Moving to a strategic scenario

- Suppose players are not allowed to make binding agreements on the design of a network.

- How should we design cost allocation protocols to minimize the efficiency loss caused by rational players that are only willing to perform update leading to an immediate reduction of their individual cost shares?

The strategic game

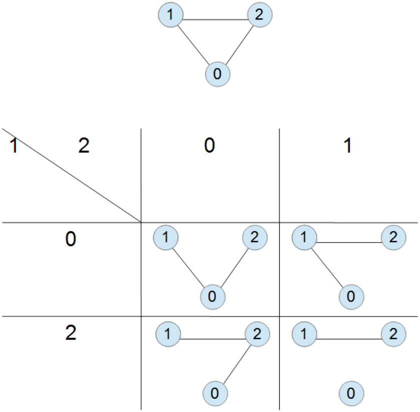
- $G = (N', E, w)$ is an undirected, connected and weighted graph, where $N' = N \cup \{0\}$ and $N = \{1, 2, \ldots, n\}$, and $w(e)$ is the cost of the power needed to send a message along the link e.

- The strategy space $\mathcal{N}_G(i)$ of every player $i \in N$ is his neighborhood in the graph. A *state* (or *strategy profile*) S is a vector $(S_1, S_2, \cdots, S_n)$

- A protocol is a vector $(c_1(w, S), \cdots, c_n(w, S))$ which, given a weighted map w and a strategy profile S, allocates a cost to the players.

- the cost of remaining disconnected from the source is larger than any finite cost that should be supported to guarantee the connection with the source

An example of game form with two players



Some properties for protocols and games

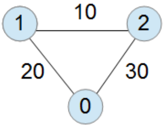
Denote by $\mathtt{con}(S)$ the set of players connected to 0 on state S, and by $T_S = \{(i, S_i) : i \in \mathtt{con}(S)\}$ the network of connected players under state S.

- A cost allocation protocol c such that $\sum_{i \in \mathtt{con}(S)} c_i(w, S) = w(T_S)$ for every strategy profile S is said budget-balanced.

A protocol is said Independent from Irrelevant Links iff for every state S and every weight functions w, w', with $w(e) = w'(e)$ for every $e \in T_S$, then $c_i(w, S) = c_i(w', S)$ for every $i \in \mathtt{con}(S)$.

The Bird protocol

Nash equilibrium $(0,1)$ is efficient (i.e. $T_{(0,1)}$ is a minimum cost spanning tree (mcst) connecting all nodes in N'), but $(2,0)$ is not.



1 \ 2	0	1
0	(20,30)	(20,10)
2	(10,30)	(∞,∞)

Better Response Dynamic and Optimality

- Given a protocol c, a strategy $x \in \mathcal{N}_G(i)$ is a better response of player i with respect to the strategy profile S if $c_i(w,(x,S_{-i})) < c_i(w,S)$.

- A Better Response Dynamic (BRD, also called Nash dynamics) (associated with a protocol c) is a sequence of states $S^0, S^1, \ldots,$ such that each state S^k (except S^0) is resulted by a better response of some player from the state S^{k-1}.

- We say that a cost allocation protocol is optimal iff every associated BRD reaches an efficient Nash equilibrium.

Theorem There is no budget-balanced and IIL protocol which is also optimal.

An optimal protocol

Let S be a state. We denote by T_S^* a mcst on the subgraph $\texttt{con}(S)$ and by S^* a state which corresponds to T_S^*

Our protocol relies on a particular set of players: we define $\hat{V}(S)$ as the set of players that do not follow T_S^* and that if they unilaterally change their strategy to follow it, then the set of connected players remains unchanged.

We proved that if at least one player in $\texttt{con}(S)$ does not follow the optimal strategy S^*, then $\hat{V}(S) \neq \emptyset$.

An optimal protocol

If $i \in N \setminus \texttt{con}(S)$, then $c_i(w, S) = +\infty$,

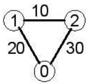
If $i \in \texttt{con}(S) \setminus \hat{V}(S)$, then $c_i(w, S) = w(i, S_i^*)$,

If $i \in \hat{V}(S)$, then $c_i(w, S) = w(i, S_i^*) + \frac{\Delta(S)}{|\hat{V}(S)|}$

where $\Delta(S) = w(T_S) - w(T_S^*)$ (actually here, we can take any cost function $w(i, S_i^*) + g_i(S)$ such that (i) $g_i(S) > 0$ and (ii) $\sum_{i \in \texttt{con}(S)} g_i(S) = \Delta(S)$.

Theorem Any state S which does not correspond to a mcst is not a Nash equilibrium.

Example



1 \ 2	0	1
0	(20, 30) $\hat{V}(S)=\{2\}$	(20, 10)
2	(30, 10) $\hat{V}(S)=\{1\}$	(∞, ∞)

Generalized ordinal potential game

$\Phi(S) = (|N \setminus \mathtt{con}(S)|, |\hat{V}(S)|, \sum_{i \in \hat{V}(S)} |E_S(i)|)$
where $E_S(i) = \{j \in N' : w(i,j) < w(i, S_i)\}$.

Lemma Let S and S' be two states which only differ on the strategy of a player i. If $c_i(S') < c_i(S)$ then $\Phi(S') \prec \Phi(S)$ ($\Phi(S')$ is lexicographically strictly smaller than $\Phi(S')$).

Theorem `BRD` always converges after at most mn^2 rounds, where m is the number of edges of the graph.

Conclusions

- We have studied cost allocation protocols for connection situations in a strategic setting.

- we have analyzed properties for protocols in relation the the convergence of the best reply dynamics to efficient Nash equilibria.

- the inherent limitations of the optimal protocols proposed in this paper is that it depends on the choice of an *a priori* selected mcst.