition for the existence of solutions to problem (3): if $\Gamma(x) \neq \emptyset$, then $E \leq [[C]]$
By Theorem 1, if $E = [[C]]$ then the solution to IBP is unique, therefore $\Gamma(x) = \{[c] - [x]\}$ and problem (3) has just this solution

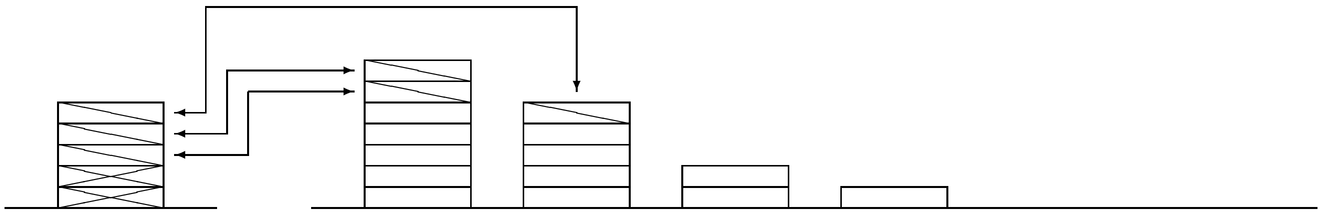
The remaining case to be discussed is $E < [[C]]$. It is trivial to observe that if $\Delta(x) = 0$, then $\Gamma(x) = \{0\}$ and existence and uniqueness are proved
Now, we are going to discuss the remaining case, i.e. $\Delta(x) > 0$, presenting a method, named *the box method*, which yields a particular solution to (3) in this case

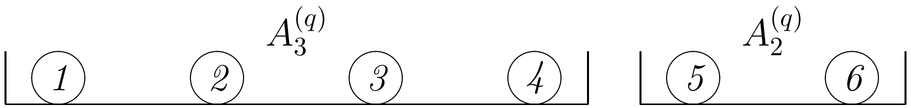
6 The box method

Example 5. *Let* $N = \{1, ..., 6\}$ *with the priority order* $3 \succ 1 \succ 2 \succ 6 \succ 5 \succ 4$, $c = (10.3, 8.5, 5.4, 4.5, 2.9, 2.8)$, $E = 17$; *Let* $x = (5.9, 4.7, 2.8, 1.9, 0.9, 0.8)$, *then* $[x] = (5, 4, 2, 1, 0, 0), \Delta(x) = 5, r = (5.3, 4.5, 3.4, 3.5, 2.9, 2.8)$

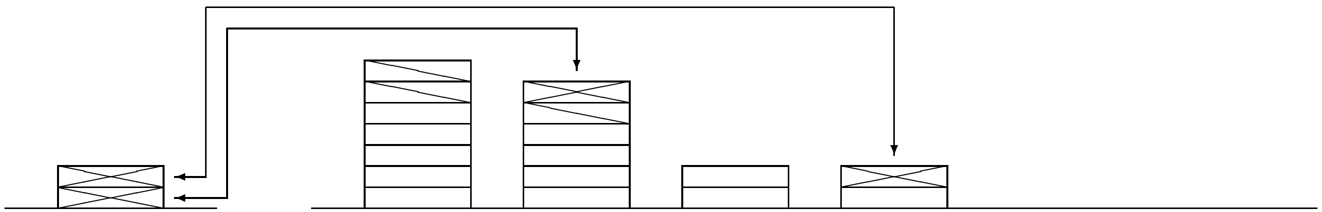


First step:
$\psi(q) = 3$ *unities are given to agents 1 and 2, so they are moved to box* $A_3^{(q)}$





Second step:
q−modified remainder is $r^{(q)} = (3.3, 3.5, 3.4, 3.5, 2.9, 2.8)$
the last $\widetilde{K} = 2$ *unities are given to agents 2 and 4 which have the largest q-modified remainders*



It follows that $u^* = [x] + d^* = (7, 6, 2, 2, 0, 0)$ *is a solution to* (IBP)
In particular, $c - u^* = r - d^* = r^* = (3.3, 2.5, 3.4, 2.5, 2.9, 2.8)$, *hence* $m(d^*) = 3.4$ *and* d^* *is the only solution*

Remark 6. *A portion of* $\Delta(x)$ *is assigned to the* $card\ A_h + card\ A_{h-1} + \ldots + card\ A_{h-q+2}$ *agents having the integer part of the remainder larger than or equal to* $h-q+2$*, moving them to* $A^{(q)}_{h-q+1}$*. This step uses*

$$(q-1)card\ A_h + (q-2)card\ A_{h-1} + \ldots + 2card\ A_{h-q+3} + card\ A_{h-q+2} = \psi(q) = \Delta(x) - \widetilde{K}$$

unities of $\Delta(x)$*, therefore* $\widetilde{K}$ *unities remain to be assigned; only some agents in* $A^{(q)}_{h-q+1}$ *may be awarded one unity of* $\widetilde{K}$*. As a particular case, if* $\Delta(x) \leq card\ A_h$ *then* $q=1$*,* $\psi(q)=0$*,* $\widetilde{K}=\Delta(x)$*,* $r^{(q)}_j = r_j$ *for every* $j \in N$ *and each agent having the integer part of the remainder equal to* h *gets at most one unity of* $\Delta(x)$*, while the agents having the integer part of the remainder smaller than* h *get nothing.*

The method used to construct the vector d^* is called *box method*: a motivation for the terminology is that each A_s, $s=1,\ldots,h$, may be regarded as a box containing some agents; the first step of the box method, namely the allotment of $\Delta(x) - \widetilde{K}$, causes the agents belonging to the boxes A_s, with $s > h-q+1$, to move to box $A^{(q)}_{h-q+1}$

7 The main results

Proposition 7. *Suppose $E \leq [[C]]$ and let x be any solution to BP. Then $\Gamma(x) \neq \emptyset$, hence (3) is solvable. In particular,*

1. *if $E = [[C]]$, then $\Gamma(x) = \{[c] - [x]\}$ and $S(x) = \{[c]\}$;*
2. *if $E < [[C]]$ and $\Delta(x) = 0$, then $\Gamma(x) = \{0\}$ and $S(x) = \{[x]\}$;*
3. *if $E < [[C]]$ and $\Delta(x) > 0$, then the vector d^* belongs to $\Gamma(x)$ and solves problem (3); moreover, $[x] + d^*$ belongs to $S(x)$ and solves IBP*

Definition 8. *Let x be any solution to BP and assume $E < [[C]]$ and $\Delta(x) > 0$; the vector $b_x = [x] + d^*$ is called the box solution to IBP related to x*

8 The $ICEL$ solution

When $x = CEL(N, c, E)$, both the minimal loss procedure and the box method convey the CEL character of x to the box solution: so, the box solution b_x is called the *integer* CEL *solution* ($ICEL$) for IBP

Definition 9. *Suppose* $E \leq [[C]]$ *and let* $x = CEL(N, c, E)$. $ICEL(N, c, E)$, *or* $ICEL$, *when no confusion arises, is:*

- $[c]$ *when* $E = [[C]]$;
- $[x]$ *when* $E < [[C]]$ *and* x *is integer;*
- *the box solution* b_x *when* $E < [[C]]$ *and* x *is not an integer.*

We also recall the following proposition and corollary

Proposition 10. *Suppose* $0 \leq E \leq [[C]]$. *Then* $ICEL(N, c, E) = ICEL(N, [c], E)$, *provided* N *is endowed with the ranking* $\underset{dec}{\succ}$.

Corollary 11. *Assume* $0 \leq E \leq [[C]]$. *Then* $ICEL(N, c, E)$ *coincides with the* $\underset{dec}{\succ}$*−discrete constrained equal losses solution to* $(N, [c], E)$ *proposed in Herrero and Martinez (2004).*

When $x = CEL(N, c, E)$, the $ML(x)$ procedure and the box method are coherent with the CEL nature of the solution x

We emphasize that the procedure and the box method do work even if $x \neq CEL(N, c, E)$, possibly compensating some roughness of x or of the truncation step

There are *two ways* of finding the same set of integer solutions to problem IBP:

(a) Truncate the CEL solution and distribute the remainder among the agents according to the $ML(x)$ procedure; the procedure minimizes the infinity norm of the residual with respect to the difference between c and $[x]$

(b) the whole estate E is distributed among the agents according to the $ML(0)$ procedure; the procedure minimizes the infinity norm of the residual with respect to the claims c

The conclusion is the following:

the $ML(x)$ procedure for assigning the quantity $\Delta(x)$ is coherent with the nature of the CEL solution

9 Introducing $ICEA$

It is well known that the CEL and CEA solutions are *dual* (Aumann and Maschler, 1985)

$$CEA(N, c, E) = c - CEL(N, c, C - E)$$

Definition 12. *Let Assumption 3 hold and suppose* $0 \le E \le [[C]]$. *The* $ICEA(N, c, E)$ *solution,* $ICEA$, *when no confusion arises, to* IBP *is the vector*

$$ICEA(N, c, E) = [c] - ICEL(N, c, E^*)$$

(shortly denoted by $ICEA$*, when no confusion arises), where* $E^* = [[C]] - E$.

Here and in the following, $ICEL$ includes the boundary cases where some claims or the estate may vanish, i.e. $E \ge 0$ and $c_j \ge 0, j \in N$
When $E = [[C]]$ the estate E^* vanishes, and when $0 < c_j < 1$ then $[c_j] = 0$
$ICEA(N, c, E)$ is an integer solution to IBP, i.e. it satisfies

$$0 \le ICEA_j \le [c_j] \le c_j$$

because $ICEL$ is non-negative and rational and the integer part is monotone. Moreover, by efficiency of $ICEL$

$$\sum_{j \in N} ICEA_j(N, c, E) = \sum_{j \in N} [c_j] - \sum_{j \in N} ICEL_j(N, c, E^*) = [[C]] - E^* = E$$

Trivially, $E = 0 \Rightarrow ICEA(N, c, E) = ICEL(N, c, E) = 0$ and $E = [[C]] \Rightarrow ICEA(N, c, E) = ICEL(N, c, E) = [c]$ by Theorem 1

Let $\overset{*}{\underset{dec}{\succ}}$ be the *dual* of the order $\underset{dec}{\succ}$ defined in N

$$\text{for every } i,\ j \in N,\quad i \overset{*}{\underset{dec}{\succ}} j \ \text{ if and only if } \ j \underset{dec}{\succ} i$$

For a given integer solution z to IBP, let *maximum gap of the awards* be

$$mg\,(z) = \max\, z - \min\, z$$

Proposition 13. *Assume* $0 \leq E \leq [[C]]$. *If* z *is any integer solution to* IBP, *then*

$$mg\,(ICEA(N, c, E)) \leq \ mg\,(z)$$

10 Characterizing $ICEA$

First characterization

Assumption 14. *Consider a IBP (N, c, E), with $0 \leq E \leq [[C]]$ and N endowed with a strict total order $\triangleright$.*

Dealing with properties involving $ICEL$ or $ICEA$ solutions, we require that $\triangleright = \underset{dec}{\succ}$

Definition 15. *Let ϕ be an integer division rule. We say that ϕ* minimizes the maximum gap of the awards, *i.e. satisfies* MMG, *if for every IBP (N, c, E) satisfying Assumption 14 and for every integer solution z to it, then $mg\,(\phi(N, c, E)) \leq mg\,(z)$.*

Definition 16. *Let ϕ be an integer division rule. We say that ϕ* complies with the order $\triangleright$, *i.e. satisfies* OC, *if for every IBP (N, c, E) satisfying Assumption 14, whenever $i, j \in N$ are such that*

$$\begin{aligned} &\phi_j(N, c, E) < [c_j] \\ &\phi_i(N, c, E) = \phi_j(N, c, E) + 1 \end{aligned}$$

then $i \triangleright^* j$, where $\triangleright^*$ is the dual of $\triangleright$.

Definition 17. *(Aumann and Maschler, 1985) An integer division rule ϕ* is self-consistent, *i.e. satisfies* SC, *if for every IBP (N, c, E) satisfying Assumption 14 and for every non-empty subset T of N it is $\phi_{\big|T}(N, c, E) = \phi(T, c^T, E^T)$ where $c^T = \{c_j,\ j \in T\}$ and $E^T = \sum_{j \in T} \phi_j(N, c, E)$.*

Theorem 18. *$ICEA$ is the unique integer division rule that satisfies* OC, MMG *and* SC.

The hypotheses in the statement of Theorem 18 are independent

Second characterization (via $TEST$)

Definition 19. *Let ϕ be any integer division rule. We say that ϕ* satisfies TEST *if for every bankruptcy problem (N, c, E) satisfying Assumption 14 and such that*

$$0 < E < [[C]] \tag{4}$$

denoting $z = \phi(N, c, E)$

$$TEST: \begin{cases} (A) & \text{if, for every } s \in \mathbb{N}, \quad B_s = \{j \in N: \ z_j = s\}, \quad m = \max\{s \in \mathbb{N}: \ B_s \neq \emptyset\}, \\ & s_0 = \min\{s \in \mathbb{N}: \ \exists j \in B_s: \ z_j < [c_j]\}, \\ (B) & \text{then either } s_0 = m \text{ or } s_0 = m - 1, \\ (C) & \text{moreover if } s_0 = m - 1 \text{ then } j \rhd^* j_0\,, \text{for every } j \in B_m\,, \text{ where} \\ & j_0 = \max_{\rhd^*}\{j \in B_{s_0}: \ z_j < [c_j]\}. \end{cases}$$

$TEST$ can be implemented easily and does not require any comparison with other integer solutions

s_0 is well defined, because $\{s \in \mathbb{N}: \ \exists j \in B_s: \ z_j < [c_j]\} \neq \emptyset$ by (4) defining j_0 the maximum is taken w.r.t. the order $\rhd^*$

E does not appear explicitly in $TEST$, yet by the efficiency property

$$\begin{aligned} E &= \sum_{j \in \cup\{B_s:\, s < s_0\}} z_j + \sum_{j \in B_{s_0} \cup B_{s_0+1}} z_j = \sum_{j \in \cup\{B_s:\, s < s_0\}} [c_j] + \sum_{j \in B_{s_0}} s_0 + \sum_{j \in B_{s_0+1}} (s_0 + 1) \\ &= \sum_{j \in \cup\{B_s:\, s < s_0\}} [c_j] + s_0\, card\,(B_{s_0} \cup B_{s_0+1}) + card\,(B_{s_0+1}) \end{aligned}$$

Remark 20. *$TEST$ is designed for checking if a given solution is $ICEA$, but it is possible to use it also for computing it; in fact truncating the classical CEA solution, the possible remaining unities may be assigned following the order $\rhd^*$. Anyhow, the computational complexity of this algorithm is comparable with the computation of $ICEL$ and then determining $ICEA$ using the duality relation.*

Lemma 21. *Let ϕ be an integer division rule satisfying $TEST(A),(B)$. Then ϕ minimizes the maximum gap of the awards.*

Theorem 22. *$ICEA$ is* the unique *integer division rule that satisfies $TEST$.*

Third characterization

Definition 23. *An integer division rule ϕ satisfies* conditional full compensation *(*CFC*) if for every IBP (N, c, E) satisfying Assumption 14 and for every $i \in N$, if i belongs to the set*

$$L = \left\{ k \in N : \sum_{j \in N} \min\{[c_k], [c_j]\} \leq E \right\} \tag{5}$$

then $\phi_i(N, c, E) = [c_i]$. We say that an agent is conditionally fully compensated *if it belongs to L.*

Definition 24. *We say that an integer division rule ϕ* minimizes the reduced maximum gap of the awards, *i.e. satisfies* MRMG, *if for every IBP (N, c, E) satisfying Assumption 14, either the set L defined in* (5) *equals N or, denoting $z = \phi(N, c, E)$,*

$$rmg\,(z) \leq 1$$

where $rmg\,(z) = mg\,(z_{|N \setminus L})$ is the reduced maximum gap of the awards.

Theorem 25. *An integer division rule satisfies OC, CFC and MRMG, if and only if it satisfies $TEST$.*

Corollary 26. *$ICEA$ is* the unique *integer division rule that satisfies OC, CFC and MRMG.*

The properties in the statement of Corollary 26 are independent

Fourth characterization of $ICEA$

Recalling that for every BP (N, c, E) the CEA division rule is such that for every solution z

$$\max CEA(N, c, E) \leq \max z$$

Definition 27. *Let ϕ be an integer division rule. We say that ϕ* minimizes the maximum award, *i.e. satisfies MMA, if for every IBP (N, c, E) satisfying Assumption 14 and for every integer solution z to it, then* $\max \phi(N, c, E) \leq \max z$.

OC and MMA are not sufficient for characterizing $ICEA$

Example 28. *Consider the following bankruptcy problem: $N = \{1, 2, 3\}$, with $3 \triangleright^* 2 \triangleright^* 1$, $c = (3, 3, 3)$, $E = 7$. Then $ICEA(N, c, E) = (2, 2, 3)$, but the solution $z = (1, 3, 3)$ satisfies OC and has the same maximum value as $ICEA(N, c, E)$.*

Theorem 29. *Assume $\triangleright = \underset{dec}{\succ}$. Then $ICEA$ is* the unique *integer division rule that satisfies OC, SC and MMA.*

Corollary 30. *Let (N, c, E) be a IBP satisfying Assumption 14 and $\triangleright = \underset{dec}{\succ}$. Then $ICEA(N, c, E)$ coincides with the $\triangleright^*-$discrete constrained equal awards solution to $(N, [c], E)$ proposed in Herrero and Martinez (2004).*

$TEST$ can be adapted to $ICEL$; if ϕ and ψ are dual integer division rules, we say that ψ *satisfies* $TEST^*$ if and only if ϕ satisfies $TEST$ (i.e. $TEST$ and $TEST^*$ are dual)

By comparison with Definition 19 (where E^* replaces E), it follows that ψ satisfies $TEST^*$ if and only if for every IBP (N, c, E) satisfying Assumption 14 and (4), denoting $w = \psi(N, c, E)$

$$TEST^* : \begin{cases} (A) & \text{if, for every } s \in \mathbb{N}, \quad B_s = \{j \in N : \ w_j = [c_j] - s\}, \\ & m = \max\{s \in \mathbb{N} : \ B_s \neq \emptyset\}, \quad s_0 = \min\{s \in \mathbb{N} : \ \exists j \in B_s : \ w_j > 0\}, \\ (B) & \text{then either } s_0 = m \text{ or } s_0 = m - 1, \\ (C) & \text{moreover if } s_0 = m - 1 \text{ then } j \rhd^* j_0 \text{, for every } j \in B_m \text{, where} \\ & j_0 = \max_{\rhd^*}\{j \in B_{s_0} : \ w_j > 0\}. \end{cases}$$

Proposition 31. *An integer division rule ψ satisfies $TEST^*$ if and only if $\psi = ICEL$.*

11 The class of perm$-ICEA$ solutions

Definition 32. *We say that an integer division rule ϕ that satisfies CFC and MRMG is a* perm$-ICEA$ division rule*.*

In other words the "last unities" may be *permutated*, as OC is not required

Proposition 33. *An integer division rule ϕ is* perm$-ICEA$ *if and only if it satisfies $TEST(A),(B)$.*

The relationship with MMG and SC is less direct, however the following propositions hold

Proposition 34. *An integer division rule which satisfies MMG and SC is* perm$-ICEA$*.*

Proposition 35. *A* perm$-ICEA$ *division rule ϕ satisfies MMG; moreover, for every IBP (N,c,E) satisfying Assumption 14 and for every non-empty subset T of N there exists a* perm$-ICEA$ *division rule ψ^T such that*

$$\phi(T,c^T,E^T) = (\psi^T)_{\big|T}(N,c,E)$$

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Thanks!