Game Theory and Algorithms, Lake Como School of Advanced Studies
7-11 September 2015, Campione d'Italia

Computational complexity of solution concepts

UNIVERSITÀ DELLA CALABRIA
CAMPUS DI ARCAVACATA

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Part I: P vs NP

One Basic Question…

- The lost letter…
  - Year: 1956
  - To: John von Neumann
- …and one question

P=NP?

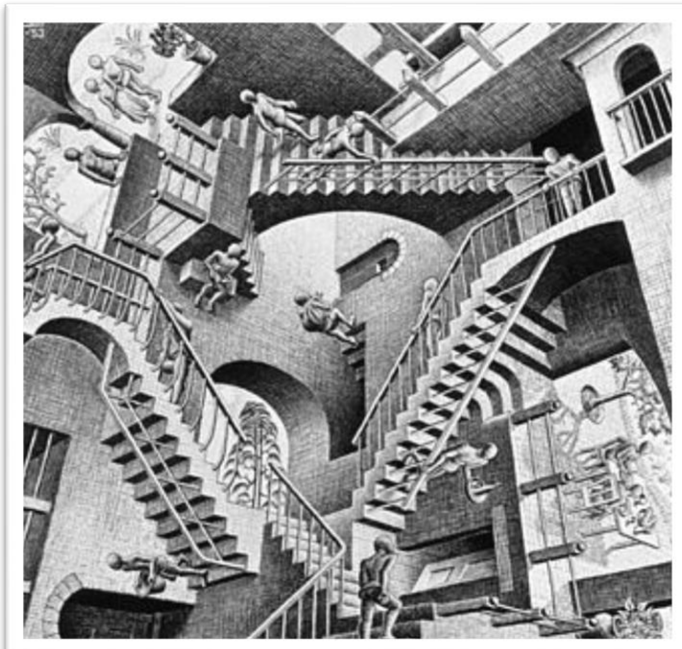
Kurt Gödel (1906-1978)

Millennium Prize Problems

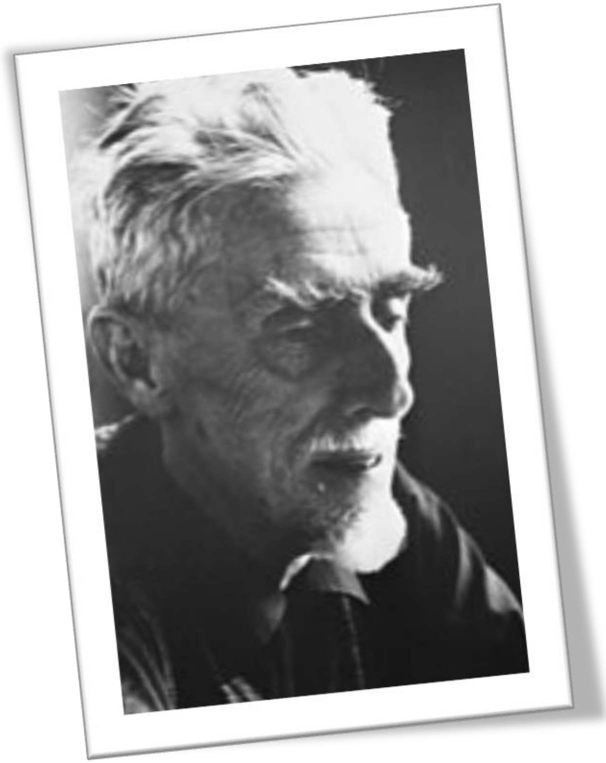
- Seven problems
  - **P versus NP**
  - Poincaré conjecture
  - Hodge conjecture
  - Riemann hypothesis
  - Yang–Mills existence and mass gap
  - Navier–Stokes existence and smoothness
  - Birch and Swinnerton-Dyer conjecture

CLAY MATHEMATICS INSTITUTE

About Clay



M. C. ESCHER
RELATIVITY, 1953



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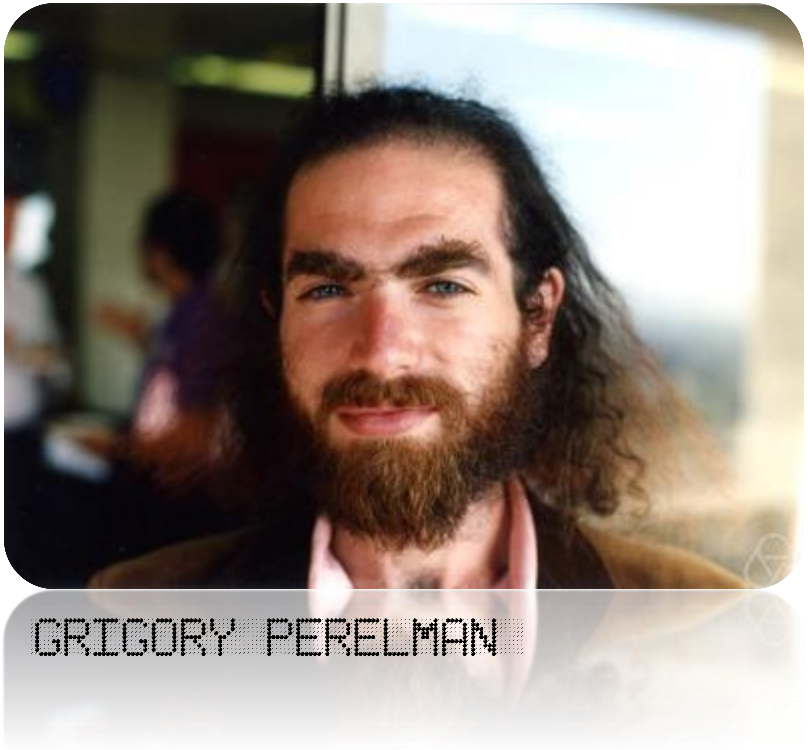
About the Problems

- One problem has been solved in 2006
  - Poincaré conjecture



- He did not want the prize…
- …and the Fields Medal

Problems

- Look for a document

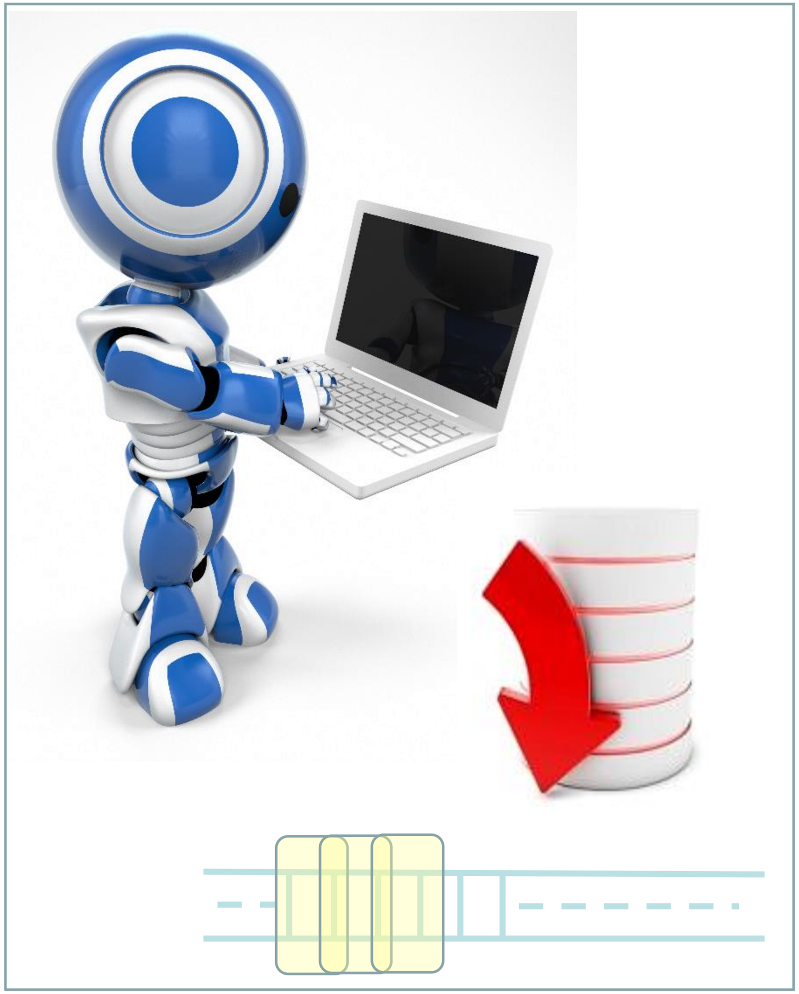
INPUT

OUTPUT

Algorithms

- Look for a document

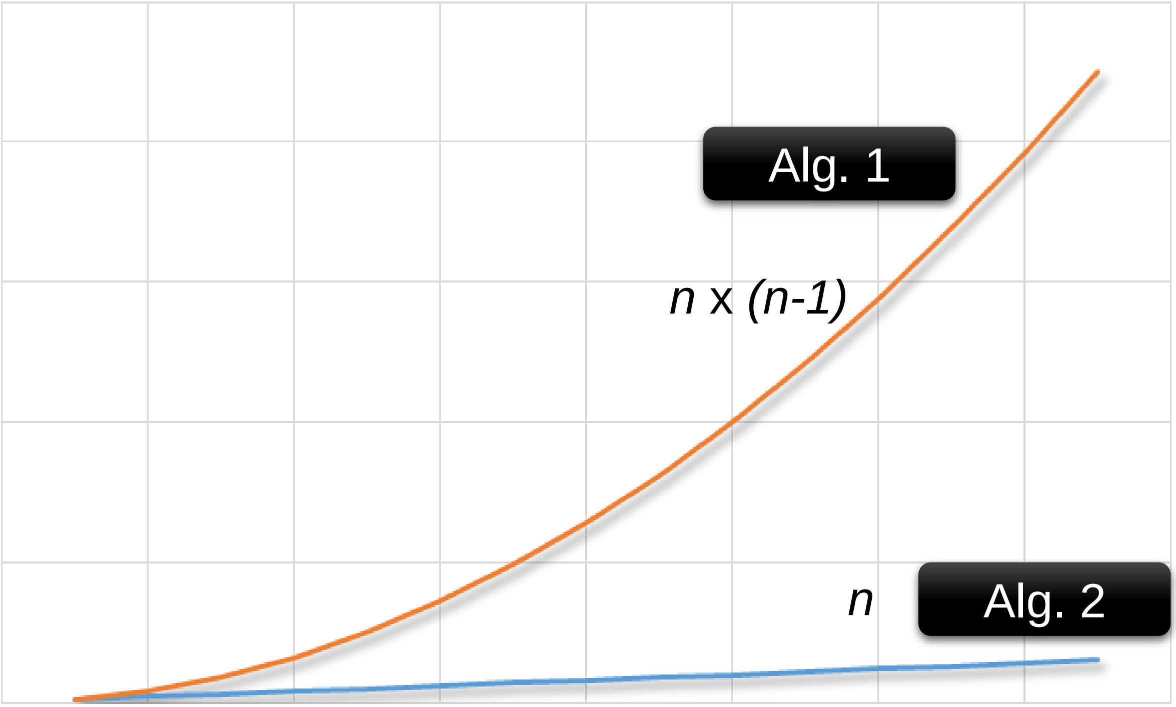
INPUT



OUTPUT

Comparison

- Algorithm 1
  - «quadratic»
- Algorithm 2
  - «linear»



The problem is in **P** (Polynomial)

Look at the Pictures

Look at the Pictures

Golden Ratio

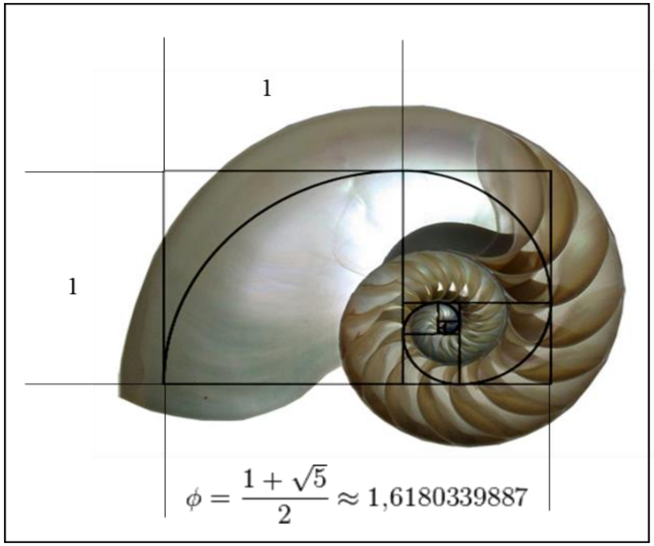
Golden Ratio

$$\frac{L}{R} = \frac{R+L}{L}$$

R=1

$$L = \frac{1+L}{L}$$

$$L^2-L-1=0$$



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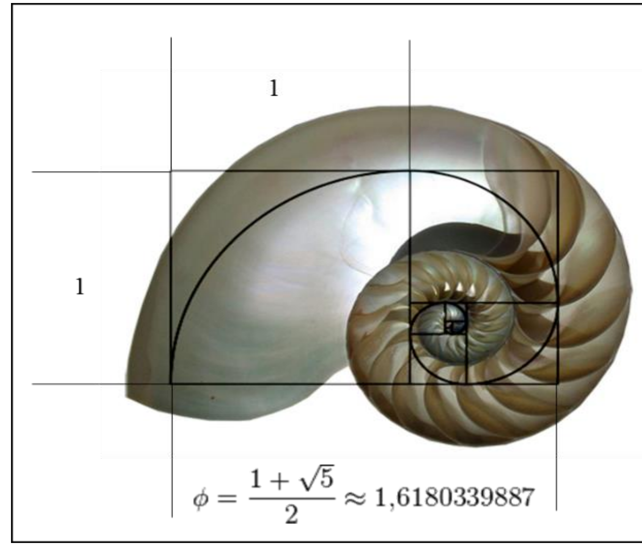


- Does there exist a solution?
- More generally, take L_1, L_2,…L_n and answer questions such as:
  - Is ithere some value for L_1 such that for all values for L_2…

Golden Ratio

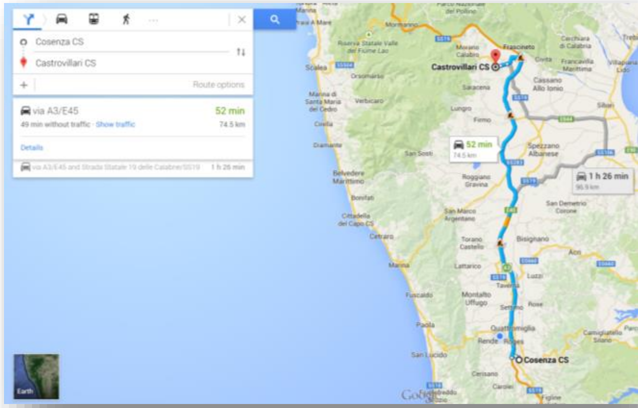
- Provably **EXP**onential
- Every algorithm takes $\mathbf{10^n}$ operations in the worst case

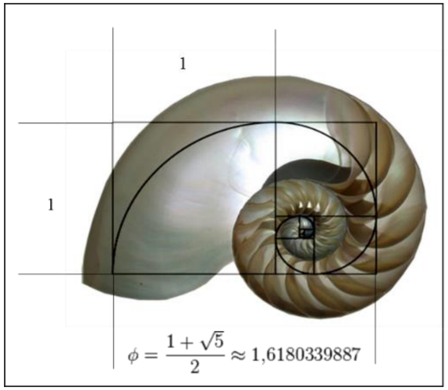
- Does there exist a solution?
- More generally, take L_1, L_2,…L_n and answer questions such as:
  - Is ithere some value for L_1 such that for all values for L_2…

Orders of Magnitude

- 10^4: There are 20,000–40,000 distinct Chinese characters
- 10^5 : 67,000 words in James Joyce's Ulysses
- 10^6 : As of August 31, 2015, Wikipedia contains approximately 4956000 articles in the English language
- 10^9: Approximate population of India in 2011
- 10^{14}: Cells in the human body
- 10^{21}: Estimated number of observable stars
- 10^{80}: Atoms in the Universe

Classes of Problems

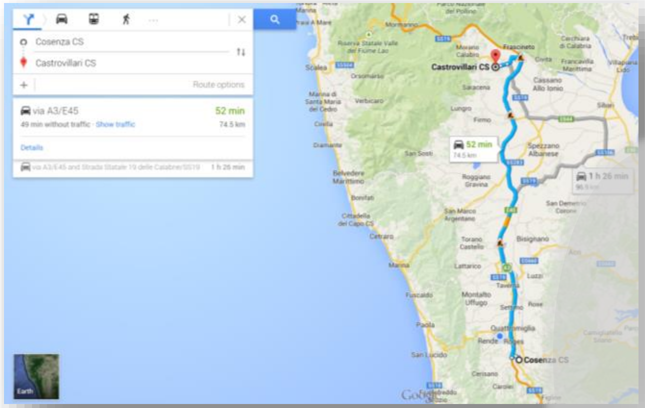


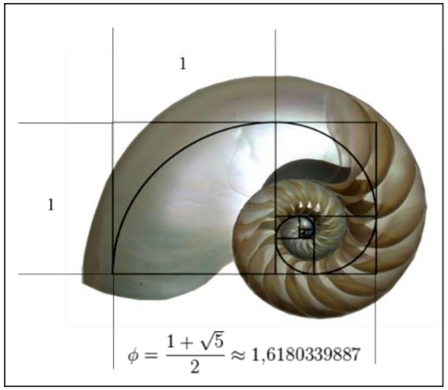


P

EXP

Classes of Problems





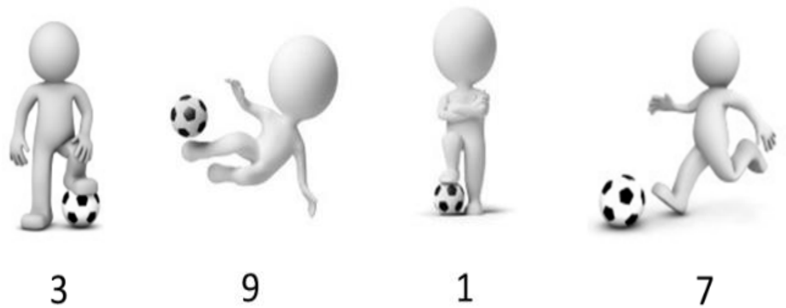
P

EXP

NP

A Simple Problem for NP

- We would like to set up two teams
- Goal
  - The teams should be «balanced»



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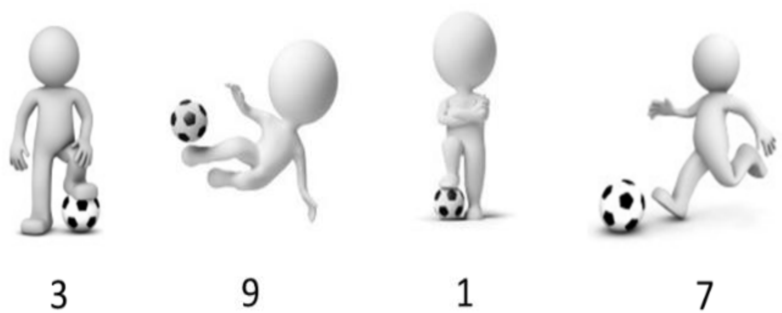

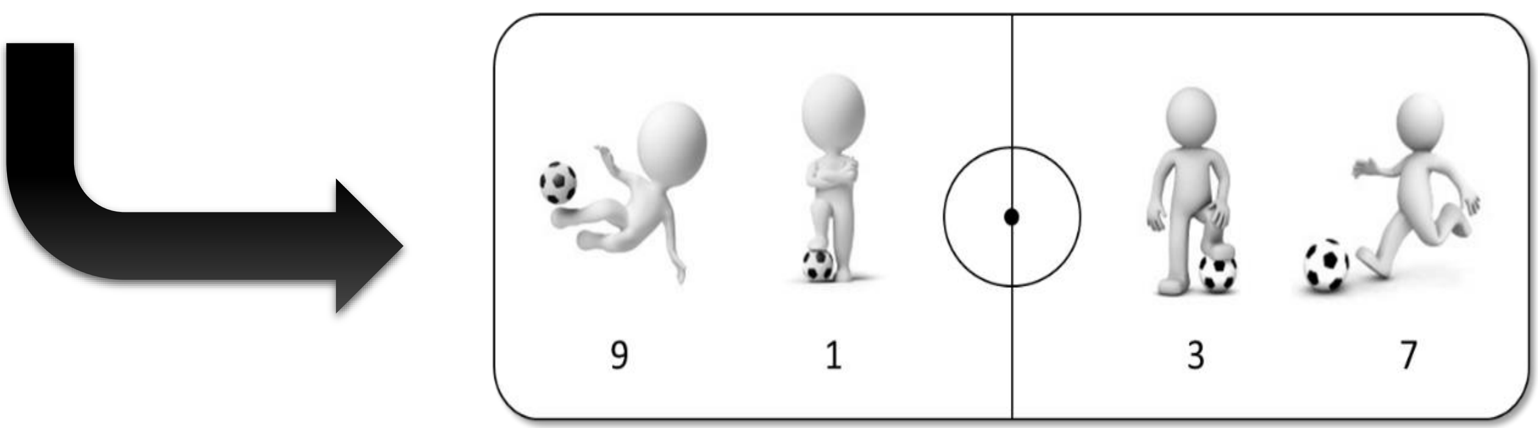

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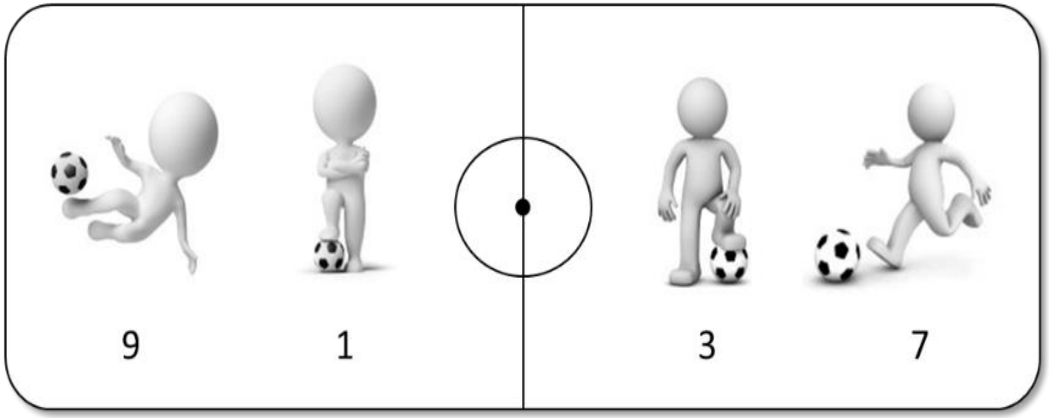
  - Consider all possible teams



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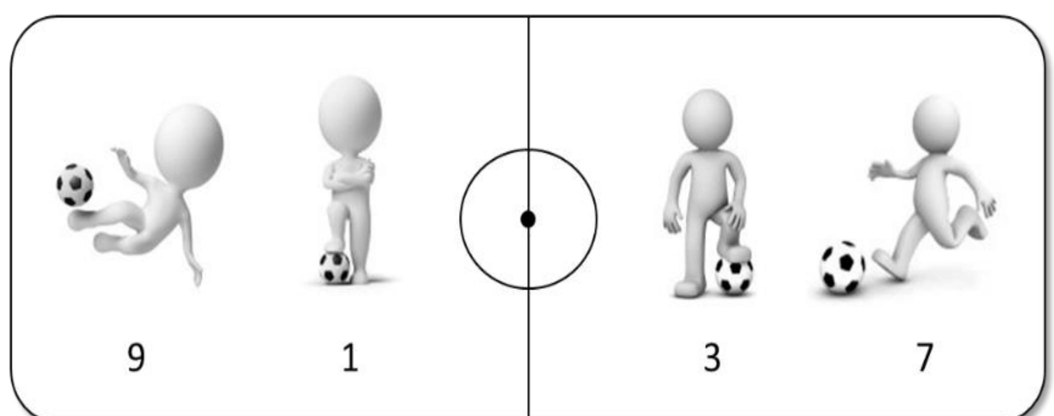
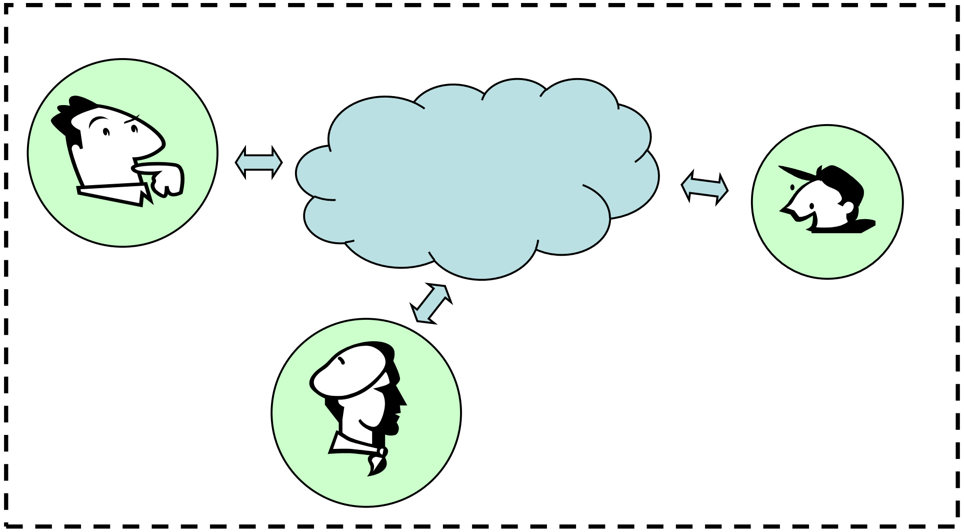


Further Examples

- Scheduling
- Planning
- Logistics
- Crypto
- …Game Theory

Part II: Nash Equilibria

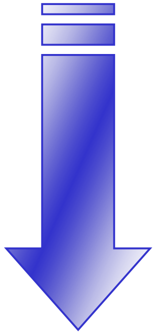
Game Theory (in a Nutshell)



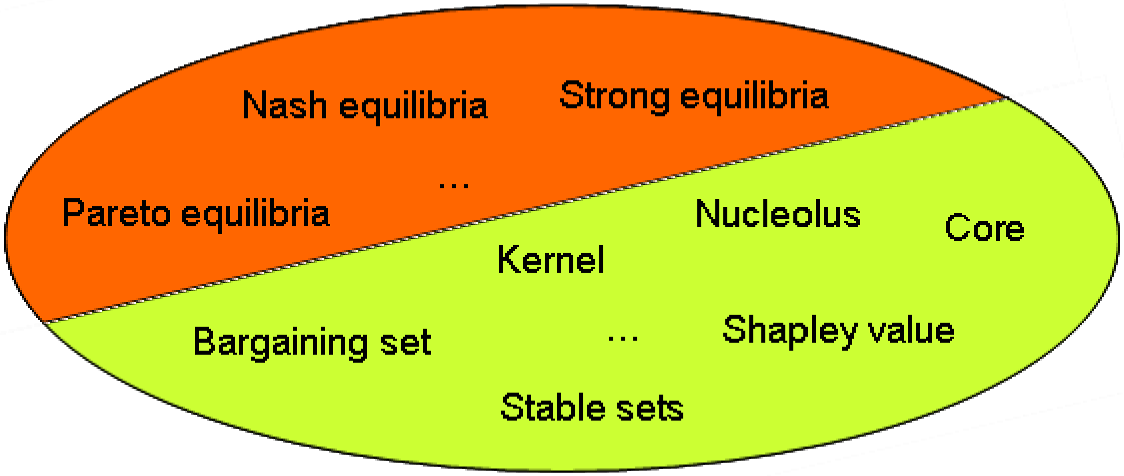
Each player:

- Has a **goal** to be achieved
- Has a set of possible **actions**
- **Interacts** with other players
- Is **rational**

Which actions have to be performed?



Solution Concepts



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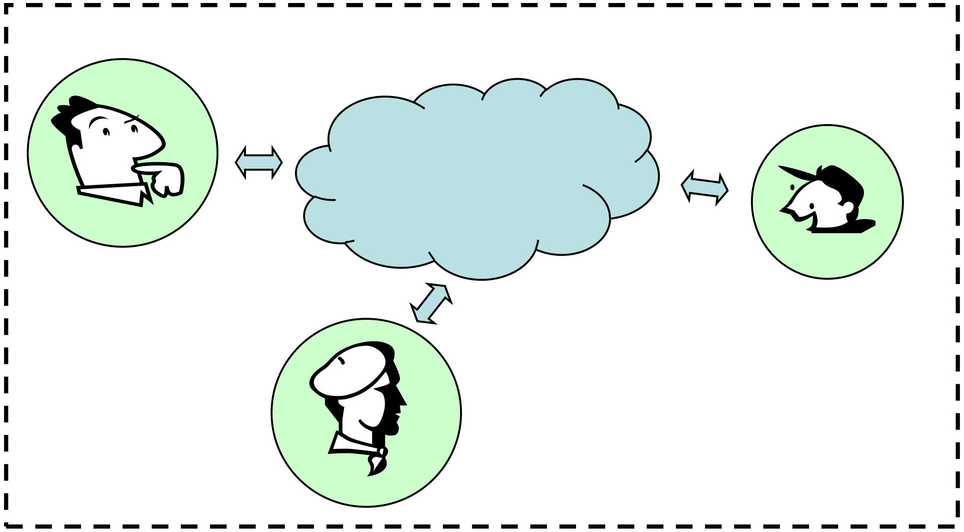

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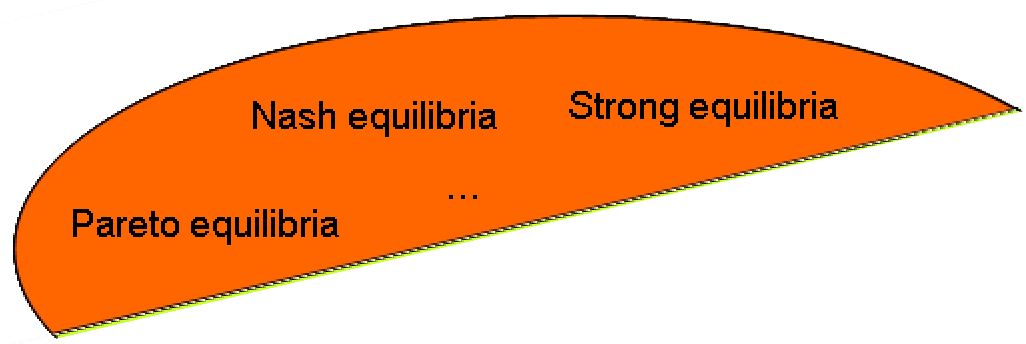

Non-Cooperative Games$_{(1/3)}$

Payoff maximization problem

Each player:

- Has a **goal** to be achieved
- Has a set of possible **actions**
- **Interacts** with other players
- Is **rational**

Bob	John goes *out*	John stays at *home*
out	2	0
home	0	1

John	Bob goes *out*	Bob stays at *home*
out	1	1
home	0	0

Non-Cooperative Games$_{(2/3)}$

Payoff maximization problem

Nash equilibria

Each player:

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home	0	1

John	Bob goes *out*	Bob stays at *home*
out	**1**	1
home	0	0

Non-Cooperative Games$_{(3/3)}$

Payoff maximization problem

Each player:

- Has a **goal** to be achieved
- Has a set of possible **actions**
- **Interacts** with other players
- Is **rational**

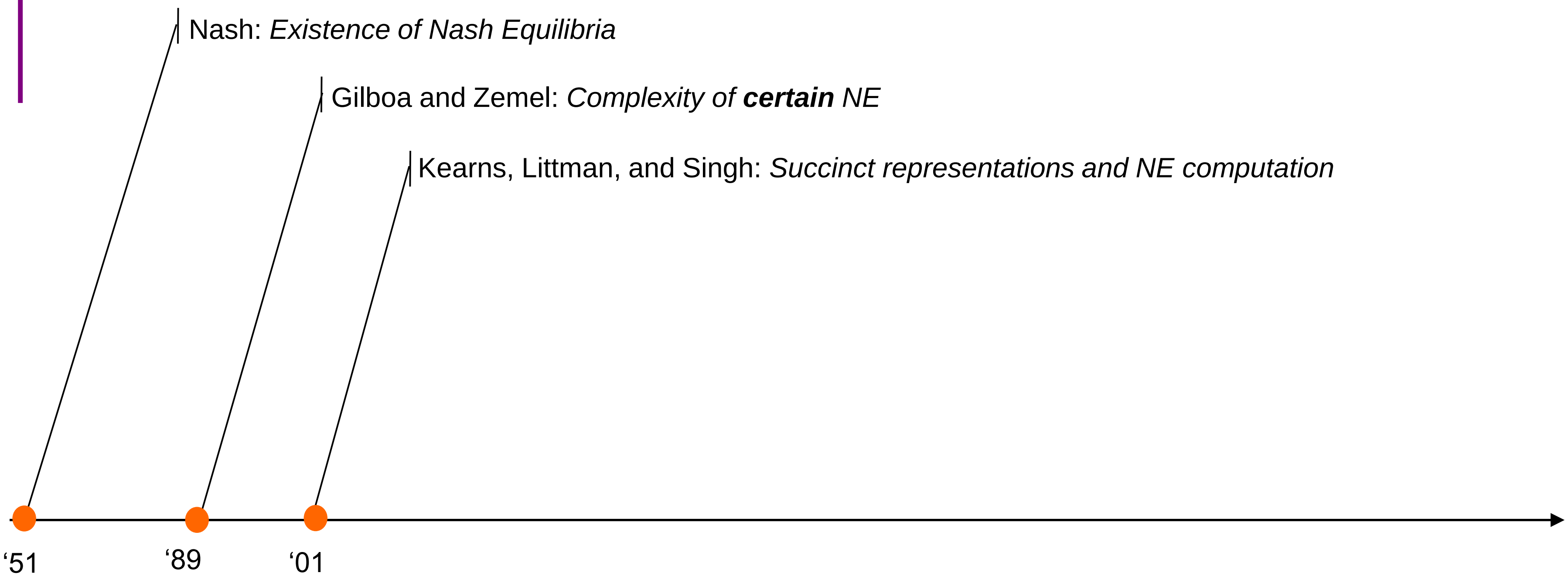
Nash equilibria

---

pure **Nash equilibria**

Every game admits a *mixed* Nash equilibrium,

- **where players chose their strategies according to probability distributions**



Nash: *Existence of Nash Equilibria*
Gilboa and Zemel: *Complexity of **certain** NE*
Kearns, Littman, and Singh: *Succinct representations and NE computation*
'51
'89
'01

Succint Game Representations

- Players:
  - Maria, Francesco
- Choices:
  - movie, opera

If 2 players, then size = 2^2

Maria	Francesco, *movie*	Francesco, *opera*
movie	2	0
opera	0	1

Succint Game Representations

- Players:
  - Maria, Francesco, Paola
- Choices:
  - movie, opera

If 2 players, then size = 2^2

If 3 players, then size = 2^3

Maria	F_{movie} *and* P_{movie}	F_{movie} *and* P_{opera}	F_{opera} *and* P_{movie}	F_{opera} *and* P_{opera}
movie	2	0	2	1
opera	0	1	2	2

Succint Game Representations

- Players:
  - Maria, Francesco, Paola, Roberto, and Giorgio
- Choices:
  - movie, opera

If 2 players, then size = 2^2

If 3 players, then size = 2^3

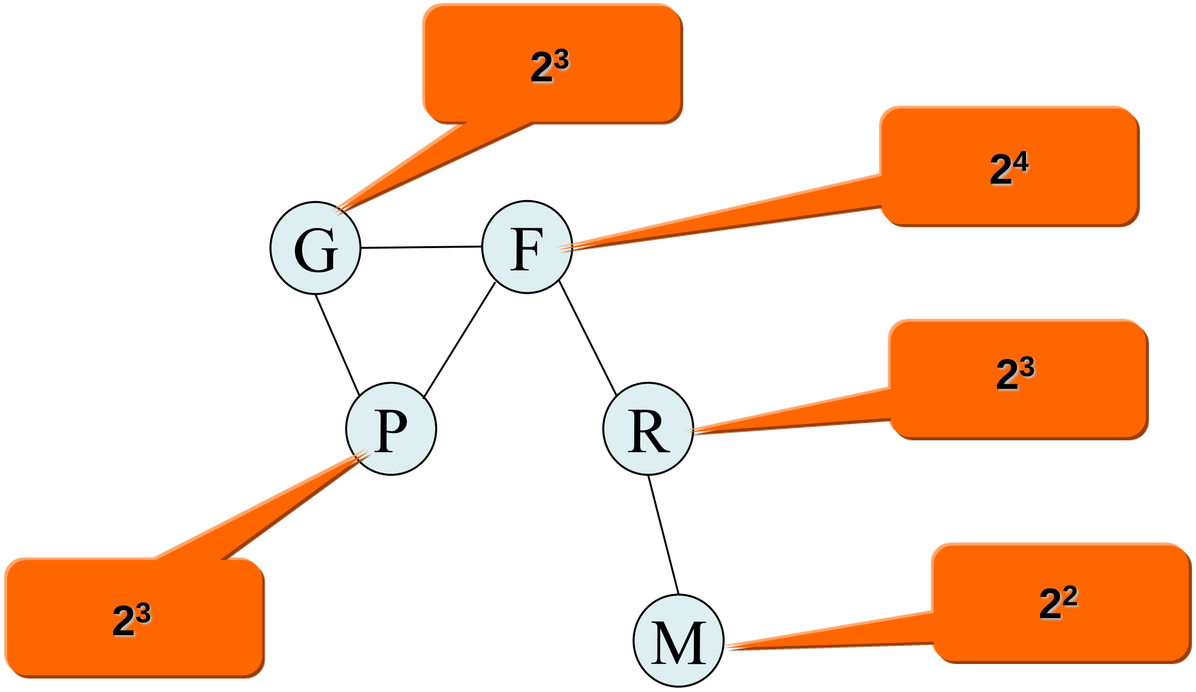
…

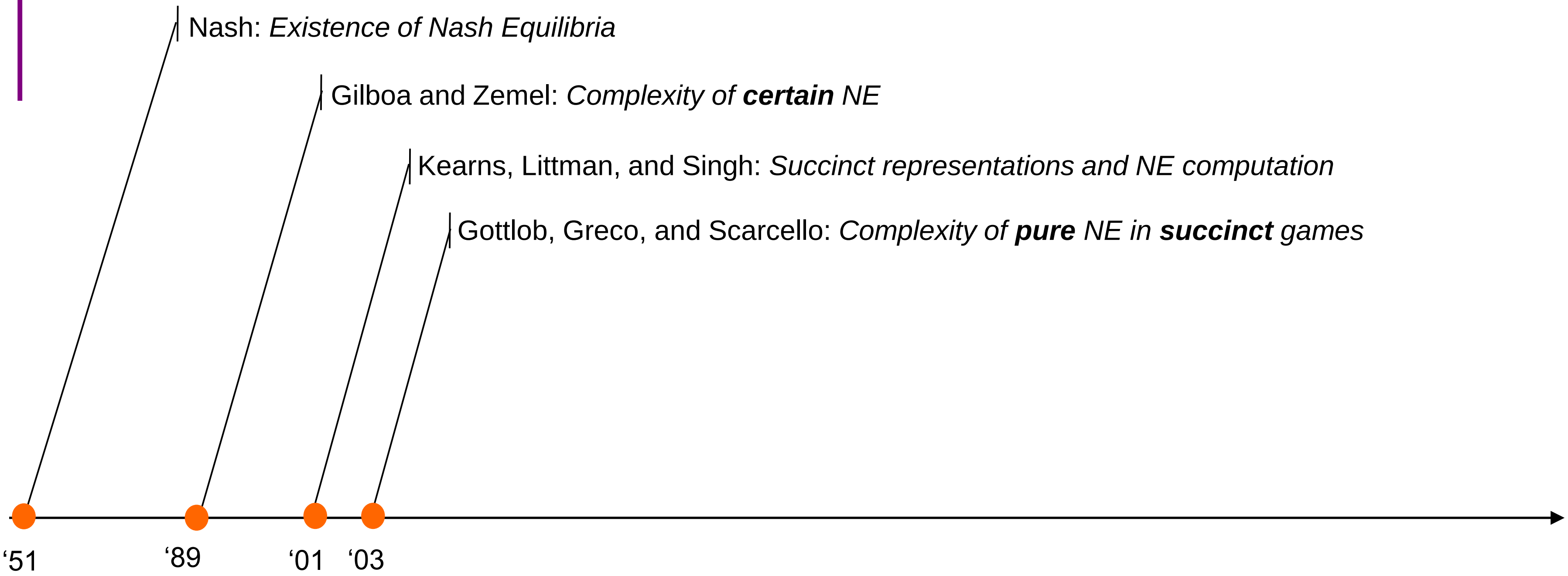
If N players, then size = 2^N

Maria	F_{movie} *and* P_{movie} *and* R_{movie} *and* G_{movie}	…………………………		
movie	2	……..	……..	……..
opera	0	……..	……..	……..

Succinct Game Representations

- Players:
  - Francesco, Paola, Roberto, Giorgio, and Maria
- Choices:
  - movie, opera

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‘51
‘89
‘01
‘03

Complexity of Pure Nash Equilibria$_{(1/3)}$

- Game Representation
  - Tables
  - Arbitrary Functions

Maria	F_{movie} *and* P_{movie}	*and* R_{movie} *and* G_{movie}	…………………………	
movie	2	……..	……..	……..
opera	0	……..	……..	……..

Complexity of Pure Nash Equilibria$_{(1/3)}$

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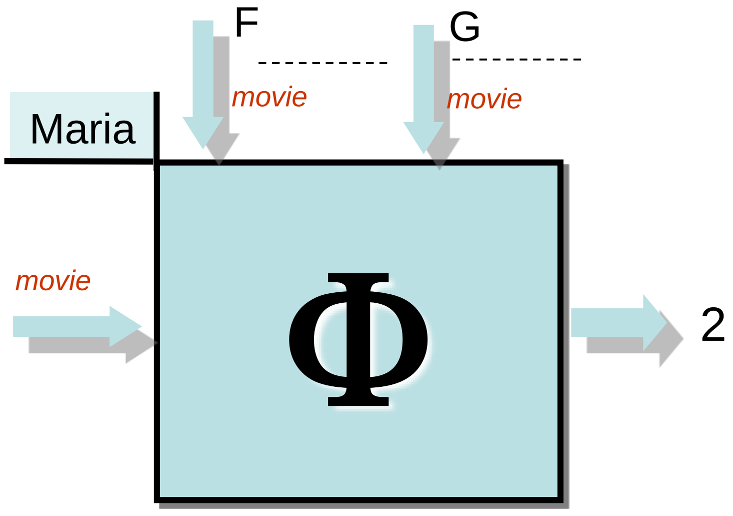

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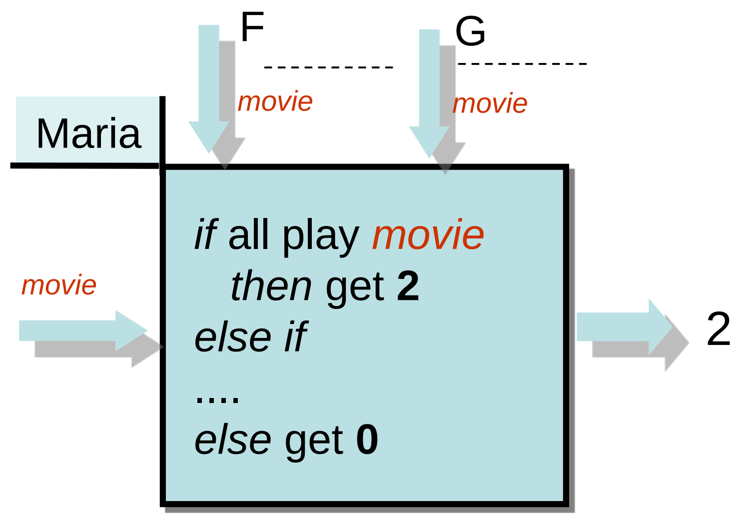

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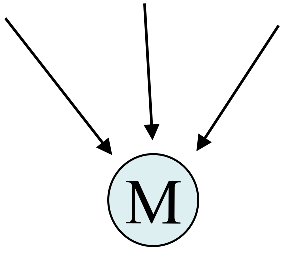
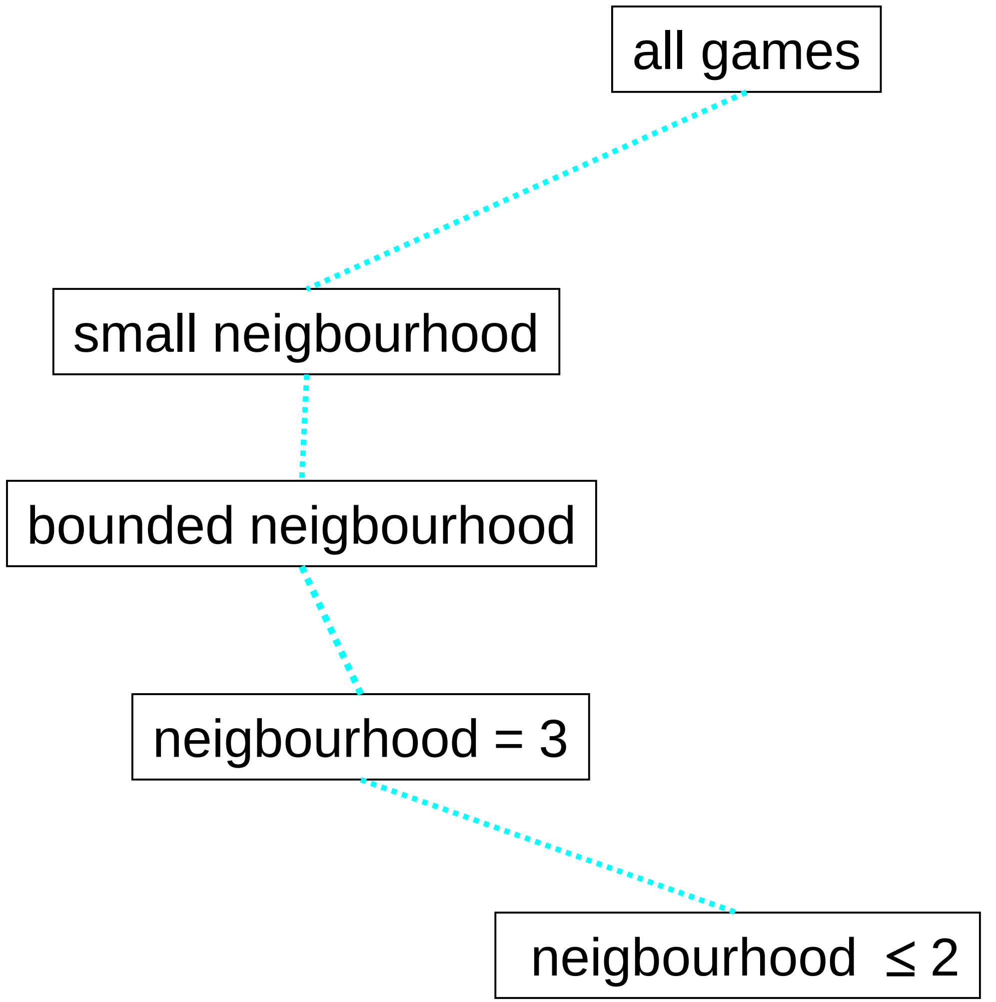
movie	2	……..	……..	……..
opera	0	……..	……..	……..

Complexity of Pure Nash Equilibria$_{(1/3)}$

- Game Representation
  - Tables
  - Arbitrary Functions
- Neighborood
  - Arbitrary
  - Small (i.e., log)
  - Bounded (i.e., constant)



Complexity of Pure Nash Equilibria$_{(3/3)}$



Complexity of Pure Nash Equilibria$_{(3/3)}$

THE BAD NEWS:

NP-c all games

NP-c small neigbourhood

NP-c bounded neigbourhood

NP-c neigbourhood = 3

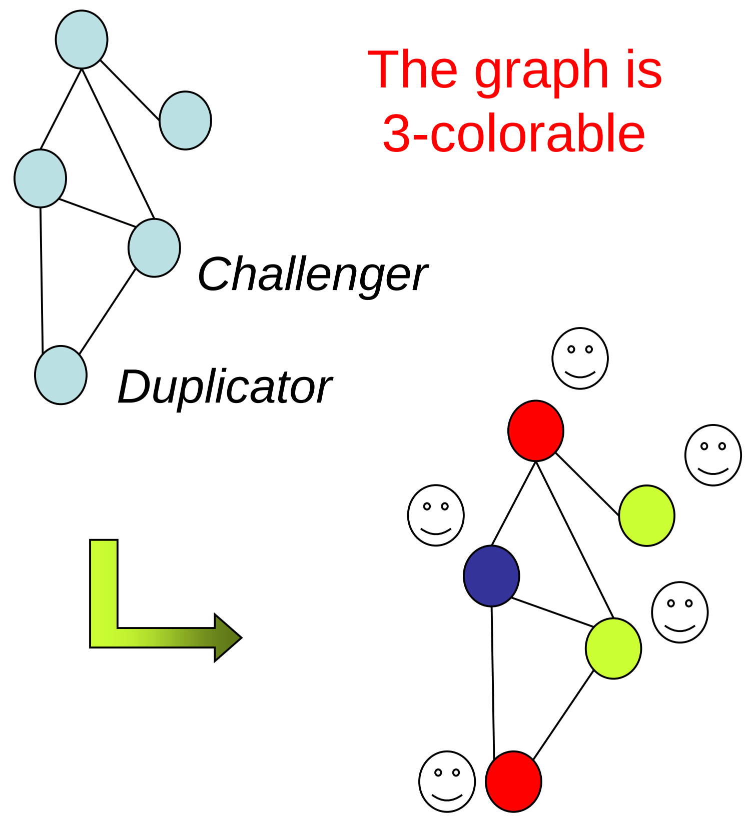
polynomial neigbourhood ≤ 2

Hard Games: NP-hardness

Theorem Deciding whether a game has pure Nash equilibrium is NP-complete. Hardness holds even if the game is in GNF, and if it has 3-bounded neighborhood.

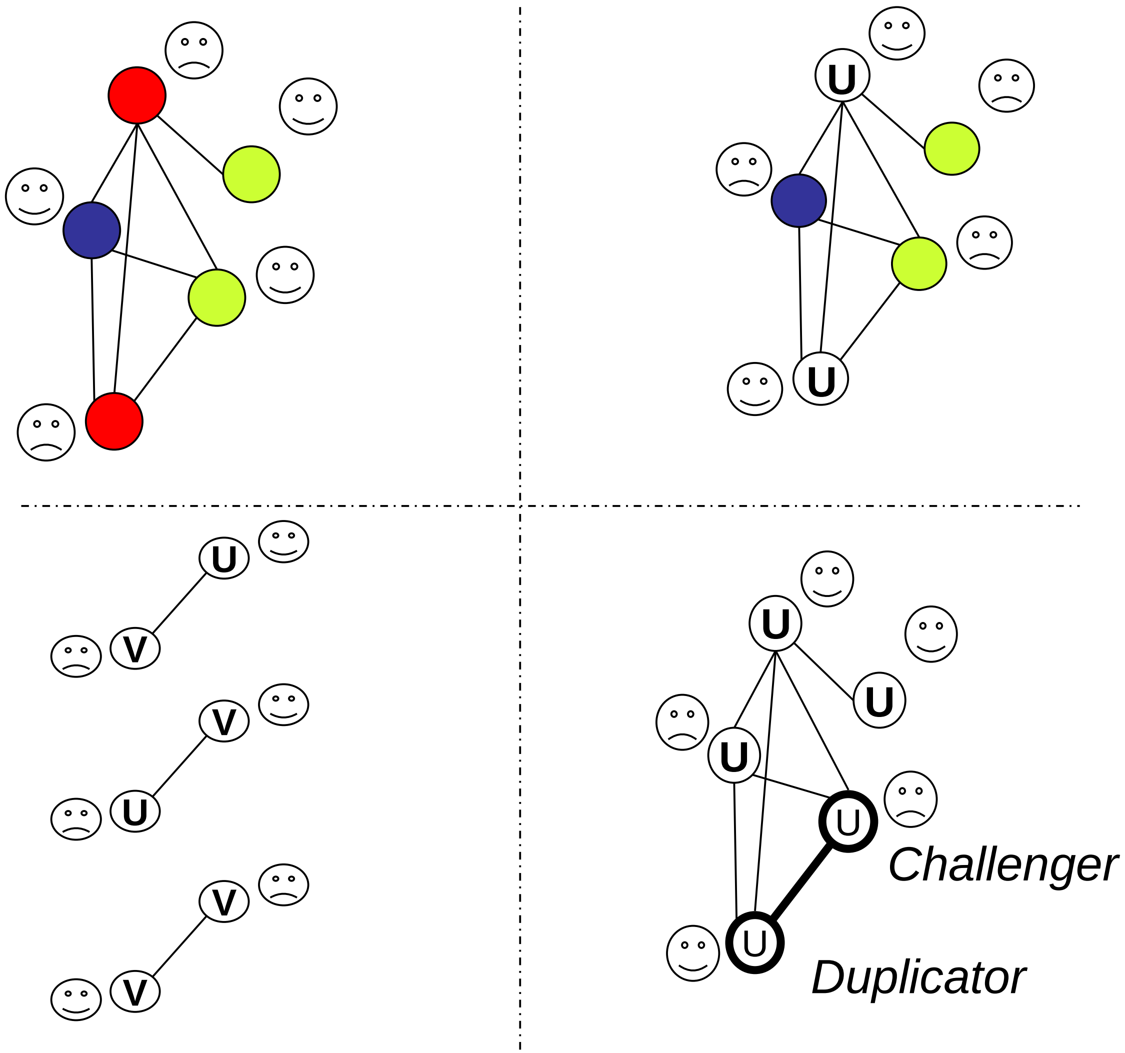
Reduction from 3-colorability

- **Players**: Nodes
- **Actions**: Colors + {U,V}
- **Utility Functions**: the players have an incentive to play a color different from the ones played by the neighbors
- ***Challenger*** and ***Duplicator*** are distinguished (connected) players such that:
  - *C* wants to play an action different from *D*;
  - *D* want to play either a color different from *C*, or the same action in {U,V}.



Hard Games: NP-hardness

The graph is **not** 3-colorable:

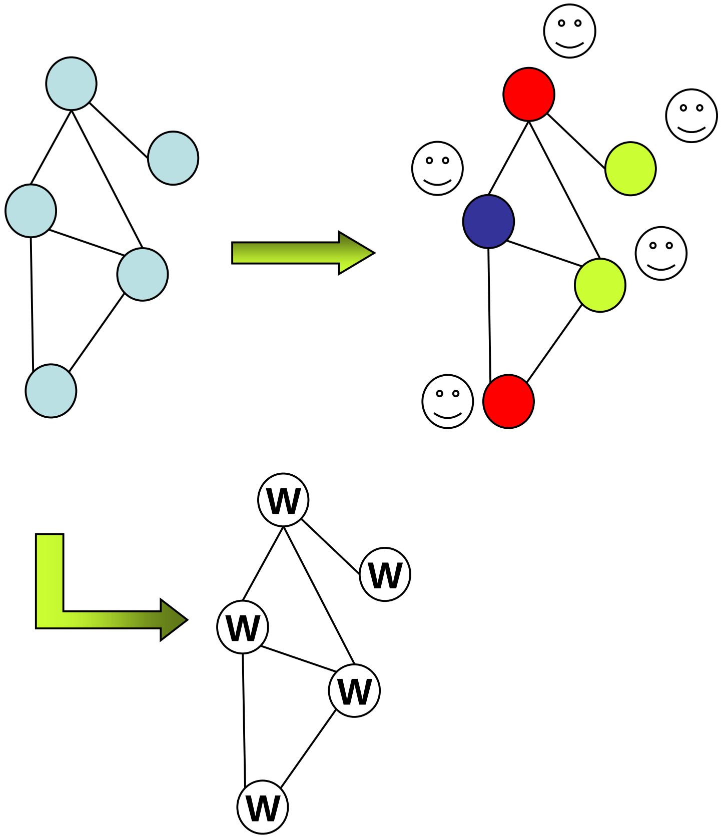


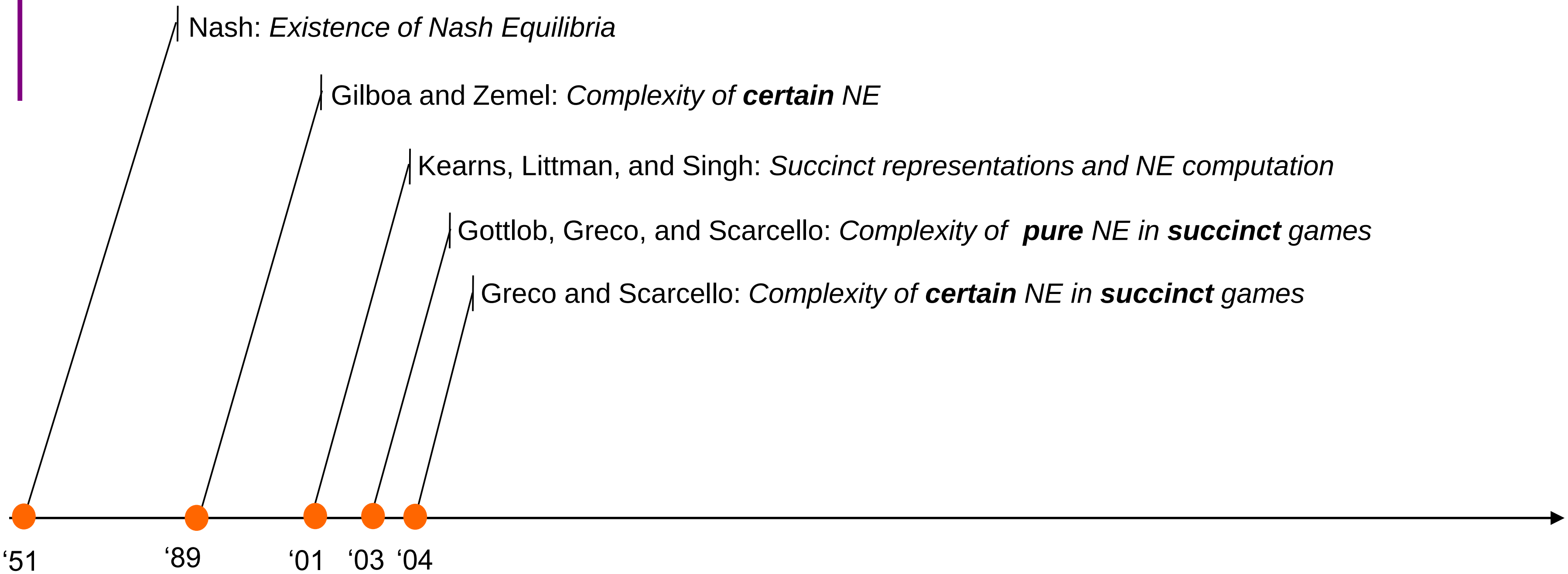
- Adjacent players playing the same color have an incentive to play U.
- The neighbors of players playing U have an incentive to play U, in their turn.
- Challenger want to plays an action different of Duplicator.
- No Nash equilibria exist.

Hard Games: coNP-hardness

Theorem Deciding whether a global strategy **x** is a Pareto (Strong) Nash equilibrium is coNP-complete. Hardness holds even if **x** is a Nash equilibrium, the game is in GNF, and if it has 3-bounded neighborhood.

- **Reduction from 3-non-colorability**
- **The same construction as above except:**
  - Each player may play also W, and has an incentive in such choice if all her neighbors play W, too.
- **There is alway a Nash equilibrium where all players play W**
  - Utility functions are such that it this equilibrium is not preferred to the equilibrium (if any) corresponding to a 3-coloring.
  - This equilibrium is Pareto iff the graph is not 3-colorable.





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'89
'01
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'04

Constrained Nash Equilibria$_{(1/3)}$

Bob	John goes *out*	John stays at *home*
out	1	0
home	0	1

John	Bob goes *out*	Bob stays at *home*
out	0	1
home	1	0

Constrained Nash Equilibria $_{(1/3)}$

- Computing "any" Nash equilibria might not be enough
  - E.g., multi-agent planning, routing protocols, etc.
- What if we ask for "certain types of equilibrium"?
  - *Bob gets at least 1*
  - *The best social welfare*
  - *Maria cannot go to the opera*
  - *…*

Bob	John goes *out*	John stays at *home*
out	1	0
home	0	1

John	Bob goes *out*	Bob stays at *home*
out	0	1
home	1	0

Constrained Nash Equilibria$_{(2/3)}$

Evaluation functions

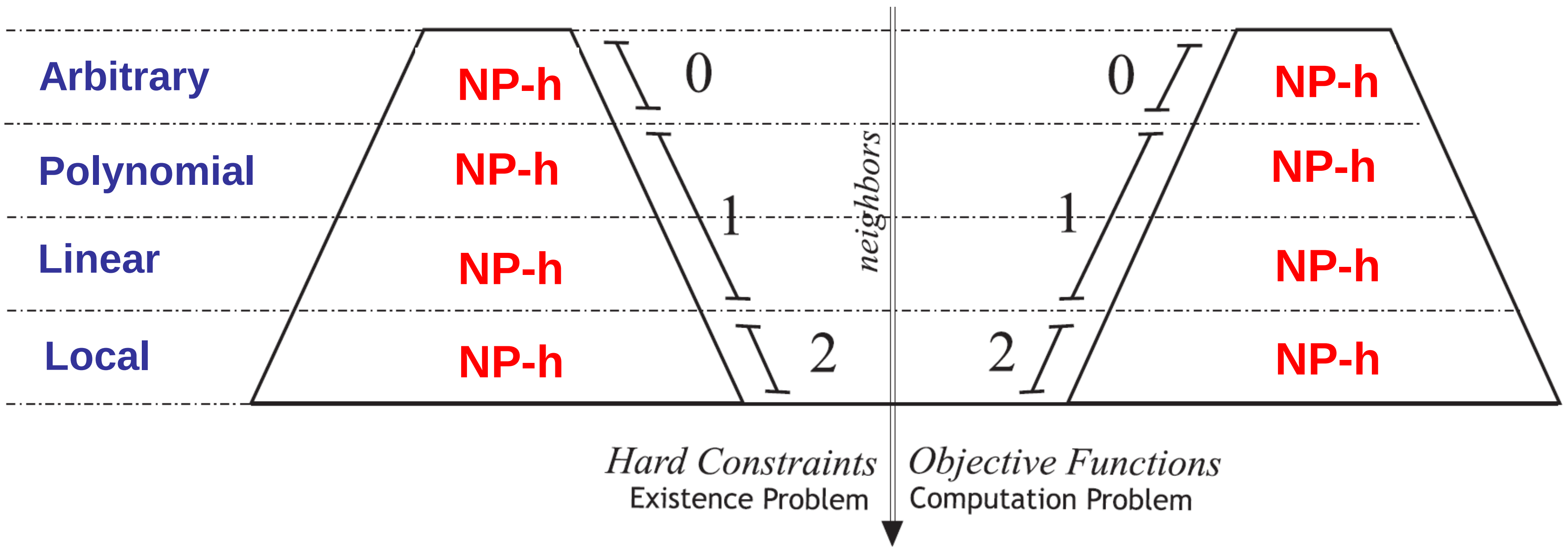
- **F_P:** polynomial-time computable functions, associating real numbers with each combined strategy of players in **P** and their neighbors

- Examples
  - *Let* $A_{\{G,P\}}$ return the minimum payoff between Giorgio and Paola
    - $A_{\{G,P\}} > 1$ is a guarantee for *G* and *P*
  - *Let* $B_{\{F,P,R,G,M\}}$ return the sum of the payoffs of all players
    - By maximizing $B_{\{F,P,R,G,M\}}$, we optimize the social welfare

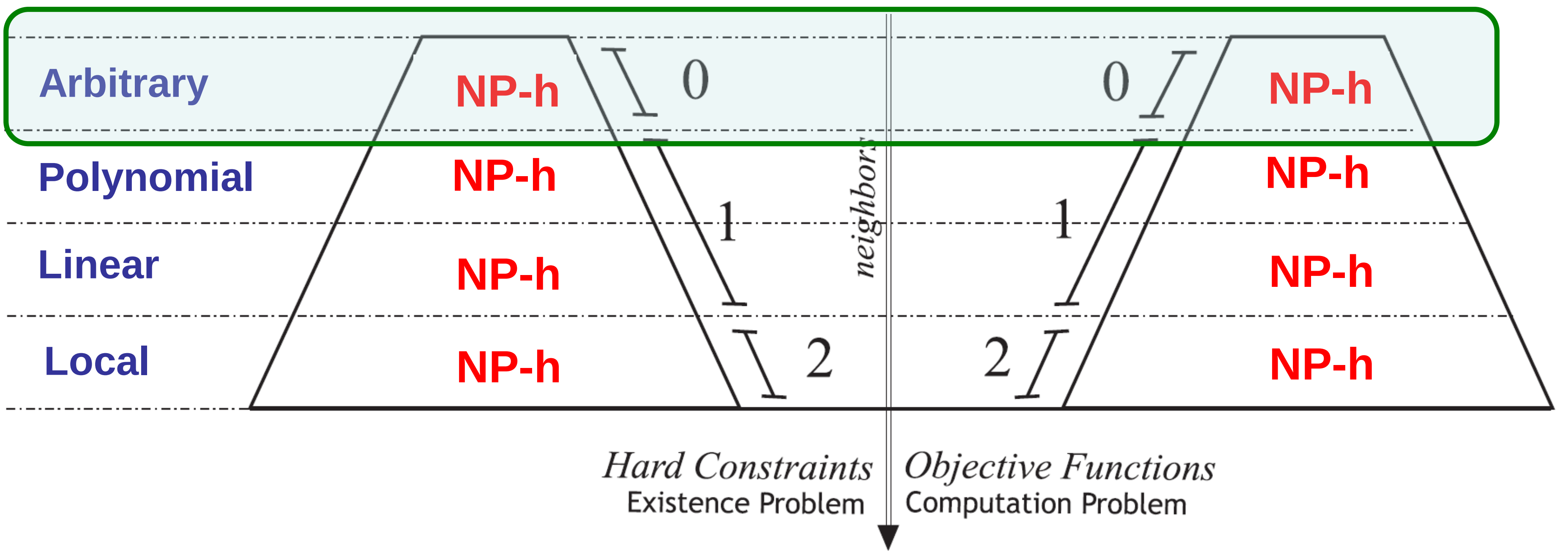
Hard Constraints

Objective Functions

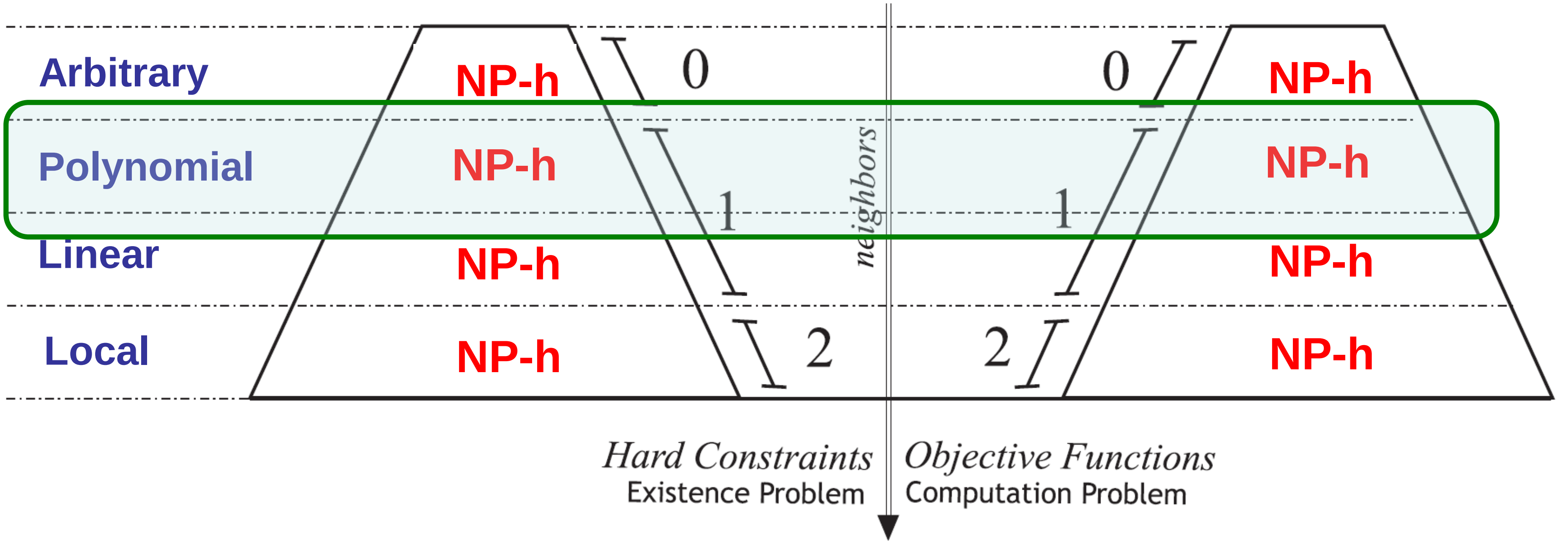
Constrained Nash Equilibria$_{(3/3)}$



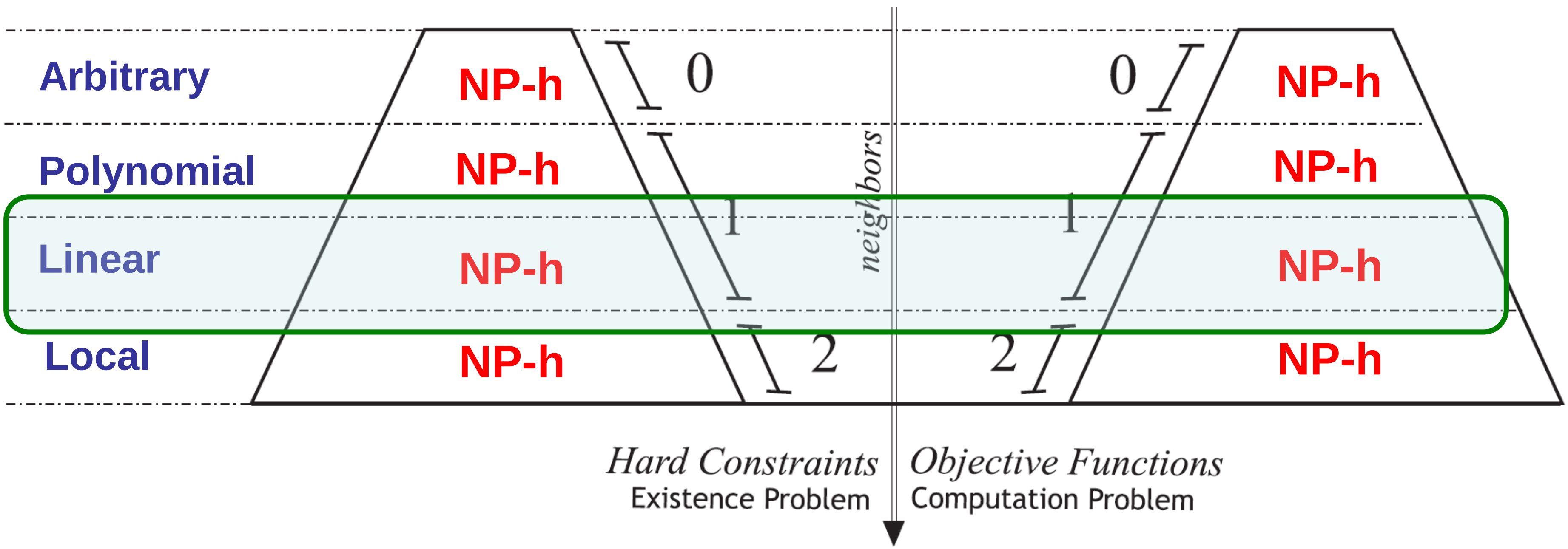
Constrained Nash Equilibria$_{(3/3)}$

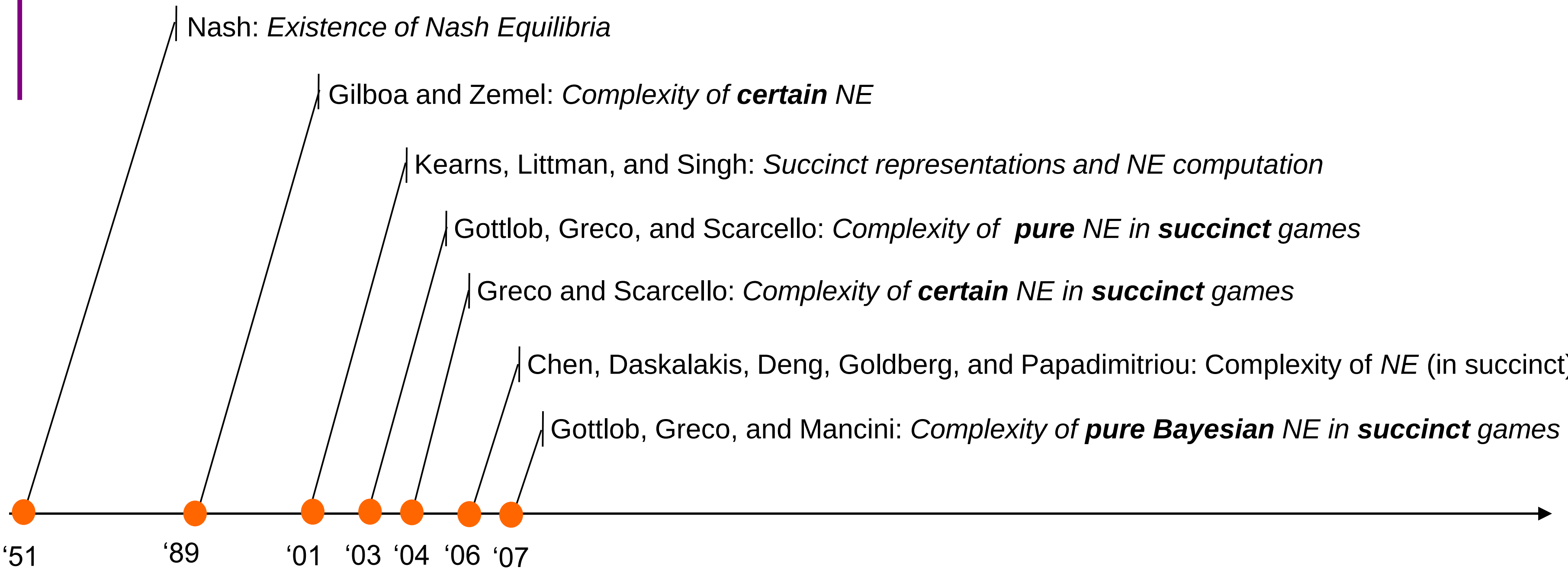


Constrained Nash Equilibria(3/3)



Constrained Nash Equilibria$_{(3/3)}$





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Greco and Scarcello: Complexity of certain NE in succinct games
Chen, Daskalakis, Deng, Goldberg, and Papadimitriou: Complexity of NE (in succinct)
Gottlob, Greco, and Mancini: Complexity of pure Bayesian NE in succinct games
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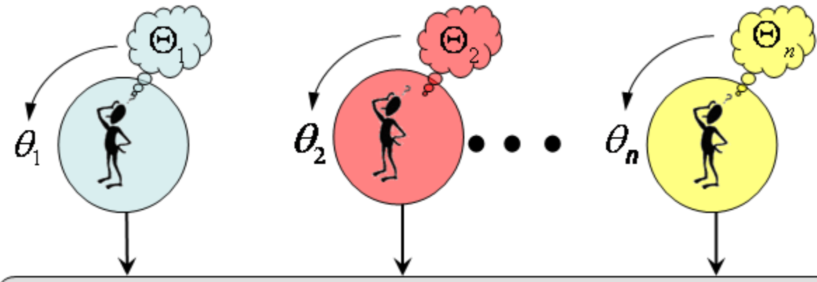
Bayesian Nash Equilibria$_{(1/3)}$



Bayesian Games (Harsany [1968])

- The **Nature** reveals a type
- Each player does not know the type of the other players, but... a probability distribution

Bob	John goes *out*	John stays at *home*
out	2	0
home	0	1

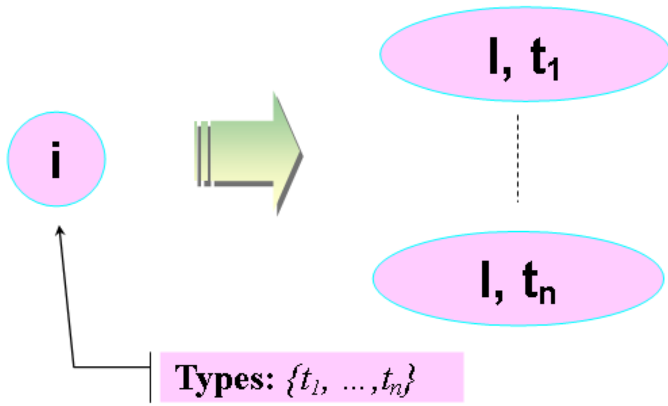
Type 1

Bob	John goes *out*	John stays at *home*
out	0	1
home	1	0

Type 2

Bayesian Nash Equilibria$_{(2/3)}$

Type-agent representation



$$v_{(i,t_i)}(\boldsymbol{a}) = \sum_{\boldsymbol{t}^{-i} \in \bigtimes_{j \in N-\{i\}} T_j} p_i(\boldsymbol{t}^{-i} | t_i)\; u_i\left(\boldsymbol{a}', \langle \boldsymbol{t}^{-i}; t_i \rangle\right)$$

BEs of the original game are in one-to-one correspondence with Nash Equilibria of the new game with complete information.

- The transformation:
  - Is feasible in polynomial time
  - Preserves the neighboord
  - Preserves the structural properties

Easy cases are preserved

Bayesian Nash Equilibria$_{(3/3)}$

PBE is **NP**-*complete, if the number of players is constant.*

PBE is **PP**-*hard, Hardness holds even for games where players have just two types.*

PBE is **PP**-*complete, in the fixed precision setting.*

PP: languages that can be decided by a nondeterministic Turing machine in polynomial time, where the acceptance condition is that a majority (more than half) of computation paths accept.

$\Rightarrow$ NP $\subseteq$ **PP** $\subseteq$ PSPACE

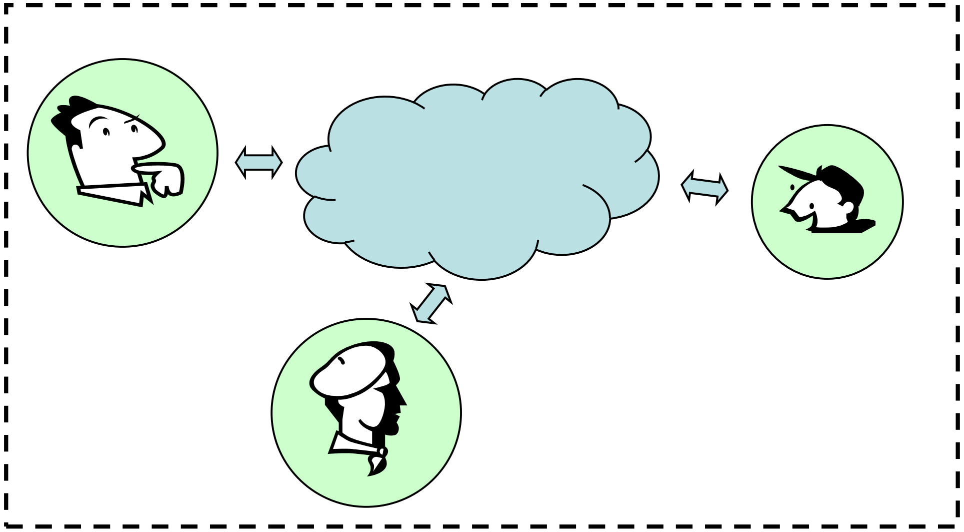
Canonical Problem: MAJ-SAT

Input: a formula Φ

Output: *true* $\Leftrightarrow$ Φ is satisfied by more than half of possible assignments

Part III: Nucleolus

Game Theory (in a Nutshell)

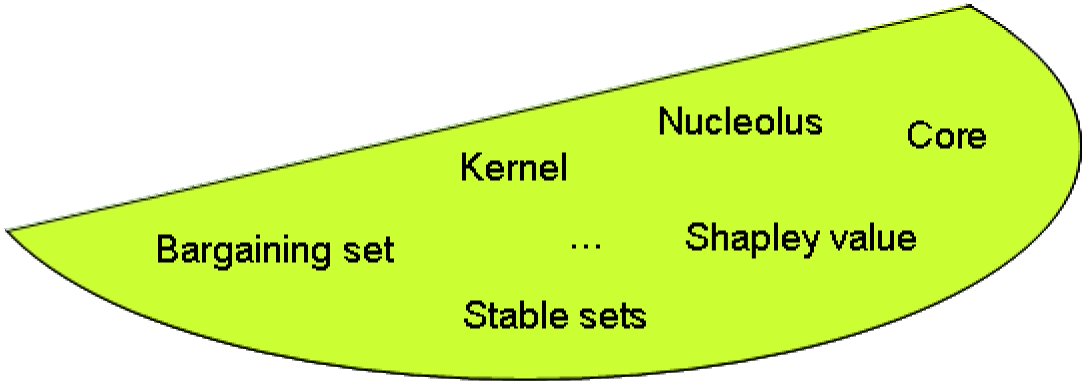


Each player:

- Has a **goal** to be achieved
- Has a set of possible **actions**
- **Interacts** with other players
- Is **rational**

Which actions have to be performed?

Solution Concepts



Cooperative Game Theory(1/2)

To perform some task

Utility distribution, if the task is performed

Jointly perform the task (with some cost)

Each player:

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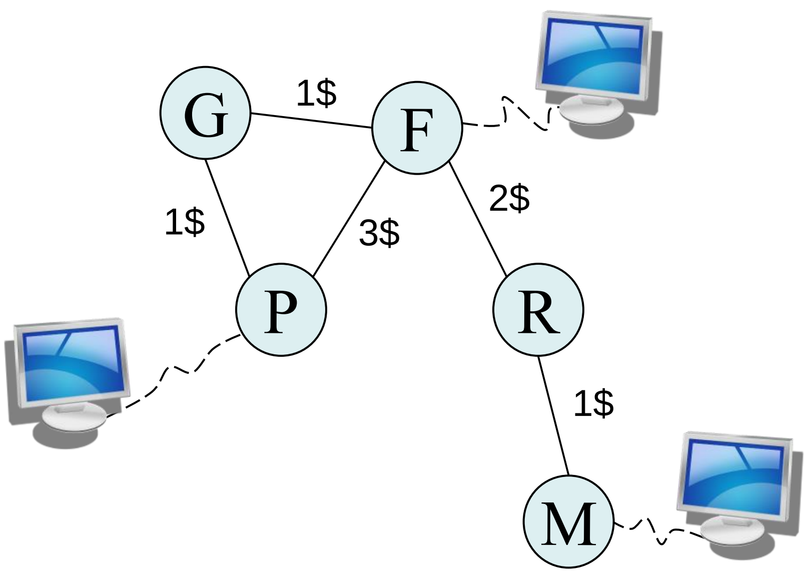

- Players get **9$**, if they enforce connectivity
- Enforcing connectivity over an edge as a cost

Cooperative Game Theory$_{(1/2)}$

To perform some task

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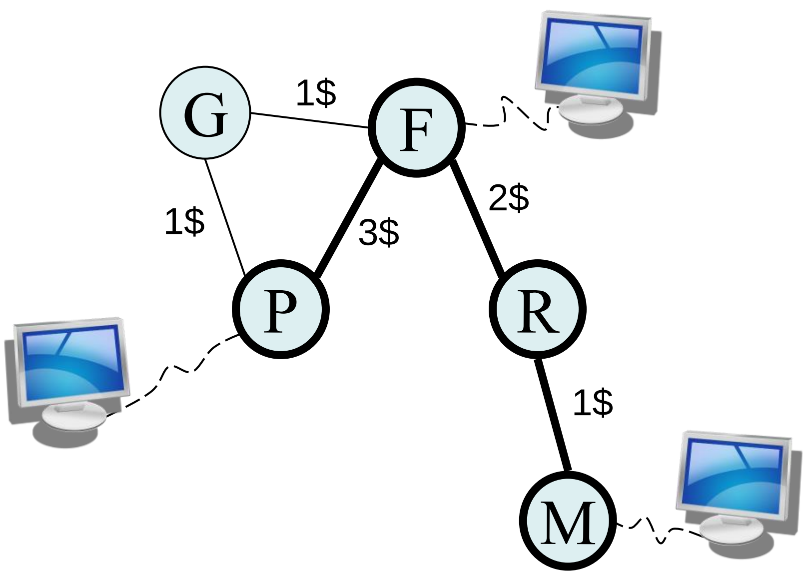

- Players get **9$**, if they enforce connectivity
- Enforcing connectivity over an edge as a cost

Coalition {F,P,R,M} gets **9$**, and pays **6$**

worth v({F,P,R,M}) = 9$ - 6$

Cooperative Game Theory$_{(1/2)}$

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Utility distribution, if the task is performed

Jointly perform the task (with some cost)

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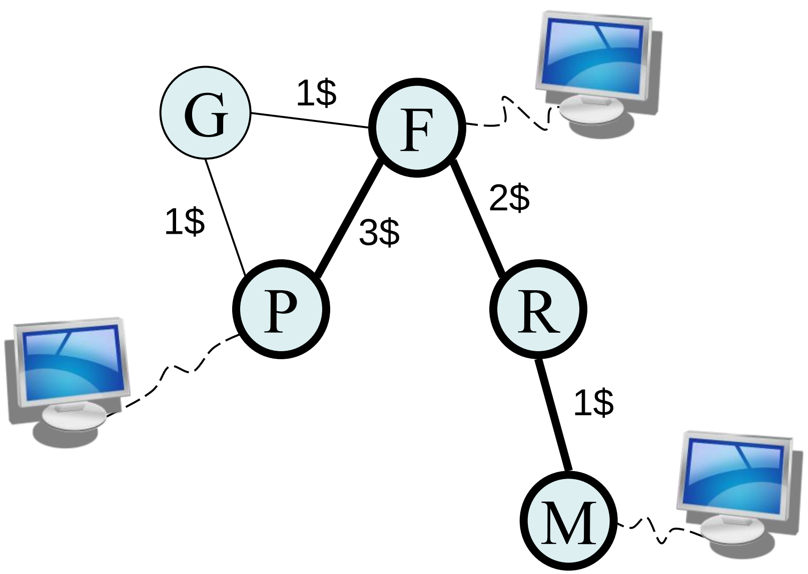
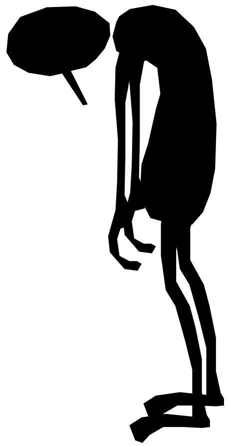


coalition	worth
{F}	0
…	0
{G,P,R,M}	0
{F,P,R,M}	**3**
{G,F,P,R,M}	4

How to distribute 9$, based on such worths?

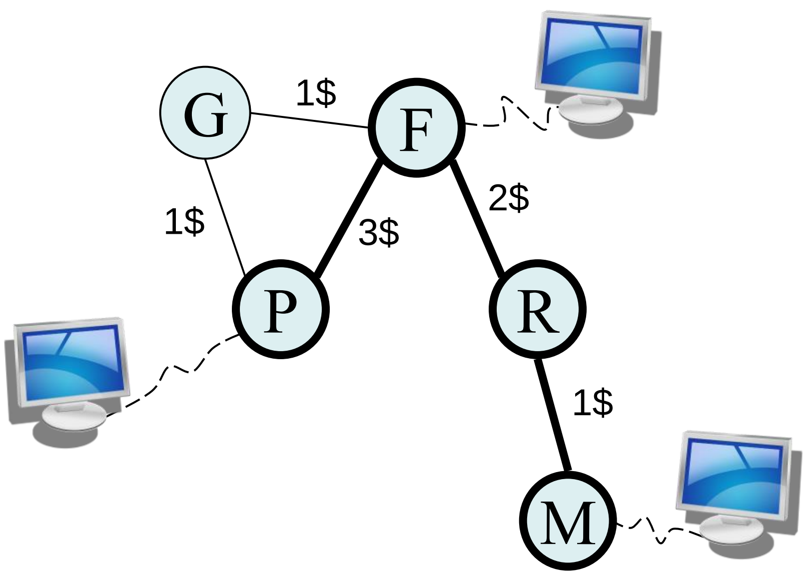
Cooperative Game Theory$_{(2/2)}$



fairness

Each player:

- Has a **goal** to be achieved
- Has a set of possible **actions**
- **Interacts** with other players
- Is **rational**



coalition	worth
{F}	0
…	0
{G,P,R,M}	0
{F,P,R,M}	**3**
{G,F,P,R,M}	4

How to distribute 9$, based on such worths?

The Model

- Players form *coalitions*
- Each coalition is associated with a *worth*
- A *total worth* has to be distributed

$$\mathcal{G} = \langle N, v \rangle,\ v : 2^N \mapsto \mathbb{R}$$

- Outcomes belong to the imputation set $X(\mathcal{G})$

$$x \in X(\mathcal{G}) \quad \begin{cases} \text{Efficiency} \\ x(N) = v(N) \\ \text{Individual Rationality} \\ x_i \geq v(\{i\}), \quad \forall i \in N \end{cases}$$

- *Efficiency*

$$x(N) = v(N)$$

- *Individual Rationality*

$$x_i \geq v(\{i\}), \quad \forall i \in N$$

The Model

- Players form *coalitions*
- Each coalition is associated with a *worth*
- A *total worth* has to be distributed

$$\mathcal{G} = \langle N, v \rangle,\ v : 2^N \mapsto \mathbb{R}$$

- **Solution Concepts** characterize outcomes in terms of
  - Fairness
  - Stability

Excess…

- How fairness/stability can be measured?

$$e(S,x) = v(S) - x(S)$$

- The excess is a measure of the dissatisfaction of S

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$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\})) = v(\{1,3\}) = v(\{2,3\}) = 1$$
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Excess…

- How fairness/stability can be measured?

$$e(S, x) = v(S) - x(S)$$

- The excess is a measure of the dissatisfaction of S

$x = (0, 0, 3) \longrightarrow e(\{1, 2\}, x) = v(\{1, 2\}) - (x_1 + x_2) = 1 - 0 = 1$

$x = (1, 2, 0) \longrightarrow e(\{1, 2\}, x) = v(\{1, 2\}) - (x_1 + x_2) = 1 - 3 = -2$

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
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…and the Nucleolus

- Arrange excess values in non-increasing order

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$x = (1, 2, 0)$ $\qquad\qquad$ $\theta(x) = (0, 0, -1, -1, -2, -2)$

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Core Imputation

$x = (1, 2, 0)$ $\theta(x) = (0, 0, -1, -1, -2, -2)$

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…and the Nucleolus

- Arrange excess values in non-increasing order

$$x^* = (1, 1, 1) \qquad \theta(x^*) = (-1, -1, -1, -1, -1, -1)$$

$$\boxed{x = (1, 2, 0) \qquad \theta(x) = (0, 0, -1, -1, -2, -2)}$$

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…and the Nucleolus

- Arrange excess values in non-increasing order

$$\boxed{x^* = (1,1,1) \qquad \theta(x^*) = (-1,-1,-1,-1,-1,-1)}$$

$$x = (1,2,0) \qquad \theta(x) = (0,0,-1,-1,-2,-2)$$

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
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…and the Nucleolus

- Arrange excess values in non-increasing order

Definition [Schmeidler]

The *nucleolus* $\mathcal{N}(\mathcal{G})$ of a game $\mathcal{G}$ is the set
$\mathcal{N}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid \nexists y \in X(\mathcal{G}) \text{ s.t. } \theta(y) \prec \theta(x)\}$

$$\boxed{x^* = (1, 1, 1) \qquad \theta(x^*) = (-1, -1, -1, -1, -1, -1)}$$

$$x = (1, 2, 0) \qquad \theta(x) = (0, 0, -1, -1, -2, -2)$$

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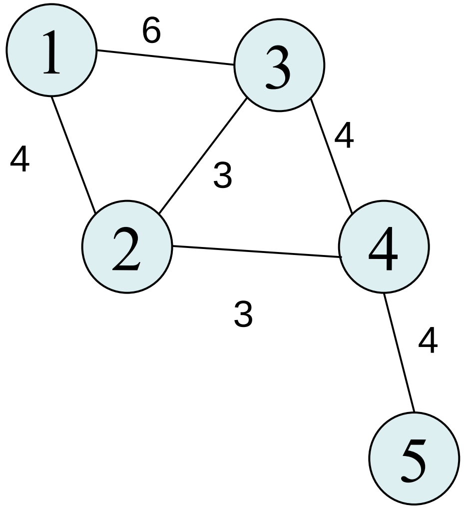
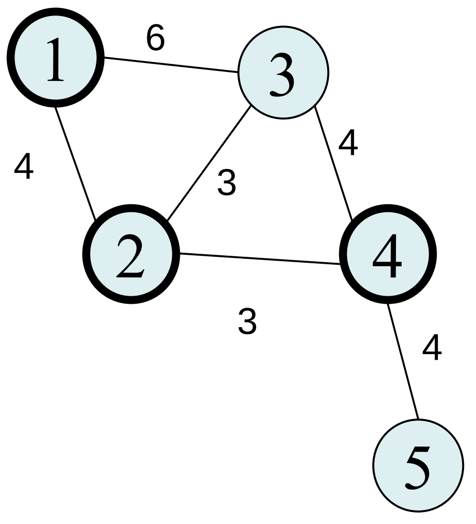
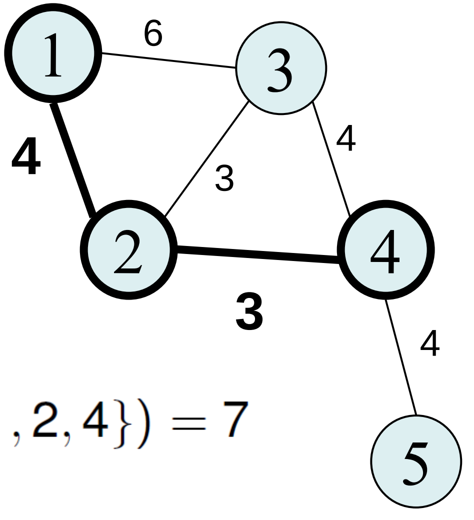
Graph Games



Graph Games

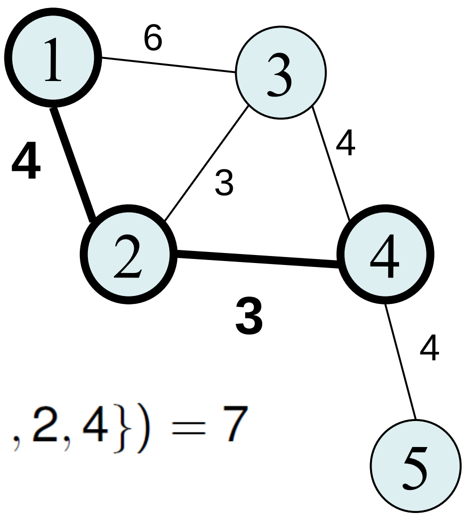


Graph Games



$v(\{1,2,4\}) = 7$

Graph Games

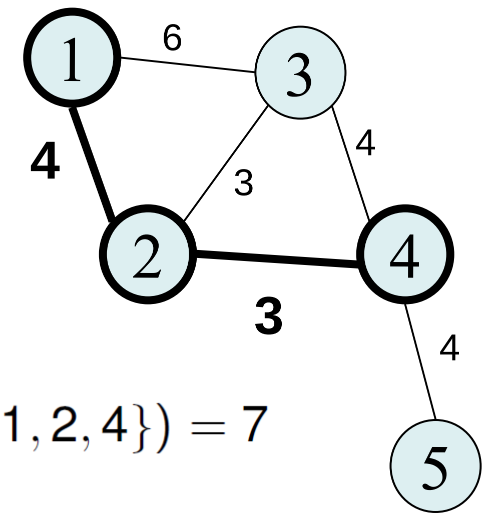


$v(\{1,2,4\}) = 7$

- *Graph Games* [Deng and Papadimitriou, 1994]
  - Computational issues of several solution concepts
  - The *(pre)nucleolus* can be computed in **P**

$$x_i^* = \frac{1}{2}\sum_{j \neq i} w_{i,j}$$

Graph Games



$v(\{1,2,4\}) = 7$

- *Graph Games* [Deng and Papadimitriou, 1994]
  - Computational issues of several solution concepts
  - The *(pre)nucleolus* can be computed in **P**

$$x_i^* = \frac{1}{2}\sum_{j \neq i} w_{i,j}$$

- *Cost allocation on trees* [Megiddo, 1978]
  - Polynomial time algorithm
- *Flow games* [Deng, Fang, and Sun, 2006]
  - Polynomial time algorithm on simple networks (unitary edge capacity)
  - **NP**-hard, in general
- *Weighted voting games* [Elkind and Pasechnik, 2009]
  - Pseudopolynomial algorithm

Kopelowitz, 1967

$$\mathtt{LP}_1 \begin{cases} \min \epsilon_1 & \\ \quad e(S,x) \leq \epsilon_1 & \forall S \subset N, S \notin W_0 = \{\varnothing\} \\ \quad x \in X(\mathcal{G}) & \end{cases}$$

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$$\mathrm{LP}_2 \begin{cases} \min \epsilon_2 & \\ e(S,x) = \epsilon_1^* & \forall S \in W_1 \\ e(S,x) \leq \epsilon_2 & \forall S \subset N, S \notin (W_0 \cup W_1) \\ x \in X(\mathcal{G}) & \end{cases}$$

where:

- $V_1 = \{x \mid (x, \epsilon_1^*)$ is an optimal solution to $\mathrm{LP}_1\}$
- $W_1 = \{S \subseteq N \mid e(S,x) = \epsilon_1^*,$ for every $x \in V_1\}$

Kopelowitz, 1967

$$\mathtt{LP}_k \left\{ \begin{array}{ll} \min \epsilon_k & \\ \quad e(S,x) = \epsilon_r^* & \forall S \in W_r, r \in \{1,\ldots,k-1\} \\ \quad e(S,x) \le \epsilon_k & \forall S \subset N, S \notin (W_0 \cup \cdots \cup W_{k-1}) \\ \quad x \in X(\mathcal{G}) & \end{array} \right.$$

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How many iterations?

An Example Computation

$N = 1, ..., n, n+1, n+2$

$v(N) = n + 2$
$v(\{i\}) = 1,\ i \in \{1, ..., n\}$
$v(\{1, ..., n\}) = n$
$v(\{n+1\}) = v(\{n+2\}) = 0$
$v(\{n+1, n+2\}) = 2$
$v(S) = -\infty,\ |\{n+1, n+2\} \cap S| \geq 1,$
$|\{1, ..., n\} \cap S| \geq 1,\ S \neq N$

$S_1, S_2, ... \subset \{1, ..., n\}$ $|S_i| > 1$
$v(S_i) = |S_i| - 1 + 2^{-i}$

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$\epsilon_1^* = 0$

$x^* = (1, \ldots, 1, x_{n+1}^*, x_{n+2}^*)$

$$\mathrm{LP}_1 \left\{ \begin{array}{l} \min \epsilon_1 \\ \quad n - x(\{1, ..., n\}) \leq \epsilon_1 \\ \quad 2 - x_{n+1} - x_{n+2} \leq \epsilon_1 \\ \quad x(\{1, ..., n\}) + x_{n+1} + x_{n+2} = n+2 \\ \quad x_i \geq 1,\ i \in \{1, ..., n\} \\ \quad \vdots \end{array} \right.$$

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The excess is constant

$$e(S_i, x^*) = v(S_i) - x^*(S_i) = -1 + 2^{-i}$$

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$e(S_i, x^*) \leq \epsilon_2$
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An Example Computation

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The excess is constant

$$e(S_i, x^*) = v(S_i) - x^*(S_i) = -1 + 2^{-i}$$

$$e(S_i, x^*) \leq \epsilon_3$$
$$\epsilon_2^* = -1 + 2^{-1} > \epsilon_3^* = -1 + 2^{-2}, \ldots >$$

Kopelowitz, 1967

$$\mathrm{LP}_k \left\{ \begin{array}{ll} \min \epsilon_k & \\ \quad e(S,x) = \epsilon_r^* & \forall S \in W_r, r \in \{1,\ldots,k-1\} \\ \quad e(S,x) \le \epsilon_k & \forall S \subset N, S \notin (W_0 \cup \cdots \cup W_{k-1}) \\ \quad x \in X(\mathcal{G}) & \end{array} \right.$$

where:

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How many iterations?

Kopelowitz, 1967

$$\mathtt{LP}_k \left\{ \begin{array}{ll} \min \epsilon_k & \\ \quad e(S,x) = \epsilon_r^* & \forall S \in W_r, r \in \{1,\ldots,k-1\} \\ \quad e(S,x) \leq \epsilon_k & \forall S \subset N, S \notin (W_0 \cup \cdots \cup W_{k-1}) \\ \quad x \in X(\mathcal{G}) & \end{array} \right.$$

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Theorem

The algorithm performs $\Omega(2^n)$ steps, in some cases.

cf. Mashler, Peleg, and Shapley, 1979

$$
\mathtt{LP}_k \begin{cases} \min \epsilon_k \\ e(S,x) = \epsilon_r^* & \forall S \in W_r, r \in \{1,\ldots,k-1\} \\ e(S,x) \leq \epsilon_k & \forall S \subset N, S \notin (W_0 \cup \cdots \cup W_{k-1}) \\ x \in X(\mathcal{G}) \end{cases}
$$

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$$\boxed{\{S \subseteq N \mid x(S) = y(S), \forall x, y \in V_{k-1}\}}$$

cf. Mashler, Peleg, and Shapley, 1979

$$
\mathrm{LP}_k \begin{cases}
\min \epsilon_k \\
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x \in X(\mathcal{G})
\end{cases}
$$

where:

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$$\boxed{\{S \subseteq N \mid x(S) = y(S), \forall x, y \in V_{k-1}\}}$$

[Kern and Paulusuma, 2003]

LP Approaches over Compact Games

$$\mathrm{LP}_k \begin{cases} \min \epsilon_k \\ \quad e(S,x) = \epsilon_r^* \qquad \forall S \in W_r, r \in \{1,\ldots,k-1\} \\ \quad e(S,x) \leq \epsilon_k \qquad \forall S \subset N, S \notin \mathcal{F}_{k-1} \\ \quad x \in X(\mathcal{G}) \end{cases}$$

where:

- $V_r = \{x \mid (x, \epsilon_r^*) \text{ is an optimal solution to } \mathrm{LP}_r\}$
- $W_r = \{S \subseteq N \mid e(S,x) = \epsilon_r^*, \text{ for every } x \in V_r\}$
- $\mathcal{F}_{k-1} = \{S \subseteq N \mid x(S) = y(S), \forall x, y \in V_{k-1}\}$

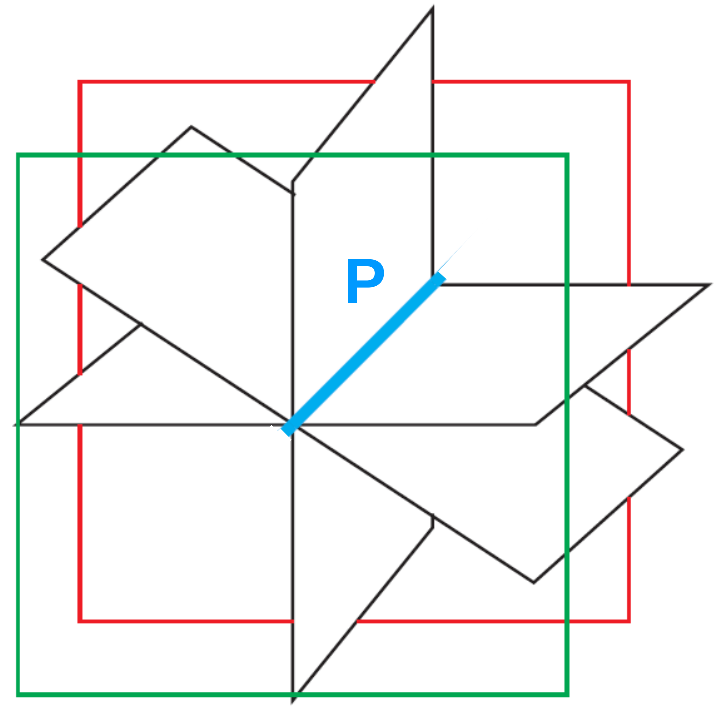
- In compact games, two problems have to be faced:
  - **(P1)** Sets W and $\mathcal{F}$ contain exponentially many elements, but we would like to avoid listing them explicitly
  - **(P2)** Translate LP (complexity) results to "succinct programs"

(P1): A Convenient Representation

$W \qquad \mathcal{F}$

fixed inequalities

equalities + implied equalities

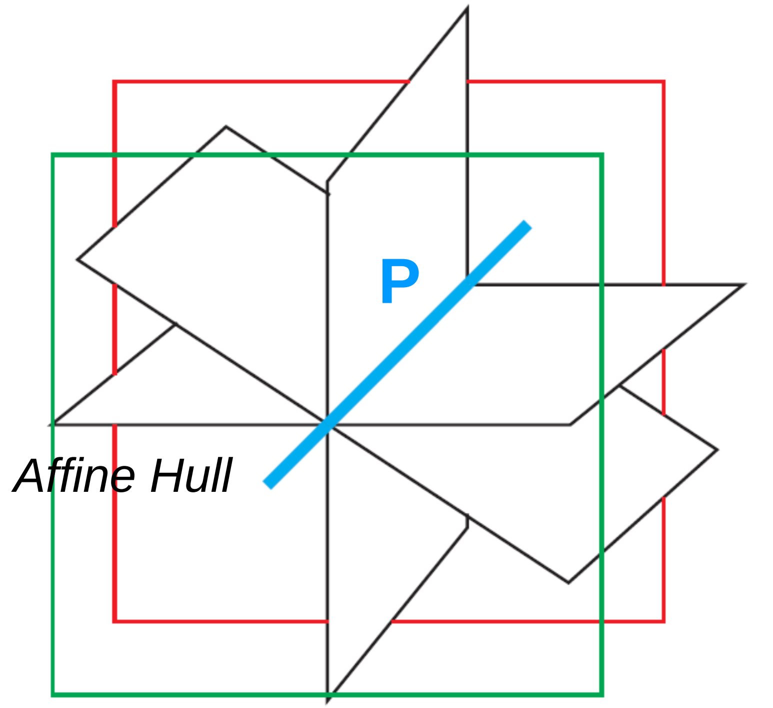


(P1): A Convenient Representation

W $\mathcal{F}$

fixed inequalities

equalities + implied equalities



Theorem

- $aff.hull(V_k) =$ *solutions for equalities over* $W_k \cup W_{k-1} \cup \cdots \cup W_1$

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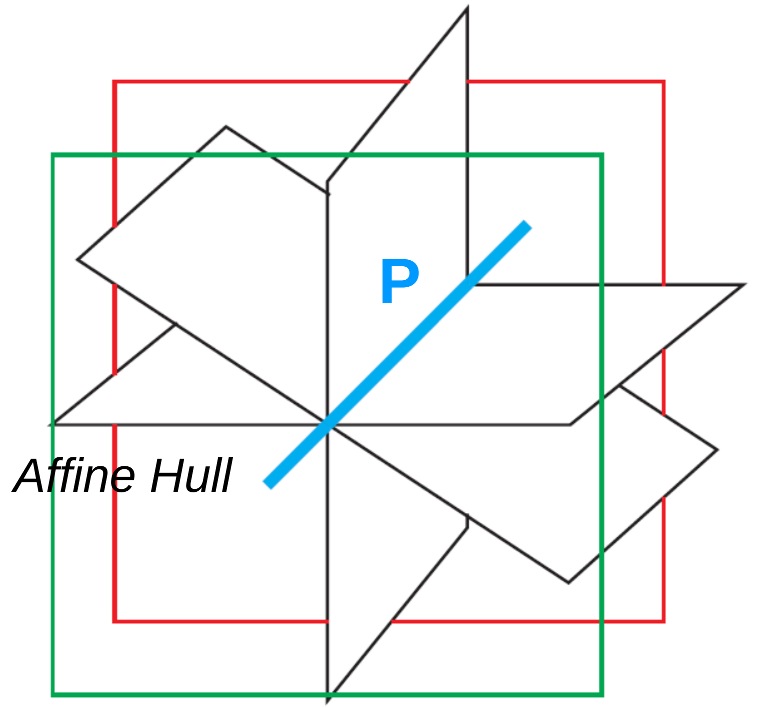

Theorem

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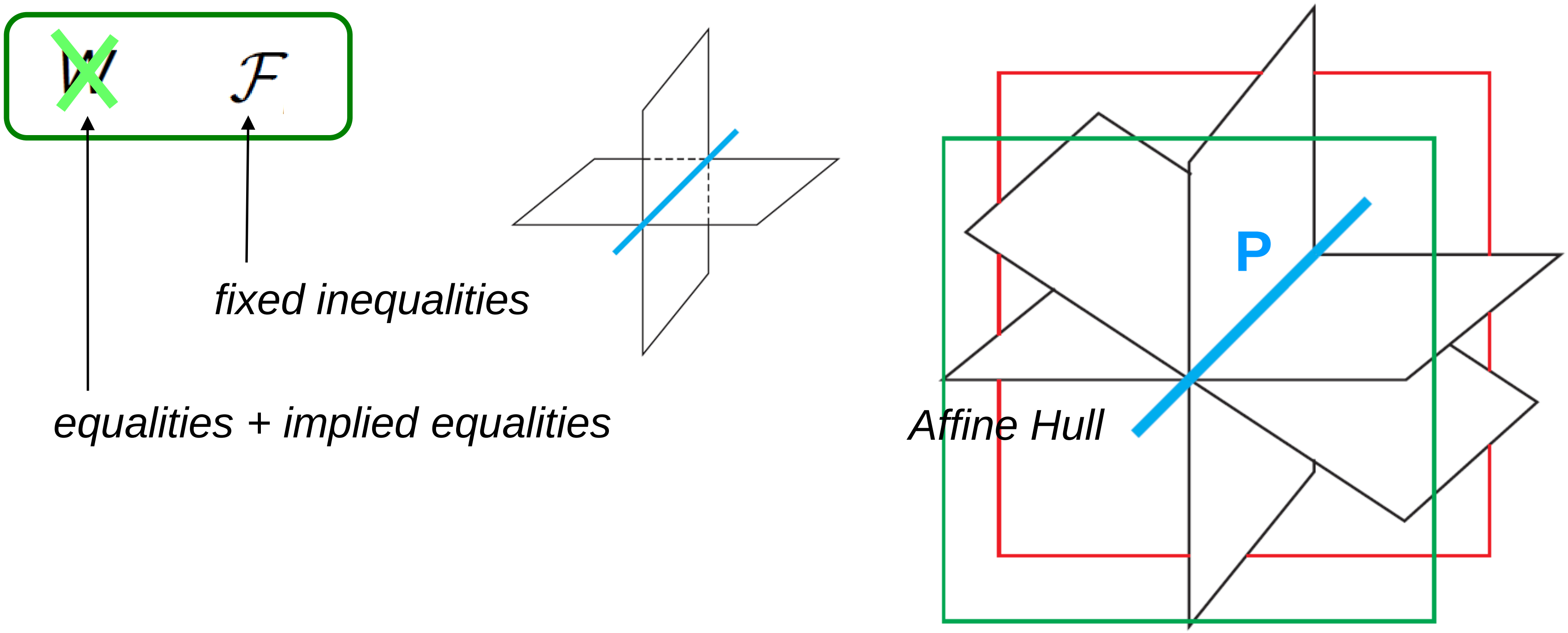
  $\{S \subseteq N \mid e(S, x) = \epsilon_k^*, \text{ for every } x \in V_k\}$

  Implied equalities

(P1): A Convenient Representation



Theorem

- $aff.hull(V_k) = $ *solutions for equalities over* $W_k \cup W_{k-1} \cup \cdots \cup W_1$
- *A basis* $\mathcal{B}_k$ *for* $aff.hull(V_k)$ *contains n vectors at most*

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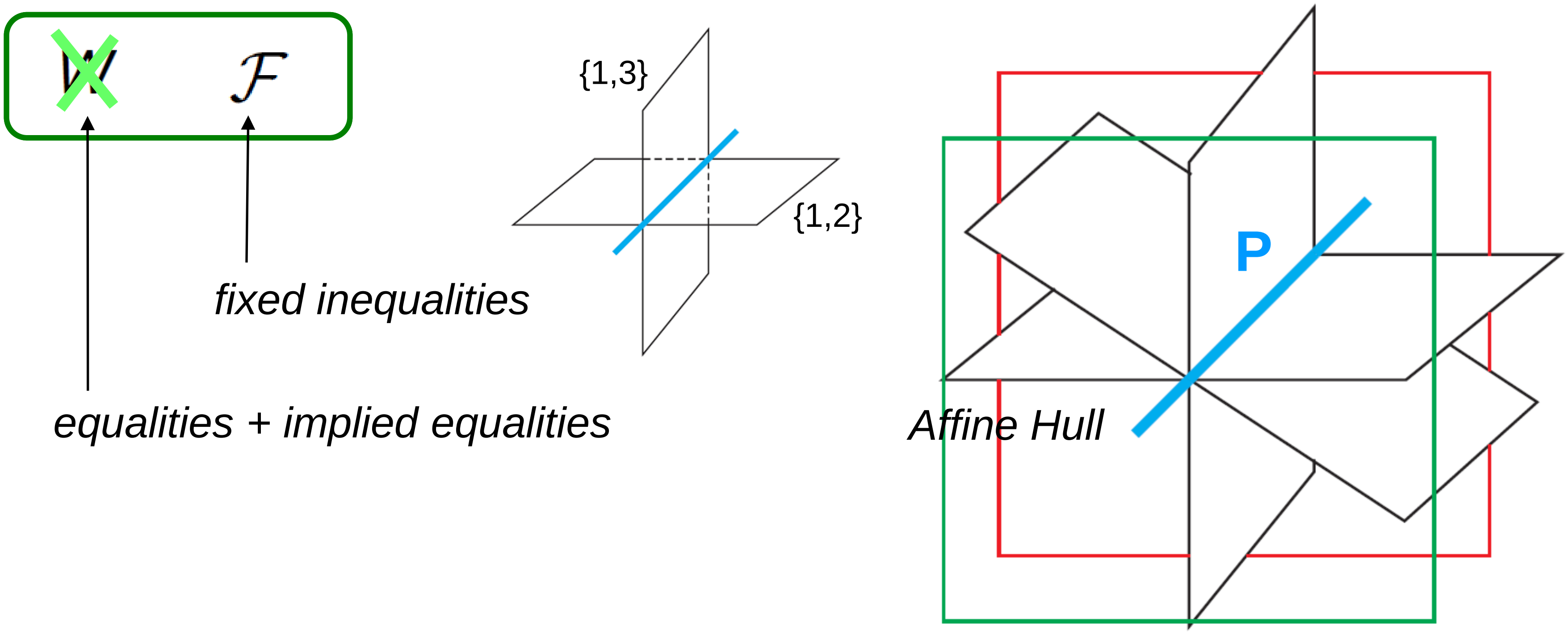

Theorem

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- *A basis $\mathcal{B}_k$ for $aff.hull(V_k)$ contains n vectors at most*
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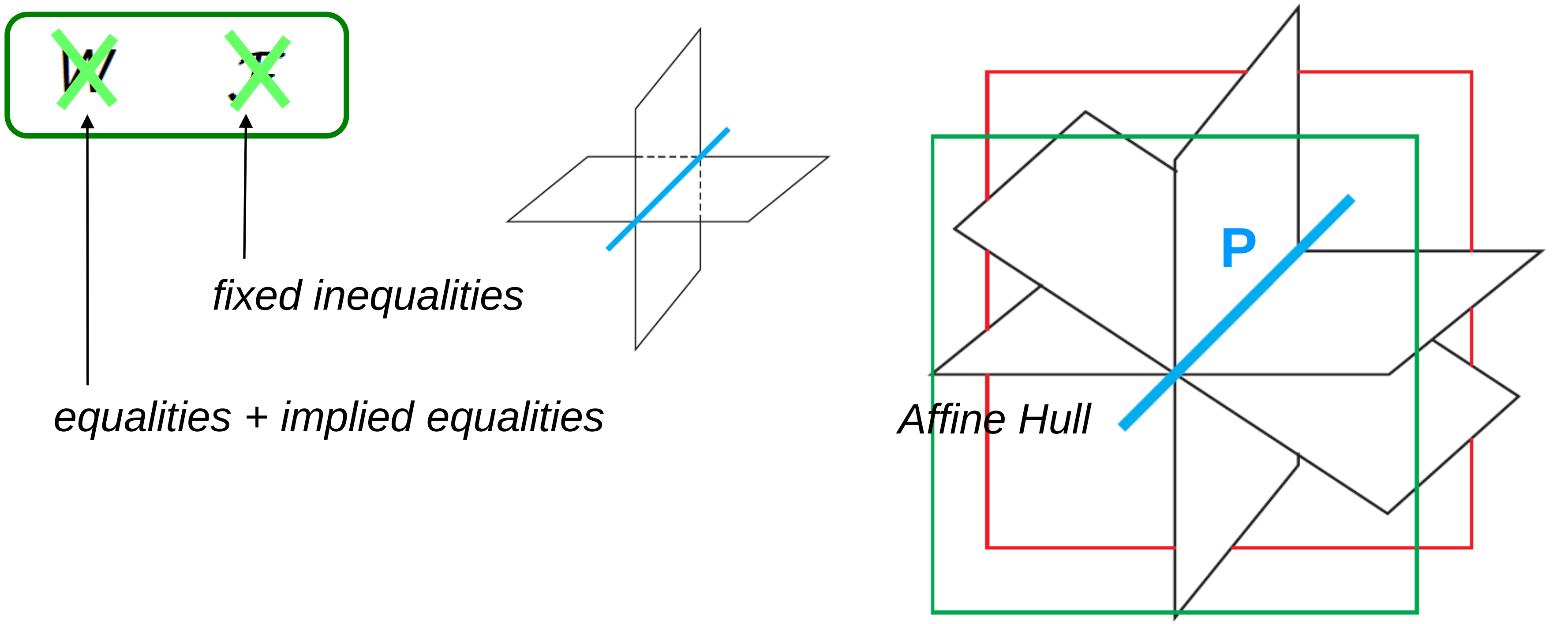

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(P1): A Convenient Representation

$$v : 2^N \mapsto \mathbb{R}$$

$$+$$

$$aff.hull$$

$$\Longrightarrow \quad \mathrm{LP}_{k+1}$$

$$\xrightarrow{\ i\ } \boxed{\mathrm{LP}} \xrightarrow{\ \textit{i-th inequality}\ }$$

Theorem

- $aff.hull(V_k)$ = solutions for equalities over $W_k \cup W_{k-1} \cup \cdots \cup W_1$
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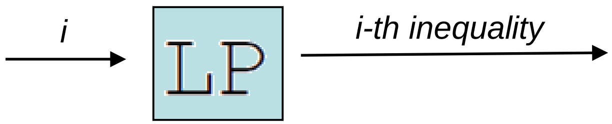
(P2) Computation Problems

- In compact games, two problems have to be faced:
  - **(P1)** Sets W and $\mathcal{F}$ contain exponentially many elements, but we would like to avoid listing them explicitly
  - **(P2)** Translate LP (complexity) results to "succinct programs"

(P2) Computation Problems



Problem	Result
MEMBERSHIP	in co-**NP**
NONEMPTINESS	in co-**NP**
DIMENSION	in **NP**
AFFINEHULLCOMPUTATION	in $\mathbf{F\Delta_2^P}$
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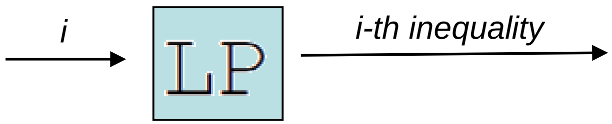
Complexity Results



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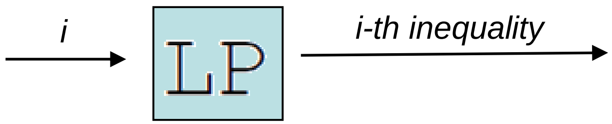

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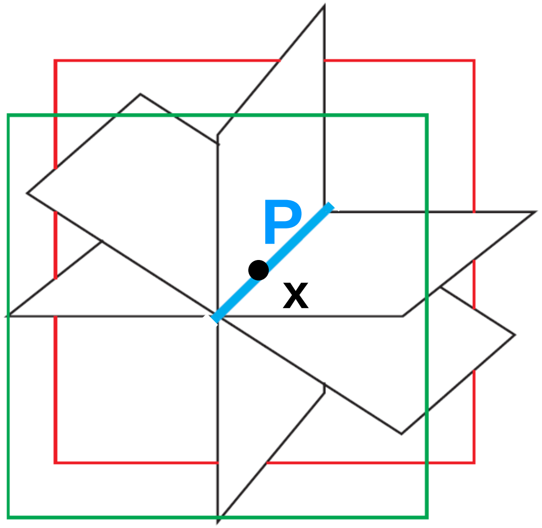
Trivial

- Given a vector **x**, we can:
  - Guess an index *i*
  - Check that the *i-th inequality* is not satisfied by **x**



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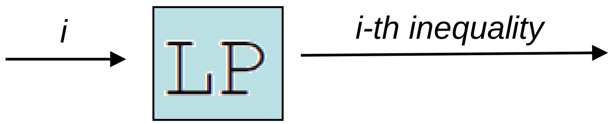

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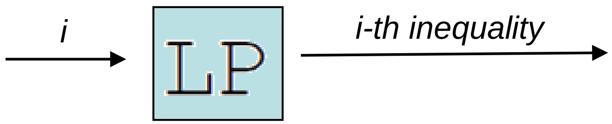

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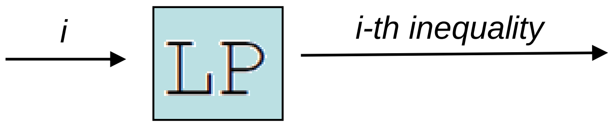
Proof

- By **Helly's theorem**, we can solve the complementary problem in **NP**:
  - Guess **n+1** inequalities
  - Check that they are not satisfiable (in polynomial time)

Complexity Results



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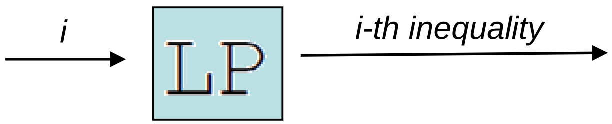

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Proof Overview

(1) The dimension is **n-k** at most, if there are at least **k** linear independent implied equalities

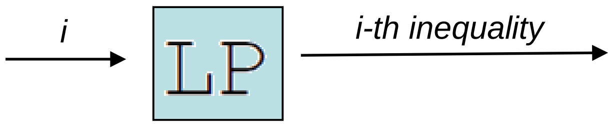
(2) In order to check that the *i-th* inequality is an implied one,

we can guess in **NP** a **support set** *W(i),* again by Helly's theorem:

- **n** inequalities + the *i-th* inequality treated as strict
- *W(i)* is not satisfiable, which can be checked in polynomial time

- Guess **k** implied equalities plus their support sets
- Check that they are linear independent

Complexity Results



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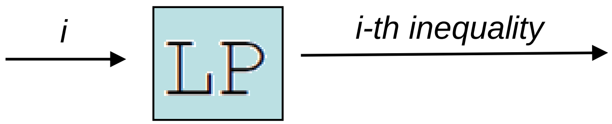

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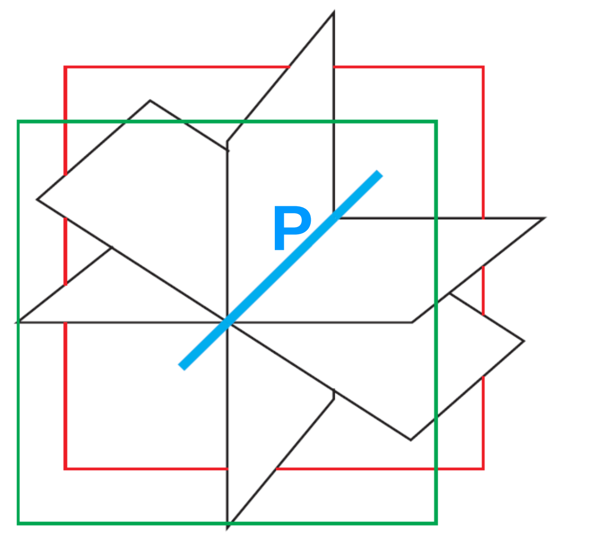
Proof

(1) Compute the dimension **n-k**, with a *binary search* invoking an **NP** oracle

(2) Guess **k** implied equalities plus their support sets



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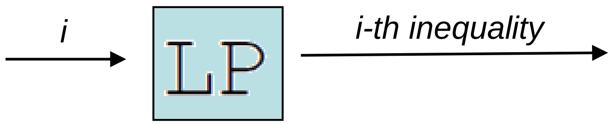

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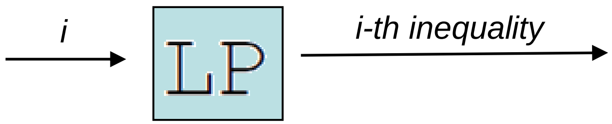

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Routine

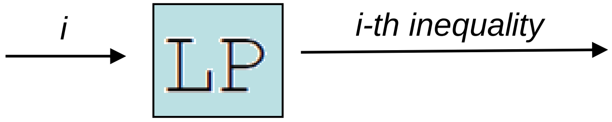
(1) Bfs can be represented with polynomially many bits

(2) LP induces a polytope and hence the optimum is achieved on some bfs.

(3) Perform a *binary search* over the range of the optimum solution:

- Add the current value as a constraint, and check satisfiability

Complexity Results



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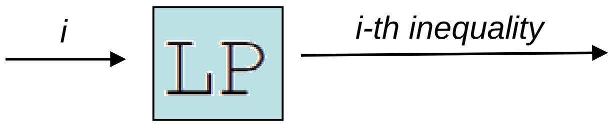

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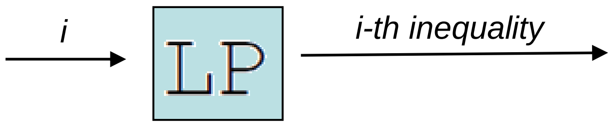
Routine

- LP induces a polytope
- Compute the lexicographically maximum bfs solution, by iterating over the various components, and treating each of them as an objective function to be optimized.

Complexity Results



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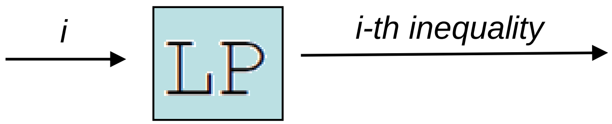

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Routine

(1) Compute the optimum value

(2) Define LP’ as LP plus the constraint stating that the objective function must equal the optimum value

(3) Compute a feasible value for LP’

Putting It All Togheter

$$
\mathrm{LP}_k \begin{cases} \min \epsilon_k \\ e(S,x) = \epsilon^*_r & \forall S \in W_r, r \in \{1,\ldots,k-1\} \\ e(S,x) \leq \epsilon_k & \forall S \subset N, S \notin \mathcal{F}_{k-1} \\ x \in X(\mathcal{G}) \\ \textit{where:} \\ \bullet\; V_r = \{x \mid (x,\epsilon^*_r) \text{ is an optimal solution to } \mathrm{LP}_r\} \\ \bullet\; W_r = \{S \subseteq N \mid e(S,x) = \epsilon^*_r, \text{ for every } x \in V_r\} \\ \bullet\; \mathcal{F}_{k-1} = \{S \subseteq N \mid x(S) = y(S), \forall x,y \in V_{k-1}\} \end{cases}
$$

$v : 2^N \mapsto \mathbb{R}$ + *aff.hull* $\longrightarrow \mathrm{LP}_{k+1}$

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M.P.S. **Compact Encoding** **Algorithms in $\mathbf{F\Delta_2^P}$**

- In compact games, two problems have to be faced:
  - **(P1)** Sets W and $\mathcal{F}$ contain exponentially many elements, but we would like to avoid listing them explicitly
  - **(P2)** Translate LP (complexity) results to “succinct programs”

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where:

- $V_r = \{x \mid (x, \epsilon_r^*)$ is an optimal solution to $\mathrm{LP}_r\}$
- $W_r = \{S \subseteq N \mid e(S,x) = \epsilon_r^*,$ for every $x \in V_r\}$
- $\mathcal{F}_{k-1} = \{S \subseteq N \mid x(S) = y(S), \forall x, y \in V_{k-1}\}$

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\+ *aff.hull* $\longrightarrow \mathrm{LP}_{k+1}$

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M.P.S. **Compact Encoding** **Algorithms in $\mathrm{F}\Delta^P_2$**

Theorem

Computing the nucleolus is feasible in $\mathbf{F\Delta_2^P}$. Thus, deciding whether an imputation is the nucleolus is feasible in $\mathbf{\Delta_2^P}$.

Checking Problem

Theorem

Deciding whether an imputation is the nucleolus is $\mathbf{\Delta_2^P}$*-hard. Thus, it is* $\mathbf{\Delta_2^P}$*-complete.*

Checking Problem

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Proof (Reduction for Graph Games: *The cost of individual rationality!*)

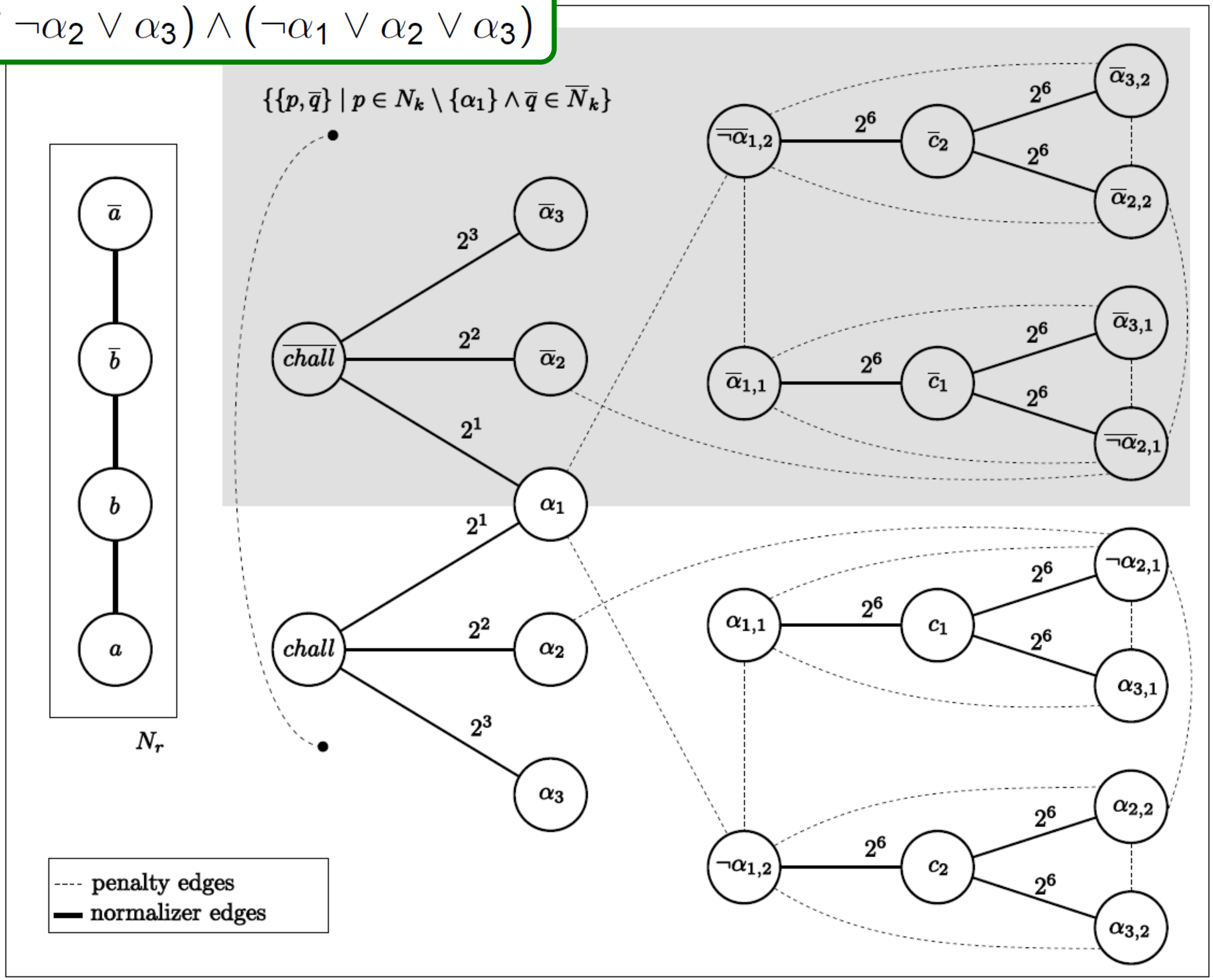
- Deciding the truth value of the least significant variable in the lexicographically maximum satisfying assignment

$$\boxed{\hat{\phi} = (\alpha_1 \vee \neg\alpha_2 \vee \alpha_3) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \alpha_3)}$$

$\alpha_1 < \alpha_2 < \alpha_3$

Overview of the Reduction

$$\hat{\phi} = (\alpha_1 \vee \neg\alpha_2 \vee \alpha_3) \wedge (\neg\alpha_1 \vee \alpha_2 \vee \alpha_3)$$



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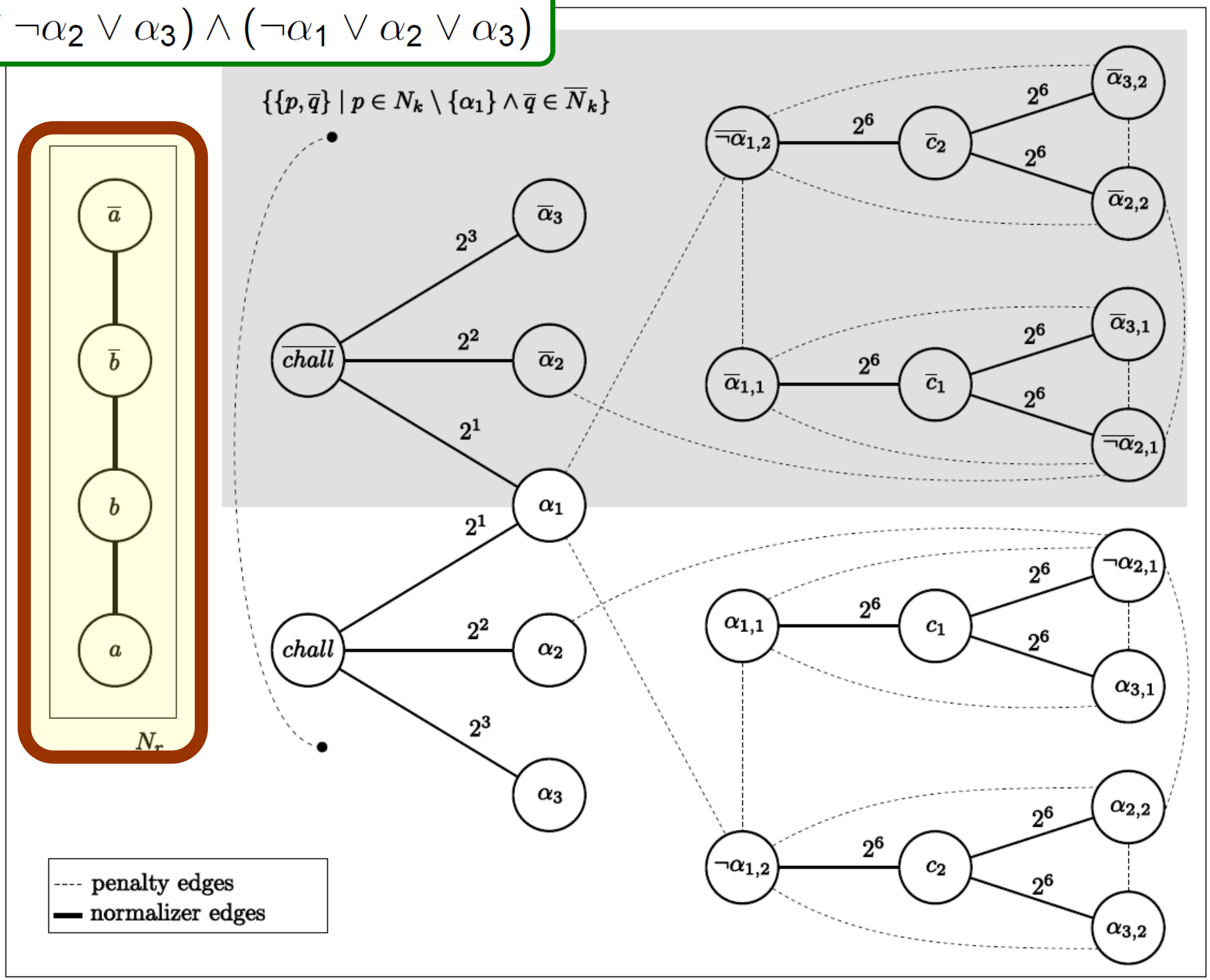

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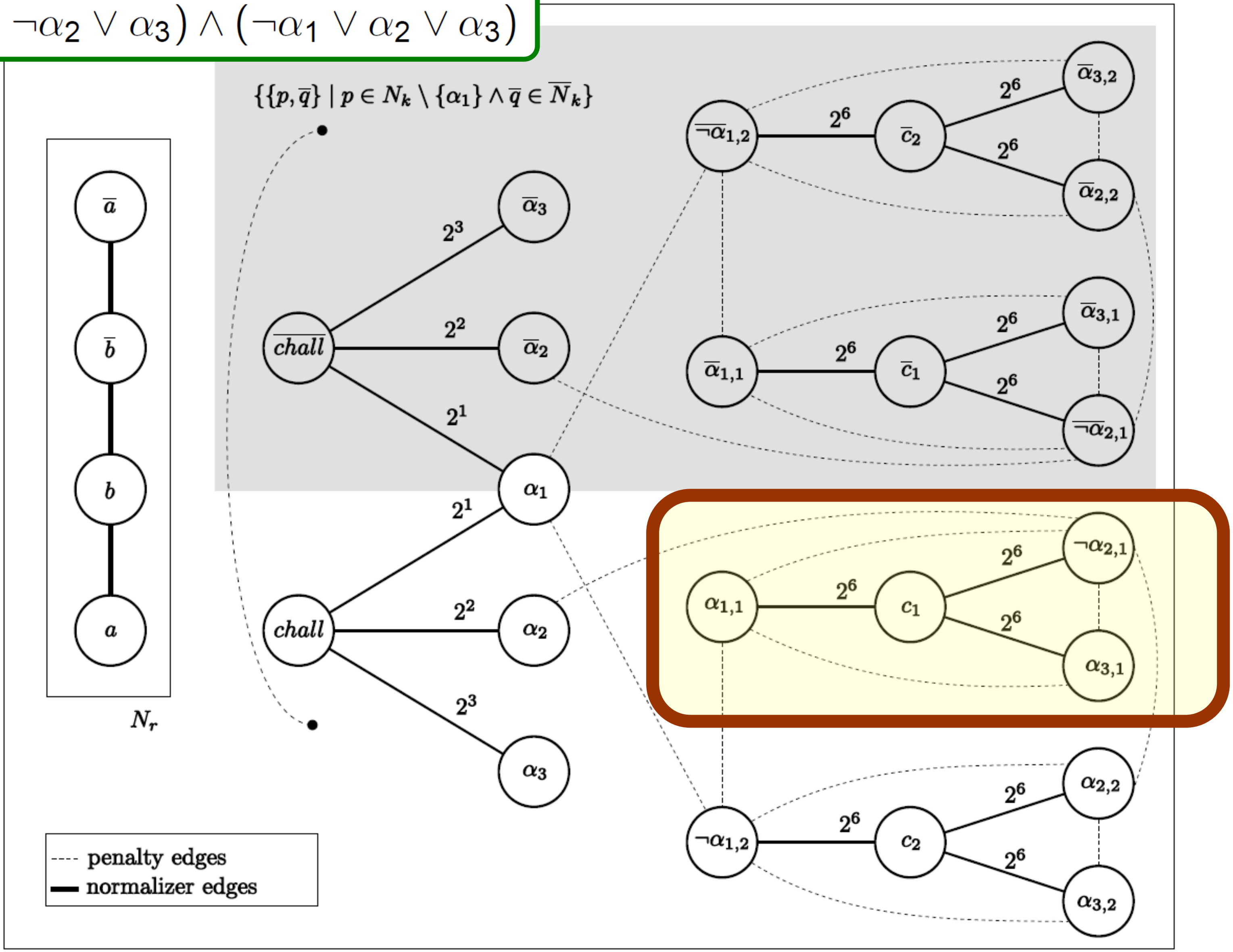

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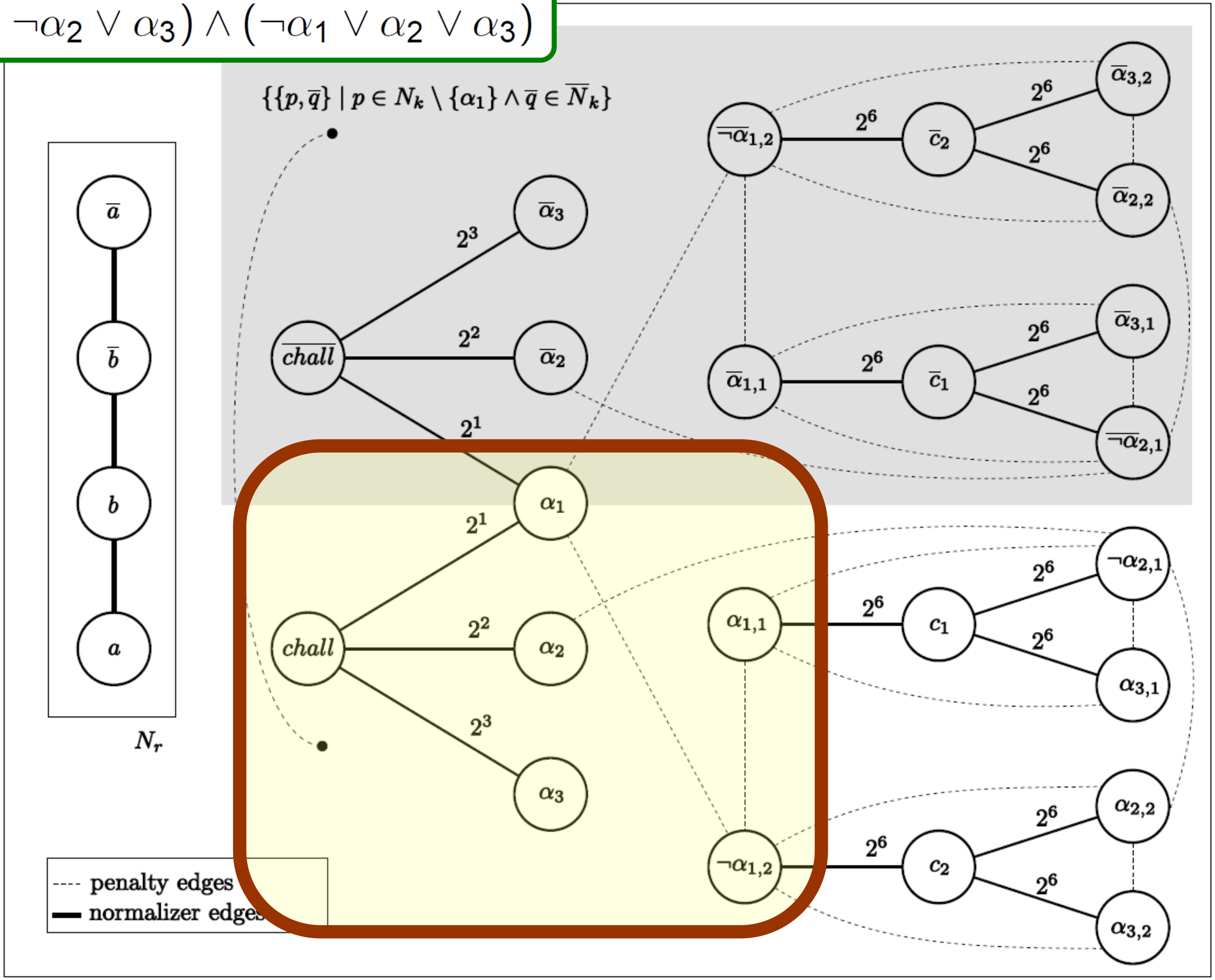

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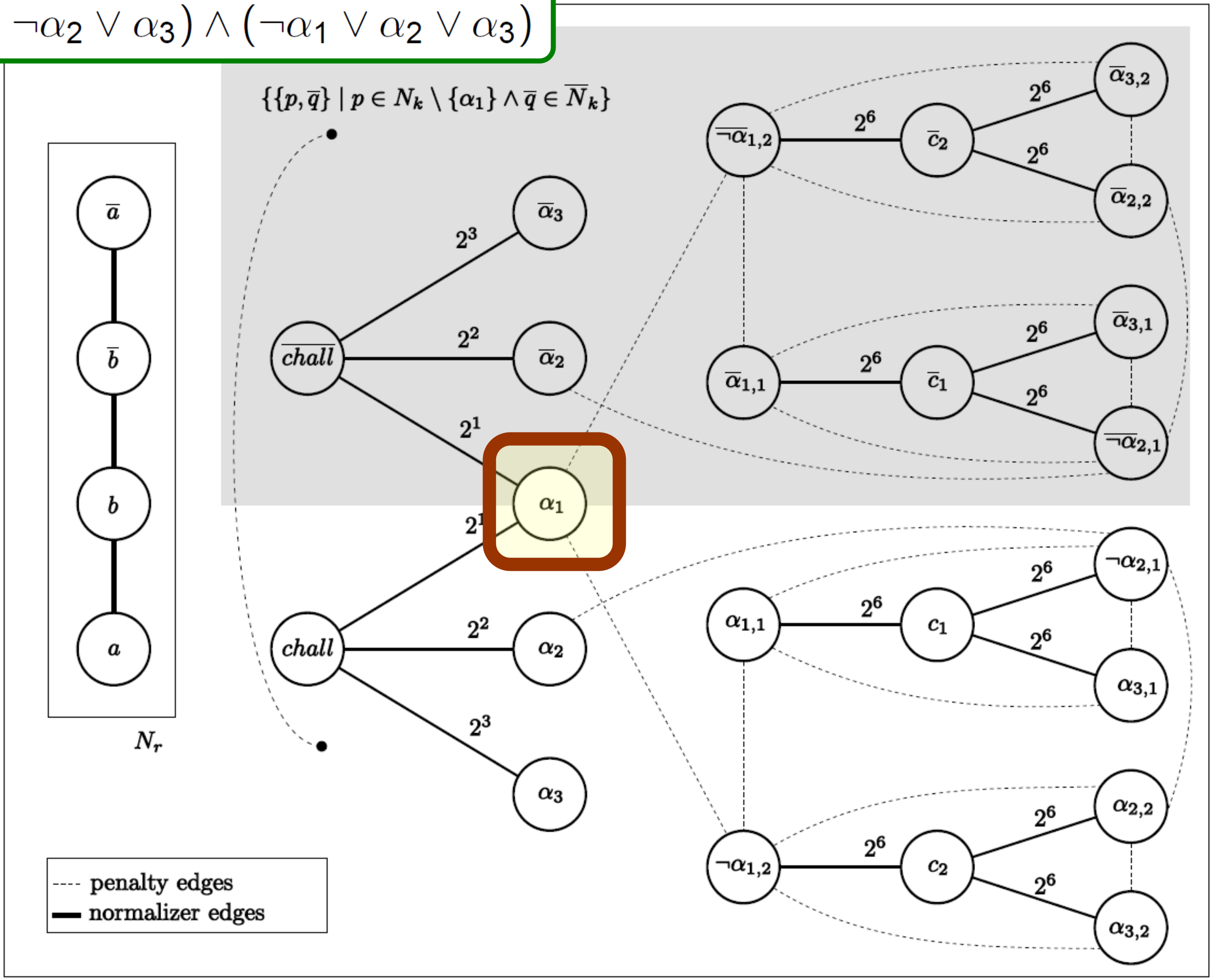

Thank you!