Equilibrium efficiency

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Outline

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Strategic games

$$\langle N, (A_i)_{i \in N}, (c_i)_{i \in N} \rangle$$

- $N = \{1, 2, \ldots, n\}$ is the set of players
- A_i is the strategy space of player i
- $A := A_1 \times A_2 \times \ldots \times A_n$
- $c_i : A \to \mathbb{R}$ is the *individual* cost of player $i \in N$

strategy profile (or state) $\sigma = (\sigma_1, \ldots, \sigma_n)$

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Social cost

Capture the cost of the group

$C : A \rightarrow \mathbb{R}$

Different definitions:

- Utilitarian: $C(\sigma) = \sum_{i \in N} c_i(\sigma)$
- Egalitarian: $C(\sigma) = \max_{i \in N} c_i(\sigma)$
- Median cost over N, OWA, etc.

Focus on the utilitarian and the egalitarian social cost

Social optimum: a strategy profile σ^* for which $C(\sigma^*)$ is minimum

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Prisoner's dilemma

	Silent	Betray
Silent	2 / 2	1 / 10
Betray	10 / 1	5 / 5

Prisoner's dilemma

	Silent	Betray
Silent	4	11
Betray	11	10

Utilitarian social cost

Optimizing individual costs can harm the social cost

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Utilitarian social cost

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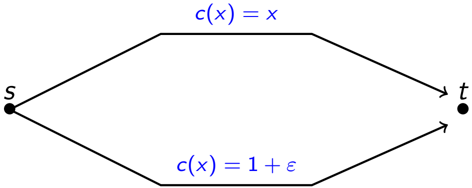
Pigou 1920

A continuum of players, each one being an infinitesimal part of a flow of quantity 1

Travel from s to t at the least possible cost



Equilibrium: all players on the upper link $\rightarrow$ social cost $1 * 1$

Social optimum: evenly divided between the two links $\rightarrow$ social cost $\frac{1}{2} * \frac{1}{2} + \frac{1}{2} * (1 + \varepsilon) \approx 0.75$

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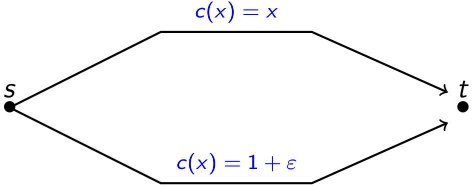

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Price of anarchy

Optimizing individual costs can harm the social cost: by how much?

Γ =game ; $\gamma \in \Gamma$ = instance of the game

NE_γ = set of Nash equilibria of instance γ

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Restrictions and extensions

PoS/PoA for mixed or pure Nash equilibria

PoS/PoA can be extended to solution concepts other than Nash equilibria

Strong price of anarchy

Strong equilibrium σ: there is no group of players who can jointly deviate and improve their individual costs

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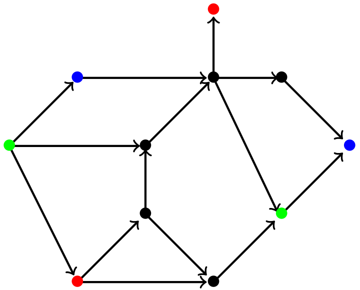
Connection game



a network $\mathcal{G} = (V, E)$ with a weight w_{xy} on every arc xy, each player wants to create a link between her source and her destination

if ℓ_{xy} players use arc xy then they share the weight of xy: every user pays w_{xy}/ℓ_{xy}

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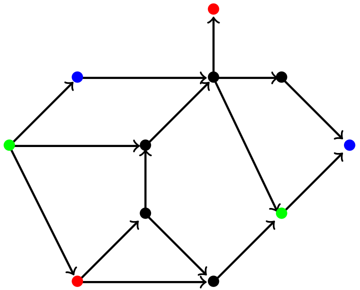

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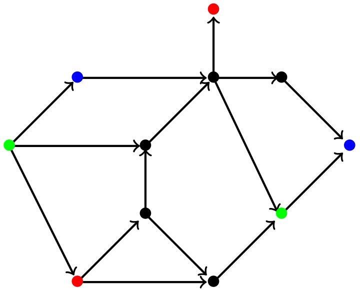

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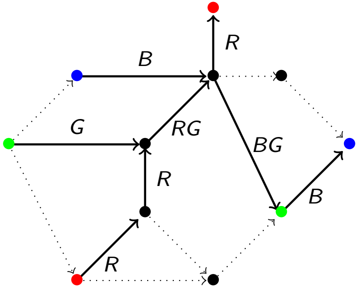
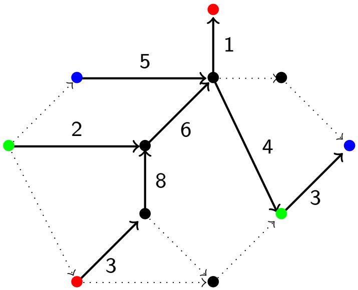


R=red
B=blue
G=green

Connection game



cost of player blue = 10
cost of player red = 15
cost of player green=10

Connection game: Nash equilibria

Proposition

Every instance of the connection game has a pure Nash equilibrium

Proof

It is a congestion game (each arc is a resource)

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$$\phi(\sigma) = \sum_{xy \in E} \sum_{k=1}^{\ell_{xy}(\sigma)} \frac{w_{xy}}{k} = \sum_{xy \in E} w_{xy} * H(\ell_{xy}(\sigma))$$

where $H(i) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i}$ (Harmonic number)

$\ell_{xy}(\sigma)$: number of players using arc xy under strategy profile σ

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The price of anarchy of the connection game is at most n (number of players)

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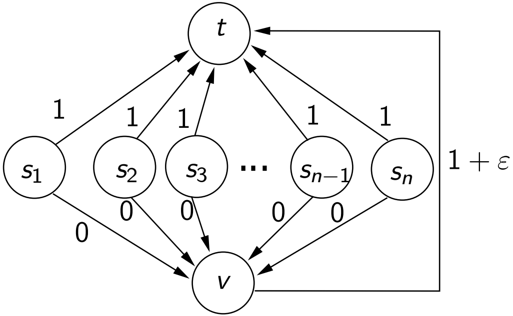

worst Nash eq.: every player i plays $\{(s_i, t)\}$ $C(\sigma) = n$

best strategy profile: every player i plays $\{(s_i, v), \{(v, t)\}$
$C(\sigma^*) = 1 + \varepsilon$

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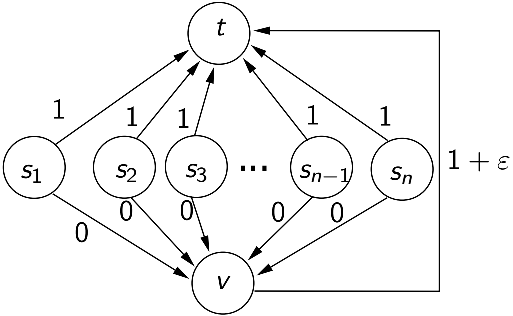

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The price of stability of the connection game is at most $H(n)$

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Take a social optimum σ_0 with potential $\Phi(\sigma_0)$ and let the players do improving deviations until a Nash equilibrium σ_r is reached

$$\sigma_0 \to \sigma_1 \to \sigma_2 \to \ldots \to \sigma_r$$

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$\phi(\sigma) = \sum_{xy \in E} w_{xy} * H(\ell_{xy}(\sigma))$

$C(\sigma) = \sum_{xy \in E'} w_{xy}$ where $E' =$ arcs used by at least one player

$$C(\sigma) \leq \phi(\sigma) \leq H(n)C(\sigma)$$

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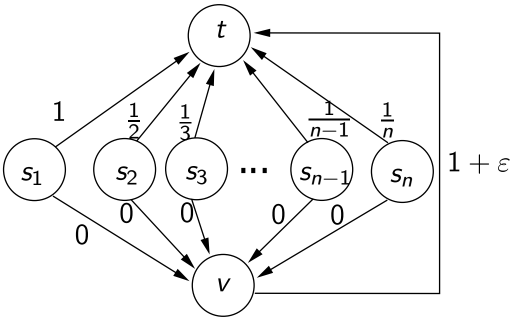

Only one Nash eq.: every player i plays $\{(s_i, t)\}$ $\qquad$ $C(\sigma) = H(n)$

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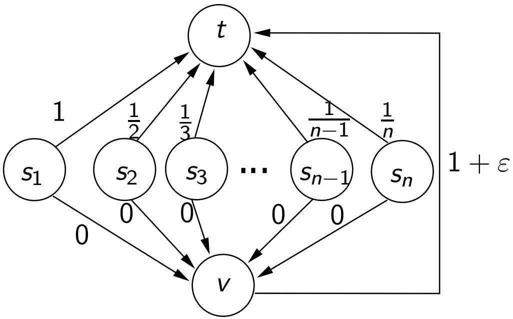

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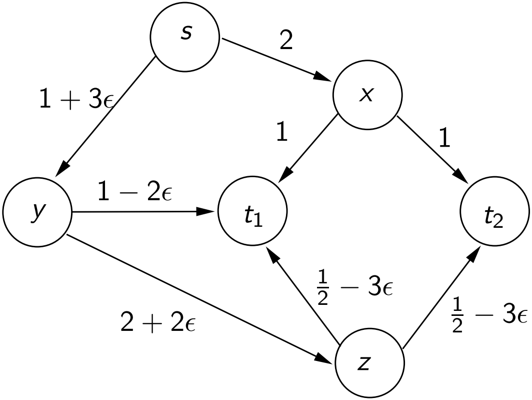
Connection game: strong equilibria

If the players have the same souce-destination pair then a strong equilibrium exists (symmetric instance)

- All players use an $s - t$ path of minimum weight

A non-symmetric instance with no strong equilibrium



2 players with source-destination pairs $s - t_1$ and $s - t_2$, respectively

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Set cover game

A set $\mathcal{E}$ of n elements $\{e_1, \ldots, e_n\}$ (the players)

A set $\mathcal{S}$ of m sets $\{S_1, \ldots, S_m\}$ such that

- $S_j \subseteq \mathcal{E}, \forall j$
- $\bigcup_{j=1}^{m} S_j = \mathcal{E}$
- each set S_j has a weight $w_j \in \mathbb{R}_+$

Each player e_i selects a set where it appears

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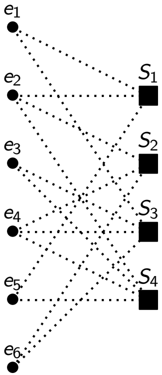
The set cover game is a special case of the connection game



$S_1 = \{e_1, e_2, e_5\}$, $S_2 = \{e_2, e_4, e_6\}$, $S_3 = \{e_1, e_3, e_4, e_6\}$, $S_4 = \{e_2, e_3, e_4, e_5\}$

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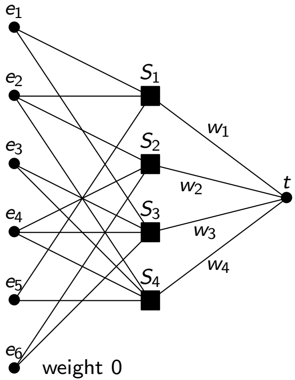

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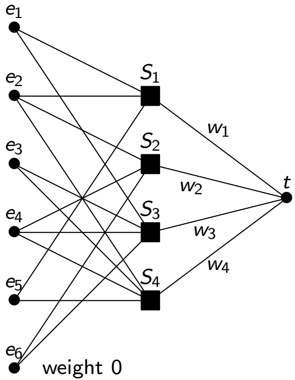

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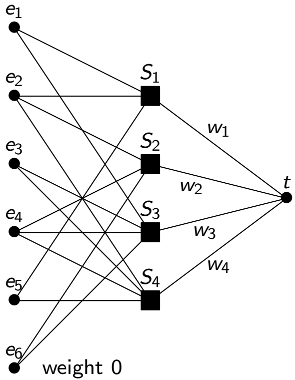

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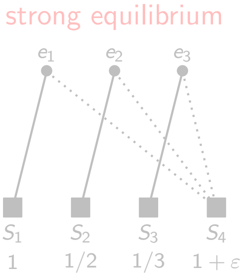

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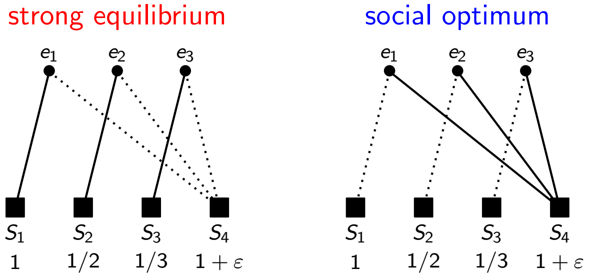

A property

Lemma

In a strong equilibrium σ we know that for every set $S_j = \{e_1, \cdots, e_{|S_j|}\}$ of weight w_j and such that $c_1(\sigma) \geq c_2(\sigma) \geq \cdots \geq c_{|S_j|}(\sigma)$ we have:

$$\forall i,\ c_i(\sigma) \leq w_j / i$$

Proof by contradiction

Suppose $c_1(\sigma) \geq \cdots \geq c_i(\sigma) > w_j/i$ for some $1 \leq i \leq |S_j|$

If players $e_1, \ldots, e_i$ jointly select S_j then their individual cost becomes w_j/ℓ_j where $\ell_j \geq i$, so it is smaller than w_j/i

σ is not a strong equilibrium

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σ is not a strong equilibrium

Strong price of anarchy

Let σ and σ^* be a strong equilibrium and a social optimum, respectively

Let X be the indexes of the sets selected in a social optimum σ^*

By the previous Lemma, for every S_j such that $j \in X$ it holds that:

$$\sum_{e_i \in S_j} c_i(\sigma) \leq H(|S_j|)w_j \leq H(n)w_j$$

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Finally:

$$C(\sigma) = \sum_i c_i(\sigma) \leq \sum_{j \in X} \sum_{e_i \in S_j} c_i(\sigma) \leq H(n) \sum_{j \in X} w_j = H(n)C(\sigma^*)$$

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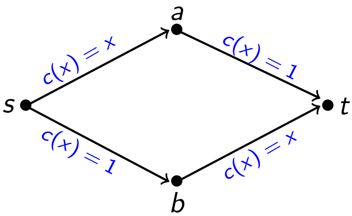
Table of contents

Optimization

Goal: alleviate the price of anarchy

Braess paradox (1968)



Half of the flow in each path → social cost = 3/2

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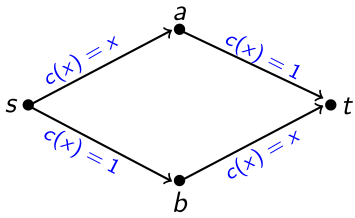

Half of the flow in each path → social cost $= 3/2$

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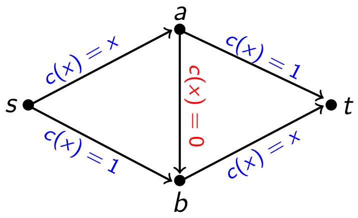

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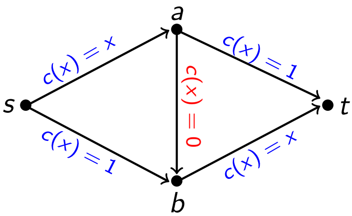

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Stackelberg games

Aim

Control (impose a strategy to) some players in order to decrease the social cost

- relevant when the price of anarchy/stability is high

A **leader** imposes a strategy to a subset of players

The other players, called **followers**, are free to pursue their individual interest

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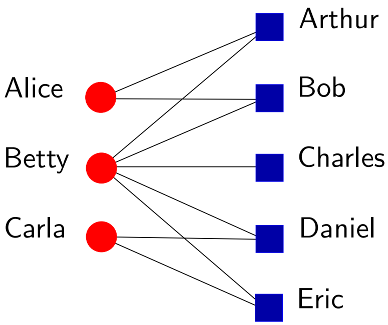
Matrimonial agency

Every node is a player who search for a partner in his neighborhood

Strategy of a player: designate a neighbor in the graph

A pair is formed when two neighbors designate each other

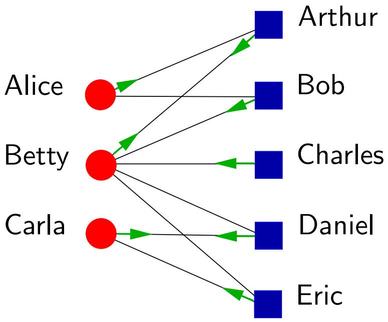
Social welfare: number of pairs



Example

Two pairs: Betty-Arthur and Carla-Daniel

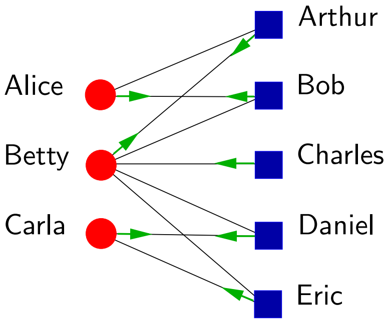
Suboptimal Nash equilibrium



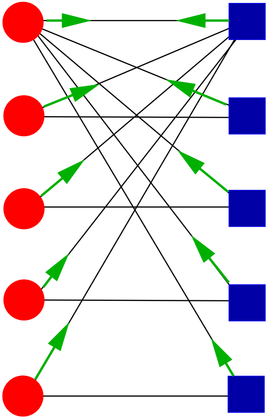
Example

3 pairs: Alice-Bob,
Betty-Arthur and Carla-Daniel

Optimal Nash equilibrium



A high price of anarchy



Obs1

At least one pair in every Nash equilibrium

Obs2

At most n pairs with $2n$ nodes

Price of anarchy n

A little help

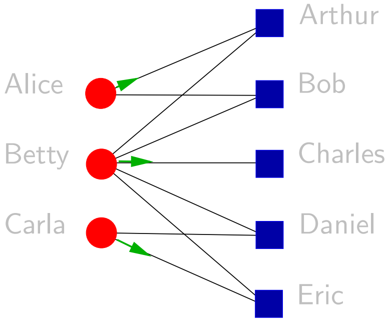
Goal: force some players' strategy such that a maximum number of pairs is formed

Idea: build a maximum matching and force some players to follow it

Big help

Alice plays Arthur
Betty plays Charles
Carla plays Eric

Every Nash equilibirum induces the pairs Alice-Arthur, Betty-Charles and Carla-Eric



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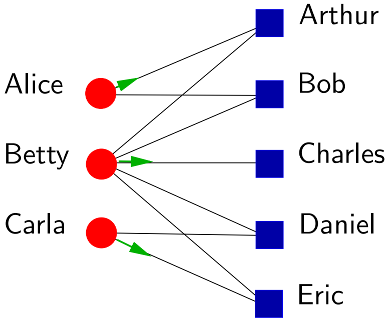


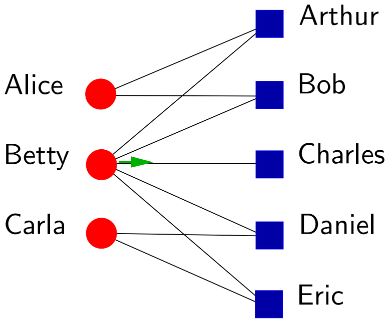
Help

a little help

Betty plays Charles

Every Nash equilibrium induces 3 pairs

- Betty-Charles
- Alice with Bob or Arthur
- Carla with Daniel or Eric



Price of optimum

Force a minimum number of players such that every induced equilibrium is socially optimal

Help

a little help

Betty plays Charles

Every Nash equilibrium induces 3 pairs

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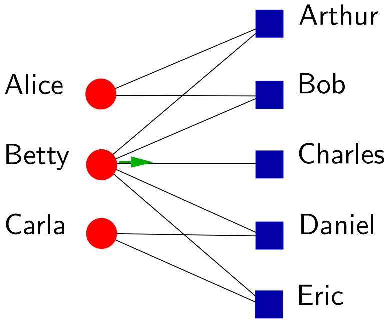
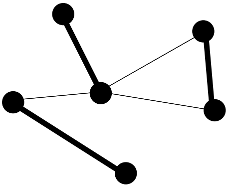


Price of optimum

Force a minimum number of players such that every induced equilibrium is socially optimal

Perfect matching

Set of non adjacent edges which touch all nodes



Finding a maximum size matching can be done in polynomial time

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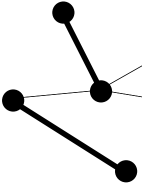
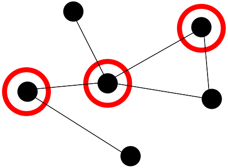


Finding a maximum size matching can be done in polynomial time

Vertex cover

Subset of nodes, adjacent to every edge



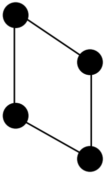
NP-hard to find a vertex cover of minimum size ; 2-approximation

Smallest help

Focus on graphs G admitting a perfect matching (not necessarilly bipartite)

Theorem

Forcing a **minimum** number of nodes such that every induced equilibrium is optimal is at least as difficult as the minimum vertex cover problem



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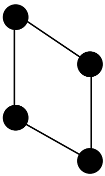

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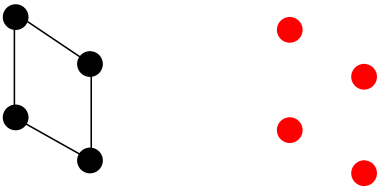

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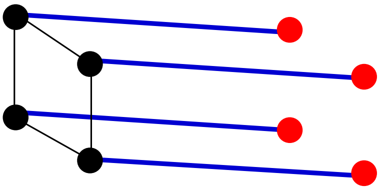

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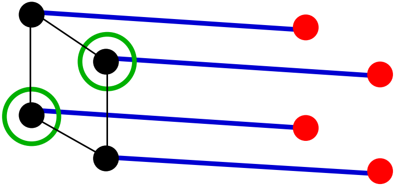

An algorithm

Forcing a **minimum** number of nodes in a graph G admitting a perfect matching

1. Build a maximum matching M of G
2. Let G' be the graph G without its leaves
3. Build a vertex cover C of G'
4. Force the nodes of C to follow M

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A sat game

A set of n variables $\{x_1, \ldots, x_n\}$

- each variable is set to T or F (true or false)
- two possible literals per variable: x_i and $\bar{x}_i$

A set of m clauses $\{C_1, \ldots, C_m\}$

Each clause C_j

- is a disjunction of 2 literals
- has a weight w_j

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Example

4 variables x_1, x_2, x_3, x_4 and 3 clauses

$$\begin{aligned} C_1 &= x_1 \vee x_2, & w_1 &= 4 \\ C_2 &= \bar{x}_3 \vee x_4, & w_2 &= 2 \\ C_3 &= x_2 \vee \bar{x}_4, & w_3 &= 1 \end{aligned}$$

All clauses satisfied when $(x_1, x_2, x_3, x_4) = (T, T, F, F)$

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A sat game

Each variable is controlled by a player with strategy space $\{T, F\}$

Under strategy profile σ, player x_i receives a **reward** from clause C_j

$$\rho_i(\sigma, C_j) = \begin{cases} 0 & \text{if } x_i \notin C_j \text{ or } x_i \text{ does not satisfy } C_j \\ w_j & \text{if } x_i \in C_j \text{ and } x_I \text{ is the only variable to satisfy } C_j \\ w_j/2 & \text{if } x_i \in C_j \text{ and } x_i \text{ satisfies } C_j \text{ together with another variable } x_{i'} \end{cases}$$

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When $\sigma = (T, T, F, F)$, $u_1(\sigma) = 2$, $u_2(\sigma) = 5/2$, $u_3(\sigma) = 2$, and $u_4(\sigma) = 1/2$

Social utility

$U(\sigma) = \sum_{i=1}^{n} u_i(\sigma)$ (utilitarian)

$U(\sigma) =$ total weight of the clauses satisfied by σ

Example

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Price of anarchy

Proposition

The price of anarchy of the sat game is 2/3

Proof

Take a Nash equilibrium σ: $u_i(\sigma) \geq u_i(\bar{\sigma}_i, \sigma_{-i})$, $\forall i$

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$\mathrm{U}(\sigma) = \sum_{i=1}^{n} u_i(\sigma) \geq \sum_{i=1}^{n} u_i(\bar{\sigma}_i, \sigma_{-i}) = 2\left(\sum_{j=1}^{m} w_j - \mathrm{U}(\sigma)\right)$

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Social optimum: $\mathrm{U}(\sigma^*) \leq \sum_{j=1}^{m} w_j$

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$$\bar{x}_1 \vee \bar{x}_2$$
$$\bar{x}_3 \vee \bar{x}_4$$

$w_j = 1, \forall j$

$\sigma = (T, T, T, T, T, T)$ is a Nash equilibrium with $\mathrm{U}(\sigma) = 4$

$\sigma^* = (F, T, F, T, T, T)$ is a social optimum with $\mathrm{U}(\sigma^*) = 6$

Price of anarchy

$$x_1 \vee x_2$$
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Improved price of anarchy

Change the reward function as follows

$$\rho'_i(\sigma, C_j) = \begin{cases} 0 & \text{if } x_i \notin C_j \text{ or } x_i \text{ does not satisfy } C_j; \\ w_j & \text{if } x_i \in C_j \text{ and } x_i \text{ is the only variable to satisfy } C_j; \\ w_j/3 & \text{if } x_i \in C_j \text{ and } x_i \text{ satisfies } C_j \text{ together with another variable } x_{i'}. \end{cases}$$

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Proposition

With σ', the price of anarchy of the sat game is $3/4$

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$ZERO(\sigma) =$ weight of clauses satisfied by 0 variable
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Improved price of anarchy

Proposition

With σ', the price of anarchy of the sat game is $3/4$

Proof

$ZERO(\sigma) =$ weight of clauses satisfied by 0 variable
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Take a Nash equilibrium σ: $u_i(\sigma) \geq u_i(\bar{\sigma}_i, \sigma_{-i}), \forall i$

$$\begin{aligned} \sum_{i=1}^{n} u_i(\sigma) &\geq \sum_{i=1}^{n} u_i(\bar{\sigma}_i, \sigma_{-i}) \\ ONE(\sigma) + \frac{2}{3} TWO(\sigma) &\geq \frac{1}{3} ONE(\sigma) + 2ZERO(\sigma) \end{aligned}$$

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\sum_{i=1}^{n} u_i(\sigma) &\geq u_i(\bar{\sigma}_i, \sigma_{-i}) \\
ONE(\sigma) + \frac{2}{3} TWO(\sigma) &\geq \frac{1}{3} ONE(\sigma) + 2ZERO(\sigma) \\
\frac{2}{3}\left(ONE(\sigma) + TWO(\sigma)\right) &\geq 2ZERO(\sigma) \\
4\left(ONE(\sigma) + TWO(\sigma)\right) &\geq 3\left(ZERO(\sigma) + ONE(\sigma) + TWO(\sigma)\right) \\
4\mathrm{U}(\sigma) &\geq 3\mathrm{U}(\sigma^*)
\end{aligned}$$

The "best" possible price of anarchy

Can we design a better reward function such that the price of anarchy is $> 3/4$?

Suppose $\rho_i(\sigma, C_j) = f(\ell_j(\sigma)) * w_j$ where f is a function which solely depends on $\ell_j(\sigma)$ defined as the number of players satisfying clause C_j under strategy profile σ

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C_1 &= x_1 \vee \bar{x}_3, & w_1 &= 1 \\
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Thank you!