An introduction to cooperative games

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GAMETHEORY

NON-COOPERATIVE THEORY

- Dominant strategies
- Nash eq. (NE)
- Subgame perfect NE
- NE & refinements
  - ...

GAMES IN STRATEGIC FORM (NORMAL FORM)

- Core
- Shapley value
- Nucleolus
- \( \tau \)-value
- PMAS
  - ...

GAMES IN C.F.F. (TU-GAMES OR COALITIONAL GAMES)

- Nash sol.
- Kalai-Smorodinsky
  - ...

COOPERATIVE THEORY

- CORE
- NTU-value
- Compromise value
  - ...

No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing
Cooperative games: a simple example

Alone, player 1 (singer) and 2 (pianist) can earn 100€ and 200€ respectively.

Together (duo) 700€

How to divide the (extra) earnings?

Imputation set: $I(v) = \{x \in \mathbb{R}^2 | x_1 \geq 100, x_2 \geq 200, x_1 + x_2 = 700 \}$
COOPERATIVE GAME THEORY

Games in coalitional form
TU-game: \((N,v)\) or \(v\)
- \(N=\{1, 2, \ldots, n\}\) set of players
- \(S \subseteq N\) coalition
- \(2^N\) set of coalitions

DEF. \(v: 2^N \rightarrow \mathbb{R}\) with \(v(\emptyset)=0\) is a Transferable Utility (TU)-game with player set \(N\).

NB: \((N,v)\leftrightarrow v\)
NB2: if \(n=|N|\), it is also called \(n\)-person TU-game, game in coalitional form, coalitional game, cooperative game with side payments...

\(v(S)\) is the value (worth) of coalition \(S\)
Example

(Glove game) \( N=L \cup R, \quad L \cap R = \emptyset \)
i \in L (i \in R) possesses 1 left (right) hand glove
Value of a pair: 1€
\( v(S) = \min\{|L \cap S|, |R \cap S|\} \) for each coalition \( S \in 2^N \setminus \{\emptyset\} \).

Example

Glove game with \( L=\{1,2\}, \ R=\{3\} \)
\( v(1,3)=v(2,3)=v(1,2,3)=1, \quad v(S)=0 \) otherwise
Q.1: which coalitions form?

**DEF.** (N, v) is a superadditive game iff

\[ v(S \cup T) \geq v(S) + v(T) \text{ for all } S, T \text{ with } S \cap T = \emptyset \]

Q.2: If the grand coalition N forms, how to divide v(N)?
(how to allocate costs?)

Many answers! (solution concepts)

**One-point concepts:**
- Shapley value (Shapley 1953)
- nucleolus (Schmeidler 1969)
- \( \tau \)-value (Tijs, 1981)
- ...

**Subset concepts:**
- Core (Gillies, 1954)
- stable sets (von Neumann, Morgenstern, ’44)
- kernel (Davis, Maschler)
- bargaining set (Aumann, Maschler)
- ..
Example

(Glove game) \((N,v)\) such that \(N=L \cup R, \quad L \cap R = \emptyset\)

\(v(S) = \min\{|L \cap S|, |R \cap S|\} \) for all \(S \in 2^N \setminus \{\emptyset\}\)

Claim: the glove game is superadditive.

Suppose \(S, T \in 2^N \setminus \{\emptyset\}\) with \(S \cap T = \emptyset\). Then

\[
v(S) + v(T) = \min\{|L \cap S|, |R \cap S|\} + \min\{|L \cap T|, |R \cap T|\}
= \min\{|L \cap S| + |L \cap T|, |L \cap S| + |R \cap T|, |R \cap S| + |L \cap T|, |R \cap S| + |R \cap T|\}
\leq \min\{|L \cap S| + |L \cap T|, |R \cap S| + |R \cap T|\}
\]

since \(S \cap T = \emptyset\)

\[
= \min\{|L \cap (S \cup T)|, |R \cap (S \cup T)|\}
= v(S \cup T).
\]
**The imputation set**

**DEF.** Let \((N,v)\) be a n-persons TU-game. A vector \(x=(x_1, x_2, \ldots, x_n)\in \mathbb{R}^N\) is called an **imputation** iff

1. **individual rational** i.e.
   \[ x_i \geq v(i) \text{ for all } i \in N \]

2. **efficient**
   \[ \sum_{i \in N} x_i = v(N) \]

[interpretation \(x_i\): payoff to player i]

\[ I(v) = \{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), x_i \geq v(i) \text{ for all } i \in N \} \]

**Set of imputations**
Example
(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2,3)=5.

\[ I(v) = \{ x \in \mathbb{R}^3 \mid x_1, x_3 \geq 0, x_2 \geq 3, x_1 + x_2 + x_3 = 5 \} \]
**The core of a game**

**DEF.** Let \((N,v)\) be a TU-game. The core \(C(v)\) of \((N,v)\) is the set
\[
C(v)=\{x\in I(v) \mid \sum_{i\in S} x_i \geq v(S) \text{ for all } S\in 2N\setminus\{\emptyset\}\}
\]

**stability conditions**

no coalition \(S\) has the incentive to split off if \(x\) is proposed

**Note:** \(x \in C(v)\) iff

1. \(\sum_{i\in N} x_i = v(N)\) efficiency
2. \(\sum_{i\in S} x_i \geq v(S)\) for all \(S\in 2N\setminus\{\emptyset\}\) stability

**Bad news:** \(C(v)\) can be empty

**Good news:** many interesting classes of games have a non-empty core.
Example

(N,v) such that

N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3,
v(1,3)=1
v(2,3)=4
v(1,2,3)=5.

Core elements satisfy the following conditions:

\[ \begin{align*}
  x_1, x_3 &\geq 0, \quad x_2 \geq 3, \quad x_1 + x_2 + x_3 = 5 \\
  x_1 + x_2 &\geq 3, \quad x_1 + x_3 \geq 1, \quad x_2 + x_3 \geq 4 \\
  5 - x_3 &\geq 3 \iff x_3 \leq 2 \\
  5 - x_2 &\geq 1 \iff x_2 \leq 4 \\
  5 - x_1 &\geq 4 \iff x_1 \leq 1
\end{align*} \]

\[ C(v) = \{ x \in \mathbb{R}^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0, 4 \geq x_2 \geq 3, \quad x_1 + x_2 + x_3 = 5 \} \]
Example
(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3, v(1,3)=1
v(2,3)=4
v(1,2,3)=5.

\[ C(v) = \{ x \in \mathbb{R}^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0, 4 \geq x_2 \geq 3, \ x_1 + x_2 + x_3 = 5 \} \]
Example (Game of pirates) Three pirates 1, 2, and 3. On the other side of the river there is a treasure (10€). At least two pirates are needed to wade the river...

\((N, v), N=\{1, 2, 3\}, v(1)=v(2)=v(3)=0, v(1, 2)=v(1, 3)=v(2, 3)=v(1, 2, 3)=10\)

Suppose \((x_1, x_2, x_3) \in C(v)\). Then

efficiency \(x_1 + x_2 + x_3 = 10\)

\[ \begin{aligned}
\text{stability} & \quad x_1 + x_2 \geq 10 \\
& \quad x_1 + x_3 \geq 10 \\
& \quad x_2 + x_3 \geq 10
\end{aligned} \]

\[ 20 = 2(x_1 + x_2 + x_3) \geq 30 \]  
Impossible. So \(C(v) = \emptyset\).

Note that \((N,v)\) is superadditive.
Example
(Glove game with $L=\{1,2\}, \ R=\{3\}$)
$v(1,3)=v(2,3)=v(1,2,3)=1, \ \ \ \ v(S)=0$ otherwise

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 1 &\quad x_2 &= 0 \\
  x_1 + x_3 &\geq 1 &\quad x_1 + x_3 &= 1 \\
  x_2 &\geq 0 \\
  x_2 + x_3 &\geq 1 \\
  x_1 &= 0 \quad \text{and} \quad x_3 = 1
\end{align*}
\]

So $C(v) = \{(0,0,1)\}$. 

\[
\text{I(v)} \\
(0,0,1) \quad (0,1,0) \\
(1,0,0)
\]
How to share \( v(N) \)...

- The Core of a game can be used to exclude those allocations which are \emph{not stable}.
- But the core of a game can be a bit “\emph{extreme}” (see for instance the glove game)
- Sometimes the core is \emph{empty} (pirates)
- And if it is not empty, there can be many allocations in the core (\emph{which is the best?})
An axiomatic approach (Shapley (1953))

- Similar to the approach of Nash in bargaining: *which properties an allocation method should satisfy in order to divide* \(v(N)\) *in a reasonable way?*
- Given a subset \(\mathcal{C}\) of \(G^N\) (class of all TU-games with \(N\) as the set of players) a *(point map) solution* on \(\mathcal{C}\) is a map \(\Phi: \mathcal{C} \rightarrow \mathbb{IR}^N\).
- For a solution \(\Phi\) we shall be interested in various properties...
PROPERTY 1 (SYM) Let $v \in \mathcal{G}^N$ be a TU-game.

Let $i, j \in \mathbb{N}$. If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{\mathbb{N}\setminus\{i,j\}}$, then $\Phi_i(v) = \Phi_j(v)$.

EXAMPLE

We have a TU-game $\langle\{1,2,3\},v\rangle$ s.t. $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(1, 3) = 4$, $v(2, 3) = 6$, $v(1, 2, 3) = 20$.

Players 2 and 3 are symmetric. In fact:

$v(\emptyset \cup \{2\}) = v(\emptyset \cup \{3\}) = 0$ and $v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4$

If $\Phi$ satisfies SYM, then $\Phi_2(v) = \Phi_3(v)$.
Efficiency

**PROPERTY 2 (EFF)** Let $v \in G^N$ be a TU-game.

$$\sum_{i \in N} \Phi_i(v) = v(N),$$
i.e., $\Phi(v)$ is a *pre-imputation*.

**Null Player Property**

**DEF.** Given a game $v \in G^N$, a player $i \in N$ s.t.

$$v(S \cup i) = v(S)$$

for all $S \in 2^N$ will be said to be a null player.

**PROPERTY 3 (NPP)** Let $v \in G^N$ be a TU-game. If $i \in N$ is a null player, then $\Phi_i(v) = 0$.

**EXAMPLE** We have a TU-game $({1,2,3},v)$ such that $v(1) = 0$, $v(2) = v(3) = 2$, $v(1, 2) = v(1, 3) = 2$, $v(2, 3) = 6$, $v(1, 2, 3) = 6$. Player 1 is null. Then $\Phi_1(v) = 0$. 
EXAMPLE We have a TU-game \((\{1,2,3\},v)\) such that 
\[v(1) = 0, \quad v(2) = v(3) = 2, \quad v(1, 2) = v(1, 3) = 2, \quad v(2, 3) = 6, \quad v(1, 2, 3) = 6.\]

On this particular example, if \(\Phi\) satisfies NPP, SYM and EFF we have that 
\[
\Phi_1(v) = 0 \text{ by NPP}
\]
\[
\Phi_2(v) = \Phi_3(v) \text{ by SYM}
\]
\[
\Phi_1(v) + \Phi_2(v) + \Phi_3(v) = 6 \text{ by EFF}
\]

So \(\Phi = (0,3,3)\)

But our goal is to characterize \(\Phi\) on \(G^N\). One more property is needed.
Additivitiy

PROPERTY 4 (ADD) Given \( v, w \in G^N \),

\[
\Phi(v) + \Phi(w) = \Phi(v + w).
\]

**EXAMPLE** Two TU-games \( v \) and \( w \) on \( N = \{1, 2, 3\} \),

\[
\begin{align*}
v(1) &= 3 \\
v(2) &= 4 \\
v(3) &= 1 \\
v(1, 2) &= 8 \\
v(1, 3) &= 4 \\
v(2, 3) &= 6 \\
v(1, 2, 3) &= 10
\end{align*}
\]

\[
\begin{align*}
w(1) &= 1 \\
w(2) &= 0 \\
w(3) &= 1 \\
w(1, 2) &= 2 \\
w(1, 3) &= 2 \\
w(2, 3) &= 3 \\
w(1, 2, 3) &= 4
\end{align*}
\]

\[
\begin{align*}
v+w(1) &= 4 \\
v+w(2) &= 4 \\
v+w(3) &= 2 \\
v+w(1, 2) &= 10 \\
v+w(1, 3) &= 6 \\
v+w(2, 3) &= 9 \\
v+w(1, 2, 3) &= 14
\end{align*}
\]
Theorem 1 (Shapley 1953)

There is a unique map $\phi$ defined on $G^N$ that satisfies EFF, SYM, NPP, ADD. Moreover, for any $i \in N$ we have that

$$
\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_i^\sigma (v)
$$

Here $\Pi$ is the set of all permutations $\sigma: N \rightarrow N$ of $N$, while $m_i^\sigma(v)$ is the marginal contribution of player $i$ according to the permutation $\sigma$, which is defined as:

$$
v(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\}),
$$

where $j$ is the unique element of $N$ s.t. $i = \sigma(j)$. 
**Probabilistic interpretation:** (the “room parable”)

- Players gather one by one in a room to create the “grand coalition”, and each one who enters gets his marginal contribution.
- Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

**Example**

(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2)=3,

v(1,3)=1,

v(2,3)=4,

v(1,2,3)=5.

<table>
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<tr>
<th>Permutation</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2,3</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1,3,2</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>2,1,3</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2,3,1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3,2,1</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3,1,2</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>3</td>
<td>21</td>
<td>6</td>
</tr>
<tr>
<td>φ(v)</td>
<td>3/6</td>
<td>21/6</td>
<td>6/6</td>
</tr>
</tbody>
</table>
Example

\((N,v)\) such that

\(N=\{1,2,3\}\),

\(v(1)=v(3)=0,\)

\(v(2)=3,\)

\(v(1,2)=3, v(1,3)=1\)

\(v(2,3)=4\)

\(v(1,2,3)=5.\)