Congestion games

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Outline

Table of contents

Congestion effect

- driving on the highway
  - traffic jams :(
  - empty highway :)
- price of French books in
  - Reykjavik :(
  - Paris :)

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- playing football
  - alone :(
  - with 9 kids :)
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Congestion model

$$\langle N, M, (A_i)_{i \in N}, (d_j)_{j \in M} \rangle$$

- $N = \{1, 2, \ldots, n\}$ is the set of players
- $M = \{1, 2, \ldots, m\}$ is the set of resources
- A_i is the strategy space of player i
  - A_i is a non-empty subset of 2^M
- d_j is a delay function associated with resource j
  - $d_j : \{1, 2, \ldots, n\} \to \mathbb{R}$

If x players have resource j in their strategy then the delay associated with j is $d_j(x)$

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Congestion game

Strategic game

- σ is a strategy profile and σ_i is the strategy of player i
- A is the set of all strategy profiles $(\times_{i \in N} A_i)$
- for $j \in M$, $\ell_j(\sigma) = |\{i \in N : j \in \sigma_i\}|$ is the load of resource j

– congestion model + individual **costs** of the players $(c_i)_{i \in N}$
– each player i strives to minimize her individual cost
$c_i(\sigma) = \sum_{j \in \sigma_i} d_j(\ell_j(\sigma))$

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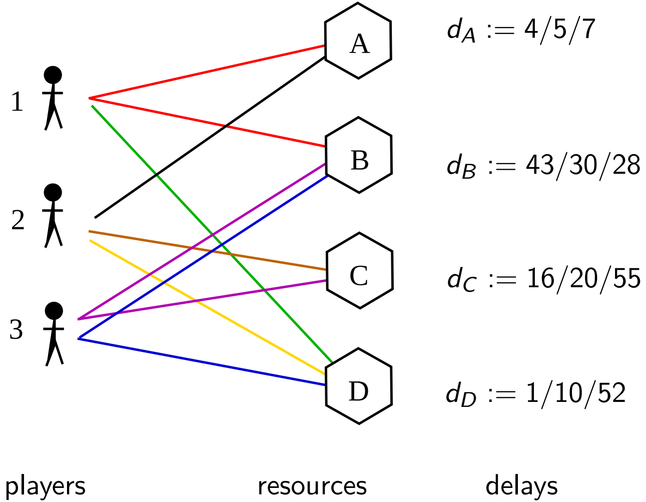
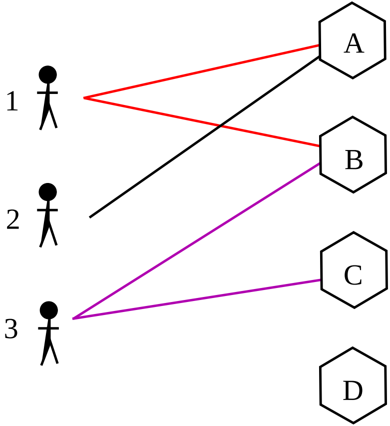
Congestion game





$d_A := 4/\mathbf{5}/7$

$d_B := 43/\mathbf{30}/28$

$d_C := \mathbf{16}/20/55$

$d_D := 1/10/52$

cost of player 1 : $5 + 30 = 35$; cost of player 2 : 5 ;
cost of player 3 : $30 + 16 = 46$

Table of contents

Pure Nash equilibrium

Theorem

Every congestion game admits a pure strategy Nash equilibrium

Proof

Every state σ is associated with a **potential function** $\Phi : A \rightarrow \mathbb{R}$

$$\Phi(\sigma) = \sum_{j \in M} \sum_{k=1}^{\ell_j(\sigma)} d_j(k)$$

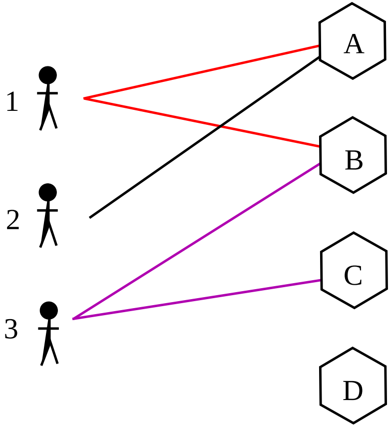
Reminder: $\ell_j(\sigma)$ is the number of players using resource j under strategy profile σ

Suppose player i modifies her strategy

The initial state is σ and the new state is σ'

$$\begin{aligned}
\Phi(\sigma) - \Phi(\sigma') &= \sum_{j \in M} \sum_{k=1}^{\ell_j(\sigma)} d_j(k) - \sum_{j \in M} \sum_{k=1}^{\ell_j(\sigma')} d_j(k) \\
&= \sum_{j \in \sigma_i \setminus \sigma'_i} d_j(\ell_j(\sigma)) - \sum_{j \in \sigma'_i \setminus \sigma_i} d_j(\ell_j(\sigma')) \\
&= c_i(\sigma) - c_i(\sigma')
\end{aligned}$$

The individual cost of player i and the potential decrease by the same amount



$c_A := \underline{4}/\underline{5}/7$

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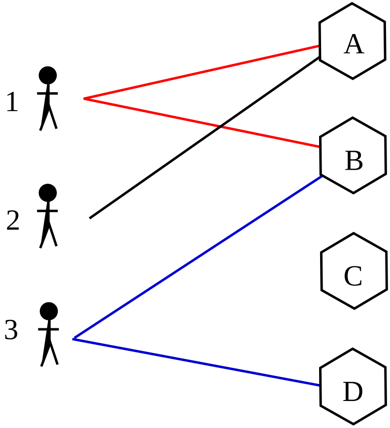
$\Phi(\sigma) = 4 + 5 + 43 + 30 + 16 = 98$

cost of player 3 : 46



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$\Phi(\sigma) = 4 + 5 + 43 + 30 + 1 = 83$

cost of player 3 : 31

If the players sequentially perform unilateral profitable deviations then we get a *finite* sequence of strategy profiles $(\sigma_0, \sigma_1, \cdots, \sigma_r)$ called *improvement path*

It must be $\Phi(\sigma_0) > \Phi(\sigma_1) > \cdots > \Phi(\sigma_r)$

If the improvement path is finite then its last state is a pure Nash equilibrium

A game has the **finite improvement property** (FIP) if every improvement path of the game is finite

Congestion games have the FIP

A state σ that minimizes $\Phi(\sigma)$ is a pure Nash equilibrium

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Finite but exponentially long improvement path

Consider a congestion game with only one player

Her strategy set is every nonempty subset of M

Using resource j induces a cost of 2^{j-1}, for all j

state	3	2	1	cost
σ_0	1	1	1	7←
σ_1	1	1	0	6
σ_2	1	0	1	5
σ_3	1	0	0	4
σ_4	0	1	1	3
σ_5	0	1	0	2
σ_6	0	0	1	1

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Each step is a *better* response, not a *best* response

However, even with *best*-response, there exists an instance having an exponentially long improvement path

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Singleton congestion games

Definition

In a *singleton* congestion game, every player selects a single resource at a time, i.e. $A_i \subseteq M$

Theorem

In a singleton congestion game, the length of every improvement path is at most n^2m

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Proof sketch

The size of $\mathcal{F} = \{d_j(k) : 1 \leq k \leq n \text{ and } 1 \leq j \leq m\}$ is at most nm

Thus $\Phi(\sigma) = \sum_{j \in M} \sum_{k=1}^{\ell_j(\sigma)} d_j(k)$, which consists of n terms, can take at most $n^2 m$ different values

The potential sctrictly decreases in an improvement path

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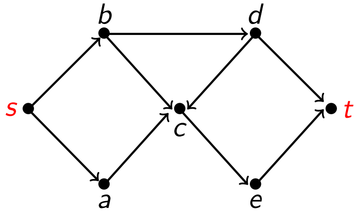
Symmetric network congestion games

A directed graph and a set of n players

A common source s and a common destination t

The cost of each arc j is $d_j(x)$ where x is the number of players using j

Every player wants to find the cheapest directed path from s to t



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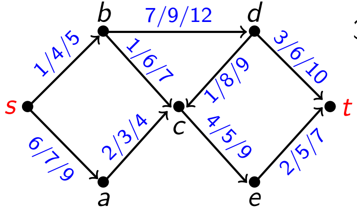


3 players

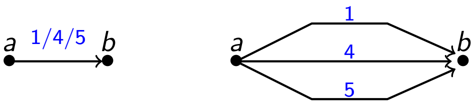
Symmetric network congestion games

Theorem

There exists a polynomial time algorithm for computing a Nash equilibrium in symmetric network congestion games when $(d_j)_{j \in M}$ is monotone non decreasing

Proof

Replace each arc *ab* by *n* arcs with costs $d_{ab}(1), \ldots d_{ab}(n)$ and capacity 1



By hypothesis $d_{ab}(1) \leq d_{ab}(2) \leq \ldots \leq d_{ab}(n)$. A flow of *n* units with minimum cost corresponds to a strategy profile σ minimizing $\phi(\sigma)$

Table of contents

Strong equilibria

Deviation from σ: some players $\Gamma \subseteq N$ change their strategy, i.e. they play $s \in A_\Gamma$ instead of σ_Γ

- *unilateral* deviation when $|\Gamma| = 1$
- *group* deviation when $|\Gamma| \geq 1$

Profitable deviation: for every $i \in \Gamma$, $c_i(s, \sigma_{-\Gamma}) < c_i(\sigma)$

Nash equilibrium: no profitable *unilateral* deviation

Strong equilibrium: no profitable *group* deviation

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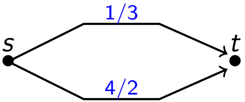
Strong equilibria in singleton congestion games

2 players and 2 resources



	top	bottom
top	3, 3	1, 4
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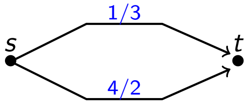

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(top,top) is a Nash equilibrium but there is no strong equilibrium

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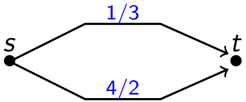

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Strong equilibria in singleton congestion games

Suppose every delay function d_j is monotone non-decreasing

Congestion has a negative impact

Theorem

Every Nash equilibrium is also a strong equilibrium

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Take a Nash equilibrium σ and suppose a coalition of players wants to deviate to σ'

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If there exists j such that $\ell_j(\sigma') > \ell_j(\sigma)$ then σ is not a Nash equilibrium

$\ell_j(\sigma) = \ell_j(\sigma'), \forall j$ because $n = \sum_{j \in M} \ell_j(\sigma) = \sum_{j \in M} \ell_j(\sigma')$

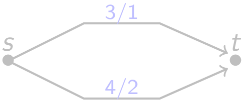
Same multi-set of individual costs $\rightarrow$ at least one player of the coalition does not decrease her individual cost

$\square$

Strong equilibria in singleton congestion games

Suppose every delay function d_j is monotone **non-increasing**

Congestion has a **positive** impact



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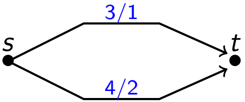

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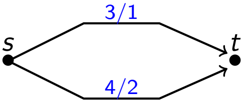

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Every instance admits a strong equilibrium

Algorithmic construction of the strong equilibrium

1. Initialisation
   - No player is assigned a resource at the beginning
   - $P =$ set of pending players ($= N$)
   - $R =$ set of pending resources ($= M$)

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   - $R =$ set of pending resources $(= M)$
2. While $P \neq \emptyset$ do
   - $\forall j \in R$, let $x_j = |\{i \in P : j \in A_i\}|$
   - $j^* = \arg\min_{j \in R} d_j(x_j)$
   - Assign the x_{j^*} players of P to resource j^*
   - Remove these x_{j^*} players from R and remove j^* from R

Strong equilibria in singleton congestion games

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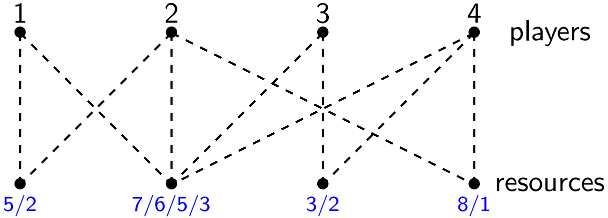
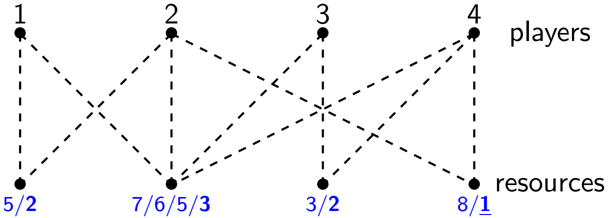
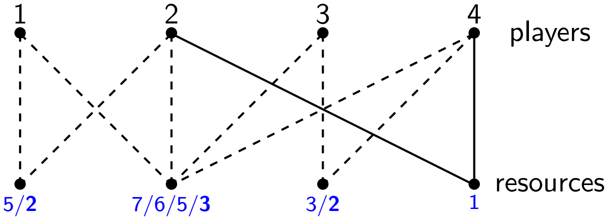


Strong equilibria in singleton congestion games



$P = \{1, 2, 3, 4\}$
$R = \{1, 2, 3, 4\}$

Strong equilibria in singleton congestion games



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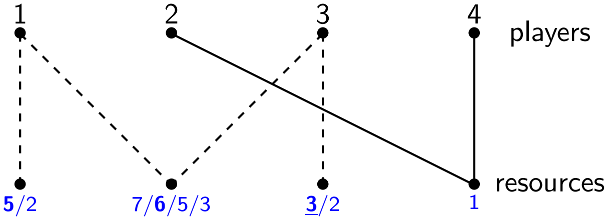

$P = \{1, 3\}$
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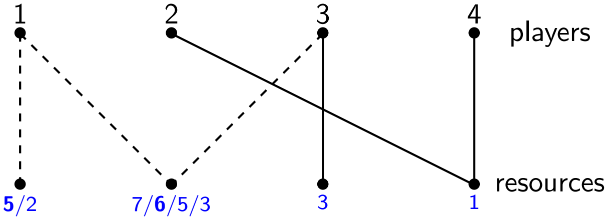
Strong equilibria in singleton congestion games



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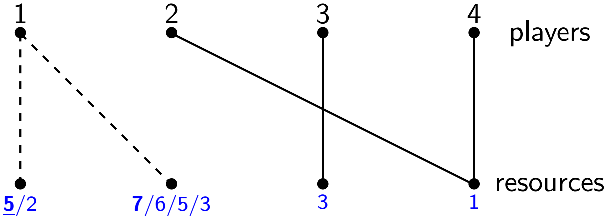
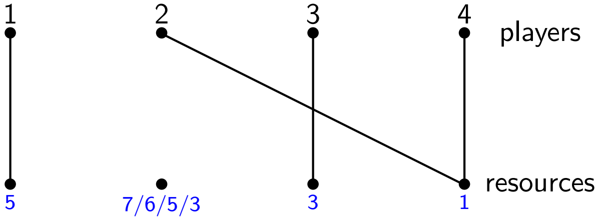
Strong equilibria in singleton congestion games



$P = \{1\}$
$R = \{1, 2\}$

Strong equilibria in singleton congestion games



$P = \emptyset$
$R = \{2\}$

Strong equilibria in singleton congestion games

Proof

The first group of players assigned by the algorithm gets the lowest possible cost, they have no incentive to deviate, and no extra player can join them on their resource

The second group of players assigned by the algorithm gets the second-lowest possible cost, they have no incentive to deviate, and no extra player can join them on their resource

etc.

Table of contents

Payoff maximization instead of cost minimization

So far $d_j(x)$ was interpreted as: if x players use resource j then each of them incurs a cost of $d_j(x)$

Moreover $c_i(\sigma) = \sum_{j \in \sigma_i} d_j(\ell_j(\sigma))$ is the cost of player i

Actually $d_j(x)$ can be interpreted as: if x players use resource j then each of them receives a **payoff** of $d_j(x)$

Accordingly, $u_i(\sigma) = \sum_{j \in \sigma_i} d_j(\ell_j(\sigma))$ is the **payoff** of player i

All previous results remain valid

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Congestion games with player-specific payoffs

Each delay function d_j is replaced by d_{ij}, a delay function that is *specific* to player i

If x players use resource j then player i receives a payoff of $d_{ij}(x)$

It can be $d_{ij}(x) \neq d_{i'j}(x)$ for $(i, i') \in N \times N$

Restrictions:

- singleton strategy (one resource at a time)
- symmetric strategy space ($A_i = M, \forall i \in N$)
- the d_{ij}'s are monotone non increasing (congestion has a negative impact)

Congestion games with player-specific payoffs

Each delay function d_j is replaced by d_{ij}, a delay function that is *specific* to player i

If x players use resource j then player i receives a payoff of $d_{ij}(x)$

It can be $d_{ij}(x) \neq d_{i'j}(x)$ for $(i, i') \in N \times N$

Restrictions:

- singleton strategy (one resource at a time)
- symmetric strategy space ($A_i = M$, $\forall i \in N$)
- the d_{ij}'s are monotone non increasing (congestion has a negative impact)

Congestion games with player-specific payoffs

The finite improvement property (FIP) is not preserved (unless there are 2 resources)

2 players and 3 resources:

	resource 1	resource 2	resource 3
1 user	$(6,5)$	$(5,6)$	$(3,3)$
2 users	$(1,4)$	$(4,1)$	$(2,2)$

the payoff of player 1 is 6 if he uses resource 1 alone

	resource 1	resource 2	resource 3
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Congestion games with player-specific payoffs

A cycle of better moves

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resource 1	$(1, 4)$	$(6, 6)$	$(6, 3)$
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Pure Nash equilibria in blue

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An instance with a cycle of **best** moves exists (3 players)

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Theorem

Every congestion game with player-specific payoffs has a pure Nash equilibrium

Proof by induction

Every instance with $n-1$ players has a pure NE

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Put player n apart and take a pure NE σ for players $1, \ldots, n-1$

Then introduce player n and let her take her best response

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Only a player with the same strategy as player n may want to deviate

- $d_{i\sigma_n}$ is non increasing $\Rightarrow$ no one wants to join resource σ_n

Suppose player i_0 moves from resource σ_n to resource j_0

Then only a player with strategy j_0, say i_1, may want to move to resource j_1

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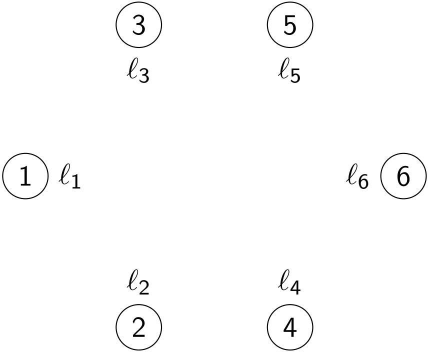
Each circle is a resource



ℓ_j is the load of resource j (number of users) in the pure Nash equilibrium reached by the $n-1$ first players

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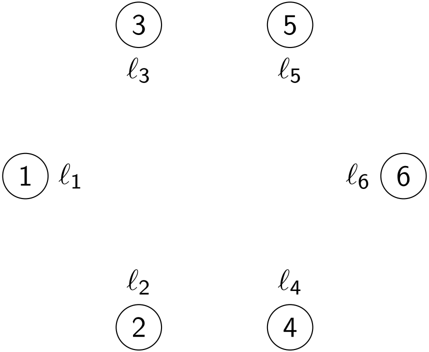

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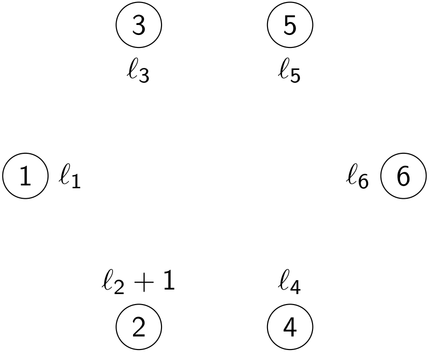
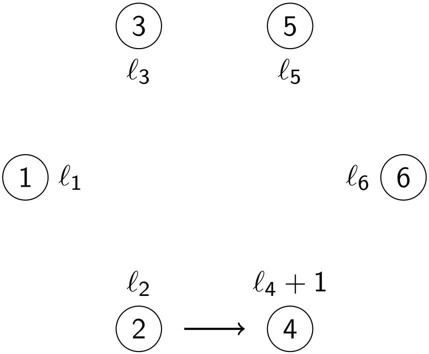
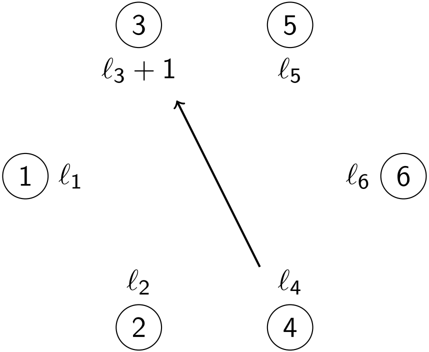
Player n is introduced and her best response is to play resource 2



A player from resource 2 may leave it



A player from resource 4 may leave it



The load vector is always $(\ell_1, \ldots, \ell_m)$ with a $+1$ somewhere

No player moves more than once

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$\square$

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Congestion game with capacities

Each resource j is associated with a capacity level κ_j

κ_j = maximum number of users that resource j may simultaneously accommodate

Each resource j has a ranking $\mathrm{pos}_j : N \to [1, n]$, prescribing the priority of accommodation of the users

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Player $i \in N_j(\sigma)$ is accommodated by resource j iff

$$|\{i' \in N_j(\sigma) : \mathtt{pos}_j(i') < \mathtt{pos}_j(i)\}| < \kappa_j$$

The delay of player $i \in N_j(\sigma)$ on resource j is:

$$d_j^i(\sigma) = \begin{cases} d_j(\min\{\ell_j(\sigma), \kappa_j\}) & \text{if } i \text{ is accommodated,} \\ +\infty & \text{otherwise.} \end{cases}$$

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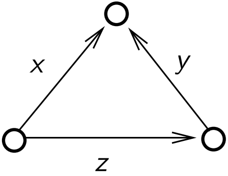

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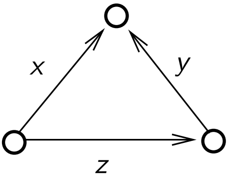
Priority is always given to the line player

Existence of a pure Nash equilibrium not guaranteed



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Theorem

A pure Nash equilibrium can be computed for any instance of symmetric singleton capacitated congestion game

Sketch of the algorithm

1. Find ℓ_j the number of players resource j accommodates
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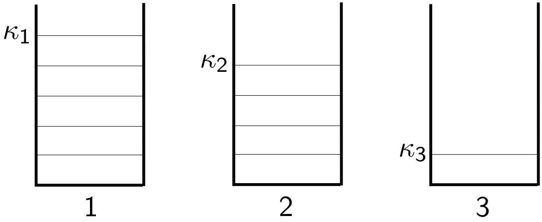
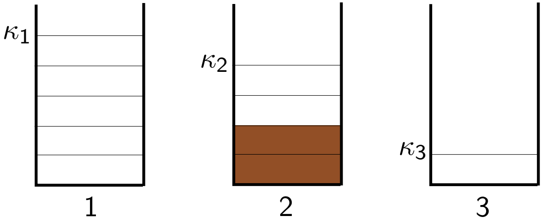


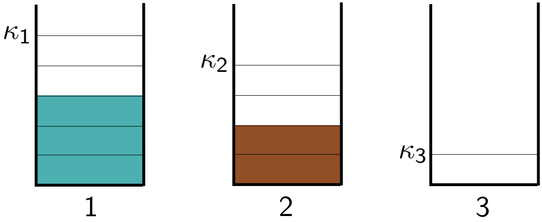
8 players

Symmetric singleton capacitated congestion games



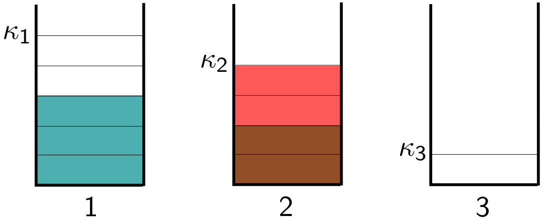
6 players

Symmetric singleton capacitated congestion games



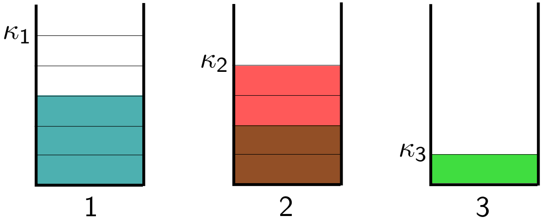
3 players

Symmetric singleton capacitated congestion games



1 player

Symmetric singleton capacitated congestion games



0 player

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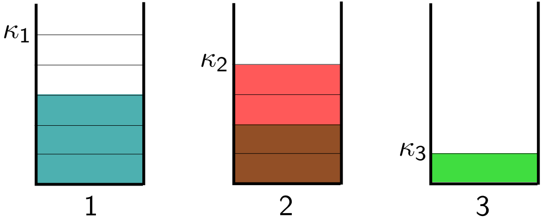
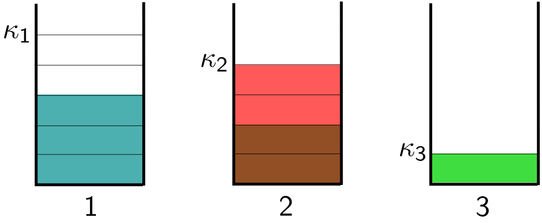


Suppose $d_1(3) \leq d_2(4) \leq d_3(1)$

Fill resource 1 with the 3 players with highest priority on resource 1

Symmetric singleton capacitated congestion games



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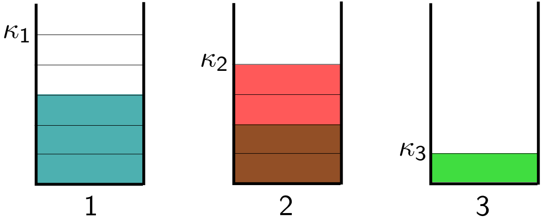
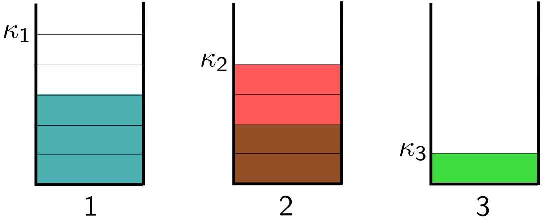


Suppose $d_1(3) \leq d_2(4) \leq d_3(1)$

Fill resource 2 with the 4 players with highest priority on resource 2 (the players assigned to resource 1 are ignored)

Symmetric singleton capacitated congestion games



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Fill resource 3 with the remaining player

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Variations in congestion games

We have seen special cases of congestion games: symmetric, singleton and monotone

- Non-atomic congestion games: a continuum of players, each of which is infinitesimally small
- Weighted congestion games
  - each player i has a weight w_i
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Thank you!