Computational Issues in Simple and Influence Games

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Framework

- Topics
  - Coalitional Game Theory
  - Decision/Voting/Social Choice Theory
  - Social Network Analysis
  - Algorithms and Complexity
- Models
  - Simple Games
  - Directed Graphs and Collective Choice Models
- Focus
  - Subfamilies amilies of simple games
  - Complexity study of some properties of simple games.

Simple Games

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  A simple game is a pair $(N, \mathcal{W})$:
  - N is a set of players,
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- Members of $N = \{1, \ldots, n\}$ are called *players* or *voters*. Any set of voters is called a *coalition*
  - N is the *grand coalition*
  - $\emptyset$ is the *null coalition*
  - the subsets of N that are in $\mathcal{W}$ are the *winning coalitions*
  - A subset of N that is not in $\mathcal{W}$ is a *losing coalition*.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N:

- *winning coalitions* $\mathcal{W}$.
- *losing coalitions* $\mathcal{L}$.
- *minimal winning coalitions* $\mathcal{W}^m$
  $\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$
- *maximal losing coalitions* $\mathcal{L}^M$
  $\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq L\}$

This provides us with many representation forms for simple games.
We concentrate on the explicit representation of those set families.

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Decision is taken without interplay of the participants

Influence Games: influence spreading model

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- An influence graph is a tuple (G, f), where:
  - $G = (V, E)$ is a labeled and directed graph, and
  - $f : V \rightarrow \mathbb{N}$ is a labeling function that quantify how influenceable each node, player or agent is.

Influence Games: influence spreading model

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  From an *initial activation* $X \subseteq V$,
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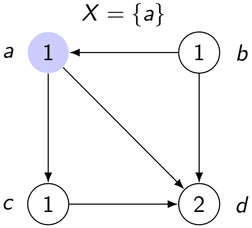
  Repeat until no more nodes are activated.



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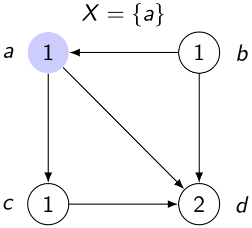

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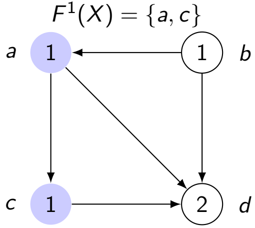

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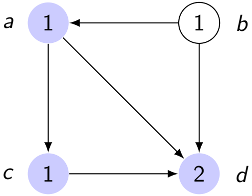
$F^2(X) = \{a, c, d\}$



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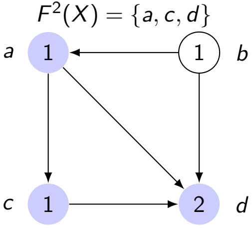

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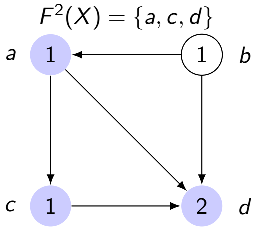
  $F(X)$ is polynomial time computable.



Influence Games

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- An influence game is a tuple (G, f, q, N), where:
  - (G, f) is an influence graph,
  - $N \subseteq V(G)$ is the set of players, and
  - $q > 0$ is an integer, the *quota*.
  - $X \subseteq V$ is winning iff $|F(X)| \geq q$.

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  - $X \subseteq V$ is winning iff $|F(X)| \geq q$.
- F is monotonic, for any $X \subseteq N$ and $i \in N$, if $|F(X)| \geq q$ then $|F(X \cup \{i\})| \geq q$, and if $|F(X)| < q$ then $|F(X \backslash \{i\})| < q$.
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- Influence games are simple games.

Participants can being influenced to adopt a new trend but have negative "initial" disposition.

Input representations

- Simple Games
  $(N, \mathcal{W})$, $(N, \mathcal{W}^m)$, $(N, \mathcal{L})$, $(N, \mathcal{L}^M)$
- Influence games
  (G, w, f, q, N)
- Weighted voting games
  $(q; w_1, \ldots, w_n)$

All numbers are integers

Problems on simple games

In general we state a property P, for simple games, and consider the associated decision problem which has the form:

Name: IsP
Input: *A simple/influence/weighted voting game* Γ
Question: *Does* Γ *satisfy property* P*?*

Four properties

A simple game (N, W) is

- strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
- proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
- a weighted voting game.
- an influence game.

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Theorem

The IsStrong *problem, when* Γ *is given in explicit losing or maximal losing form can be solved in polynomial time.*

- First observe that, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.
- Therefore, the IsStrong problem, when the input is given in explicit losing form is polynomial time solvable.

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin W$ implies $N \setminus S \in W$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in L \land N \setminus S \in L$$

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- A simple game is not strong iff

$$\exists S \subseteq N : S \in L \ \wedge \ N \setminus S \in L$$

which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in L^M : S \subseteq L_1 \ \wedge \ N \setminus S \subseteq L_2$$

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- which is equivalent to there are two maximal losing coalitions L_1 and L_2 such that $L_1 \cup L_2 = N$.

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- This Ican be checked in polynomial time, given $\mathcal{L}^M$.

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The IsStrong *problem is coNP-complete when the input game is given in explicit minimal winning form.*

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- The property can be expressed as

$$\forall S \, [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property $S \in \mathcal{W}$ can be checked in polynomial time given S and $\mathcal{W}^m$.
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection C of subsets of a finite set N. The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.

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We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m).

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- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m.
- Therefore, (N, C) has a set splitting iff (N, C^m) is not proper.

IsProper: winning forms

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

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The IsProper *problem, when the game is given in explicit winning or minimal winning form, can be solved in polynomial time.*

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The IsProper *problem, when the game is given in explicit winning or minimal winning form, can be solved in polynomial time.*

- As before, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time. Therefore, Taking into account the definitions, the IsProper problem is polynomial time solvable.

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- Which can be checked in polynomial time when $\mathcal{W}^m$ is given.

IsProper: maximal losing form

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- Therefore IsProper belongs to coNP.

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- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, W^m)$, in minimal winning form, we provide its dual game $\Gamma' = (N, \{N \setminus L : L \in W^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.

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- Given a game $\Gamma = (N, W^m)$, in minimal winning form, we provide its dual game $\Gamma' = (N, \{N \setminus L : L \in W^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.
- Besides, a game is strong iff its dual is proper

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> *Question: Is there* $S \subseteq \{1, \ldots, n\}$ *for which*

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Observe that, for any instance of the PARTITION problem in which the sum of the n input numbers is odd, the answer must be NO.

Weighted voting games

Theorem

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The IsStrong *and the* IsProper *problems, when the input is described by an integer realization of a weighted game $(q; w)$, are coNP-complete.*

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
- Observe that the weighted game with integer representation $(2; 1, 1, 1)$ is both proper and strong.

Hardness

We transform an instance $x = (x_1, \ldots, x_n)$ of PARTITION into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

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- Function f can be computed in polynomial time provided q does.
- Independently of q, when $x_1 + \cdots + x_n$ is *odd*, x is a NO input for partition, but $f(x)$ is a YES instance of ISSTRONG or ISPROPER.

IsStrong

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$.

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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are losing coalitions and $f(x)$ is not strong.

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- If S and $N \setminus S$ are losing coalitions in $f(x)$.
  If $\sum_{i \in S} x_i < s$ then $\sum_{i \notin S} x_i \geq s + 1$, $N \setminus S$ should be winning.
  Thus $\sum_{i \in S} x_i = \sum_{i \notin S} x_i = s$, and there exists a partition of x.

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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.
- When $f(x)$ is not proper

$$\exists S \subseteq N : \sum_{i \in S} x_i \geq s \wedge \sum_{i \notin S} x_i \geq s,$$

and thus $\sum_{i \in S} x_i = s$.

Influence games: $\Gamma(G)$

Influence games: $\Gamma(G)$

Let's consider a particular type of influence games.

Definition

Given an undirected graph $G = (V, E)$,
$\Gamma(G)$ is the influence game $(G, f, |V|, V)$ where,
for any $v \in V$, $f(v) = d_G(v)$.

Influence games: $\Gamma(G)$

Recall that a set $S \subseteq V$ is a *vertex cover* of a graph G if and only if, for any edge $(u, v) \in E$, u or v (or both) belong to S. From the definitions we get the following result.

Influence games: $\Gamma(G)$

Recall that a set $S \subseteq V$ is a *vertex cover* of a graph G if and only if, for any edge $(u, v) \in E$, u or v (or both) belong to S. From the definitions we get the following result.

Lemma

Let G be an undirected graph. X is winning in $\Gamma(G)$ if and only if X is a vertex cover of G, Furthermore, the influence game $\Gamma(G)$ can be obtained in polynomial time, given a description of G.

Isproper and IsStrong

Theorem

For unweighted influence games IsProper *and* IsStrong *are* coNP*-complete.*

Isproper and IsStrong

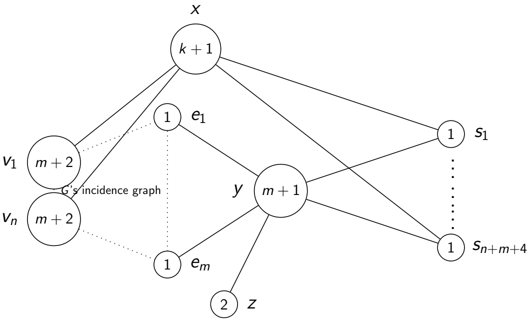
Theorem

For unweighted influence games IsProper *and* IsStrong *are* coNP*-complete.*

- Membership in coNP follows from the definitions.
- To get the hardness results, we provide reductions from problems related to Vertex Cover.
- Assume that a graph G has n vertices and m edges.

$\Delta_1(G, k)$

Let $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$.
Set $\alpha = m + n + 4$ and consider the influence graph (G_1, f_1):



$\Delta_1(G, k)$

$\alpha = m + n + 4$, $G_1 = (V_1, E_1)$ and f_1 define (G_1, f_1)

- $V_1 = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\}$.
- E_1 has edge (z, y) and
  - $(e, v_i), (e, v_j), (e, y)$, for $e = (v_i, v_j) \in E$
  - (v_i, x), for $1 \leq i \leq n$ and $(x, s_j), (y, s_j)$, for $1 \leq j \leq \alpha$.
- The labeling function f_1 is:
  $f_1(v_i) = m + 2$, $1 \leq i \leq n$; $f_1(e_j) = 1$, $1 \leq j \leq m$;
  $f_1(s_\ell) = 1$, $1 \leq \ell \leq \alpha$; and
  $f_1(z) = 2$, $f_1(x) = k + 1$, $f_1(y) = m + 1$.

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$\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$ where $q_1 = \alpha$ and $N_1 = \{v_1, \ldots, v_n, z\}$.

To prove hardness of ISPROPER, we provide a reduction from the following variation of the VERTEX COVER problem:

> *Name:* HALF VERTEX COVER
> *Input: Given a graph with an odd number of vertices n.*
> *Question: Is there a vertex cover with size* $\leq (n-1)/2$*?*

which is also NP-complete.

IsProper

- Let G be an instance of HALF VERTEX COVER with $n = 2k + 1$ vertices, for some value $k \geq 1$.
- Consider the influence game $\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$

IsProper

- Let G be an instance of HALF VERTEX COVER with $n = 2k + 1$ vertices, for some value $k \geq 1$.
- Consider the influence game $\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$
- Trivially $\Delta_1(G, k)$ can be obtained in polynomial time,

If G has a vertex cover X with $|X| \leq k$,

- $F(X \cup \{z\}) \geq q_1$.
- But as $n + 1 - |X \cup \{z\}| > k$, $F(N \setminus (X \cup \{z\})) \geq q_1$.
- Hence $\Delta_1(G, k)$ is not proper.

When all the vertex covers of G have more than k vertices,

- to have $F(Y) \geq q_1$ we need $|Y \cap \{v_1, \ldots, v_n\}| > k$, i.e., $|Y \cap \{v_1, \ldots, v_n\}| \geq k + 1$.
- For a Y, with $F(Y) \geq q_1$ we have two cases:
    - $z \in Y$, then
      $N \setminus Y \subseteq \{v_1, \ldots, v_n\}$ and $|N \setminus Y| \leq n - k - 1 = k$.
      Thus, $F(N \setminus Y) < q_1$.
    - $z \notin Y$, then
      $|N \setminus (Y \cup \{z\})| \leq k$ and $F(N \setminus Y) < q_1$
- So, we conclude that $\Delta_1(G, k)$ is proper.

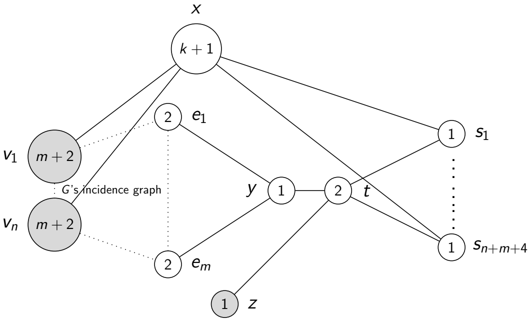
Thus the IsProper problem is coNP-hard.

To finish the proof we show hardness for the IsStrong problem. We need another problem.

> *Name:* Half independent set
> *Input: Given a graph with an even number of vertices n.*
> *Question: Is there an independent set with size* $\geq n/2$*?*

The Half independent set trivially belongs to NP. Hardness follows from a simple reduction from Half Vertex Cover.

Now we show that the complement of the HALF INDEPENDENT SET problem can be reduced to the ISSTRONG problem.
We define first an influence graph (G_3, f_3):



We associate to an input to HALF INDEPENDENT SET the game

$$\Delta_3(G) = (G_3, f_3, n + m + 5, N_3)$$

where $N_3 = V \cup \{z\}$ and (G_3, f_3) is the influence graph described before.

- When G has an independent set with size at least $n/2$, G also has an independent set X with $|X| = n/2$.
- It is easy to see that both $X \cup \{z\}$ and its complement are losing coalitions in $\Delta_3(G)$. Therefore, $\Delta_3(G)$ is not strong.

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    - When $|X \cap V| < n/2$, its complement has at least $n/2 + 1$ elements in V and thus it is winning.

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  - When $|X \cap V| > n/2$, X wins and we have to consider only those teams with $|X \cap V| = n/2$.
  - But now neither $X \cap V$ nor $V \setminus (X \cap V)$ are independent sets. Then, X or $N \setminus X$ must contain z and is winning while its complement is losing.

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  - But now neither $X \cap V$ nor $V \setminus (X \cap V)$ are independent sets. Then, X or $N \setminus X$ must contain z and is winning while its complement is losing.
- So, $\Delta_3(G)$ is strong.

Subfamilies of Influence Games

Maximum Influence Game

$\Gamma = (G, f, |V|, V)$ where $f(v) = d_G(v)$, for $v \in V$ $(\Gamma = \Gamma(G))$

Minimum Influence Game

$\Gamma = (G, 1_V, q, N)$ where $1_V(v) = 1$, for $v \in V$.

Maximum Influence games

Lemma

In a maximum influence game Γ *on a connected graph* G *the following properties hold.*

- Γ *is proper if and only if* G *is either not bipartite or a singleton.*
- Γ *is strong if and only if* G *is either a star or a triangle.*

Maximum Influence games: IsProper

Observe that in

- We know that winning coalitions of $\Gamma = \Gamma(G)$ coincide with the vertex covers of G.
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- If G is a singleton $\Gamma(G)$ is proper. Otherwise,
- If $G = (V, E)$ is bipartite, let (V_1, V_2) be a partition of V so that V_1 and V_2 are independent sets.
  Now V_1 and $V_2 = N \setminus V_1$ are winning and Γ is not proper.

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- If $G = (V, E)$ is bipartite, let (V_1, V_2) be a partition of V so that V_1 and V_2 are independent sets.
  Now V_1 and $V_2 = N \setminus V_1$ are winning and Γ is not proper.
- if Γ is not proper, then the game admits two disjoint wining coalitions i.e, two disjoint vertex covers of G, and hence both of them must be independent sets.
  Thus G is bipartite.

Maximum Influence games: IsStrong

Now we prove that Γ is not strong if and only if G has at least two non-incident edges.

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- If G has at least two non-incident edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, $\{u_1, v_1\}$ and $N \setminus \{u_1, v_1\}$ are both winning and Γ is not strong.

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- When the game is not strong, there is X such that both X and $N \setminus X$ are losing.
  For this to happen there must be an edge uncovered by X and another edge uncovered by $N \setminus X$. Thus G must have two non-incident edges.

Minimum Influence

$\Gamma = (G, 1_V, q, N)$ where $1_V(v) = 1$ for any $v \in V$.

- Observe that, if G is connected, the game has a trivial structure as any non-empty vertex subset of N is a successful team.
- For the disconnected case we can analyze the game with respect to a suitable weighted game.

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- For the disconnected case we can analyze the game with respect to a suitable weighted game.
- Assume that G has k connected components, $C_1, \ldots, C_k$. Without loss of generality, we assume that all the connected components of G have non-empty intersection with N. For $1 \leq i \leq k$, let $w_i = |V(C_i)|$ and $n_i = |V(C_i) \cap N|$.

Minimum Influence

Lemma

If a wining coalition is minimal then it has at most one node in each connected component. Minimal wining coalitions are in a many-to-one correspondence with the minimal winning coalitions of the weighted game $[q; w_1, \ldots, w_k]$.

Minimum Influence:Knapsack

We consider now two problems (all numbers are integers):

Name: KNAPSACK
Input: Given n objects, for $1 \leq i \leq n$, w_i *and* v_i, *and* k.
Question: Find a subset $S \subseteq \{1, \ldots, n\}$ *with* $\sum_{i \in S} w_i \leq k$ *and maximum* $\sum_{i \in S} v_i$.

Name: 0-1-KNAPSACK
Input: Given a finite set U, for each $i \in U$, a weight w_i, and a positive integer k.
Question: Is there a subset $S \subseteq U$ *with* $\sum_{i \in S} w_i = k$?

Both problems can be solved in pseudo polynomial time: when all the weights are at most $p(n)$.

Minimum Influence

Theorem

For unweighted influence games with minimum influence, the problems IsProper *and* IsStrong *belong to* P.

Minimum Influence: IsProper

Let $\Gamma = (G, 1_V, q, N)$ be an unweighted influence game with minimum influence.

- For the IsProper problem it is enough to check whether there is a winning coalition whose complement is also winning and answer accordingly.

Minimum Influence: IsProper

Let $\Gamma = (G, 1_V, q, N)$ be an unweighted influence game with minimum influence.

- For the IsProper problem it is enough to check whether there is a winning coalition whose complement is also winning and answer accordingly.
- We separate the connected components in two sets: those containing one player and those containing more that one player.
  Let $A = \{i \mid n_i = 1\}$ and $B = \{i \mid n_i > 1\}$.
  Let $N_A = \cup_{i \in A}(N \cap V(C_i))$ and $N_B = N \setminus N_A$.
  Let $w_A = \sum_{i \in A} w_i$ and $w_B = w_N - w_A$.

- As all the components in B have at least two vertices, we can find a set $X \subseteq N_B$ such that $|F(X)| = |F(N_B \setminus X)| = w_B$.
- If $w_B \geq q$ the game is not proper.
- If $w_B < q$ the game is proper iff the influence game Γ' played on the graph formed by the connected components belonging to A and quota $q' = q - w_B$ is proper.
  Observe that Γ' is equivalent to the weighted game with a player for each component in $i \in A$ with associated weight w_i and quota q'.
- Let α_{min} be the minimum $\alpha \in \{q', \ldots, w_A\}$ for which there is a set $S \subseteq A$ with $\sum_{i \in S} w_i = \alpha$.
  Observe that Γ' is proper if and only if $w_A - \alpha_{min} < q'$.
- The value α_{min} can be computed by solving several instances of the 0-1-KNAPSACK with weights polynomial in n.

Minimum Influence: IsStrong

Now we prove that the IsStrong problem belongs to P.

- Observe that in order to minimize the influence of the complement of a team X it is enough to consider only those teams X that contain all or none of the players in a connected component.
- Let $w_N = \sum_{i=1}^{k} w_i$, and let α_{max} be the maximum $\alpha \in \{0, \ldots, q-1\}$ for which there is a set $S \subseteq \{1, \ldots, k\}$ with $\sum_{i \in S} w_i = \alpha$.
  Note that α can be zero and thus S can be the empty set.

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  Note that α can be zero and thus S can be the empty set.
- Γ is strong iff $w_N - \alpha_{max} \geq q$.
- The value α_{max} can be computed by solving several instances of the 0-1-Knapsack problem with weights $\leq n$.

Explicit forms

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit winning or losing form.*

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Lemma

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We can obtain $\mathcal{W}^m$ and $\mathcal{L}^M$ in polynomial time.
Once this is done we write, in polynomial time, the LP

$$\begin{array}{lll} \texttt{min } q & & \\ \texttt{subject to} & w(S) \geq q & \texttt{if } S \in W^m \\ & w(S) < q & \texttt{if } S \in L^M \\ & 0 \leq w_i & \texttt{for all } 1 \leq i \leq n \\ & 0 \leq q & \end{array}$$

IsWheigthed: Minimal and Maximal

Lemma

The ISWEIGHTED *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

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- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.

IsWheigthed: Minimal and Maximal

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function Φ_{W^m} given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\wedge_{i \in S} x_i)$$

IsWheigthed: Minimal and Maximal

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

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IsWheigthed: Minimal and Maximal

By construction we have the following:

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- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.
- Further Φ_{W^m} is monotonic (i.e. *positive*)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.

- On the other hand, we can prove a similar result given (N, L^M) just taking into account that a game Γ is weighted iff its dual game Γ' is weighted.
- Thus we can compute a minimal winning representation of the dual of (N, L^M) in polynomial time and use the previous result.

IsWeighted: Influence

Open

The complexity of the IsWeighthed problem for influence graphs has not been addressed yet.

IsInfluence

Theorem

Every simple game can be represented by an influence game. Furthermore, when the simple game Γ is given by either $(N, \mathcal{W})$ or $(N, \mathcal{W}^m)$, an unweighted influence game representing Γ can be obtained in polynomial time.

IsInfluence

Theorem

Every simple game can be represented by an influence game. Furthermore, when the simple game Γ is given by either $(N, \mathcal{W})$ or $(N, \mathcal{W}^m)$, an unweighted influence game representing Γ can be obtained in polynomial time.

- It is already well known that given $(N, \mathcal{W})$, the family $\mathcal{W}^m$ can be obtained in polynomial time.
- Thus we assume in the following that the set of players and the set $\mathcal{W}^m$ are given.

IsInfluence

- Define the graph $G = (V, E)$ as
  - V contains $V_N = \{v_1, \dots, v_n\}$, one vertex for player and,
  - for $X \in \mathcal{W}^m$, a set V_X with $n + 1 - |X|$ nodes.
  - We connect vertex v_i with all the vertices in V_X whenever $i \in X$.
- For any $1 \leq i \leq n$, $f(v_i) = 1$ and, for any $X \in \mathcal{W}^m$ and any $v \in V_X$, $f(v) = |X|$.
- Observe that in the influence game $(G, f, n + 1, V_N)$ a coalition is winning iff its players form a winning coalition in Γ.
- Given $(N, \mathcal{W}^m)$ a description of $(G, f, n + 1, N)$ can be computed in polynomial time.

Opinion Leader-Follower

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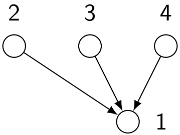

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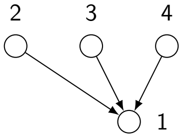

  - V is divides as L: leaders, F: followers and I: independent.

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- An initial decision vector $x \in \{0,1\}^n$ is mapped to a final decision vector $c = c^{\mathcal{M}}(x)$ for $1 \leq i \leq n$, has

$$c_i = \begin{cases} 1 & \text{if } |\{j \in P_G(i) \mid x_j = 1\}| \geq \lceil r \cdot |P_G(i)| \rceil \\ & \qquad \text{and } |\{j \in P_G(i) \mid x_j = 0\}| < \lceil r \cdot |P_G(i)| \rceil \\ 0 & \text{if } |\{j \in P_G(i) \mid x_j = 0\}| \geq \lceil r \cdot |P_G(i)| \rceil \\ & \qquad \text{and } |\{j \in P_G(i) \mid x_j = 1\}| < \lceil r \cdot |P_G(i)| \rceil \\ x_i & \text{otherwise.} \end{cases}$$

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- An initial decision vector $x \in \{0, 1\}^n$ is mapped to a final decision vector $c = c^{\mathcal{M}}(x)$.

Opinion Leader-Follower

- Generalized opinion leader-follower model (gOLF)
  A gOLF is a triple $\mathcal{M} = (G, r, q)$ where
  - $G = (V, E)$ is a two layered bipartite digraph.
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- An initial decision vector $x \in \{0,1\}^n$ is mapped to a final decision vector $c = c^{\mathcal{M}}(x)$.
- The collective decision function $C_{\mathcal{M}} : \{0,1\}^n \to \{0,1\}$ is defined as

$$C_{\mathcal{M}}(x) = \begin{cases} 1 & \text{if } |\{i \in V \mid c_i(x) = 1\}| \geq q \\ 0 & \text{otherwise.} \end{cases}$$

Opinion Leader-Follower

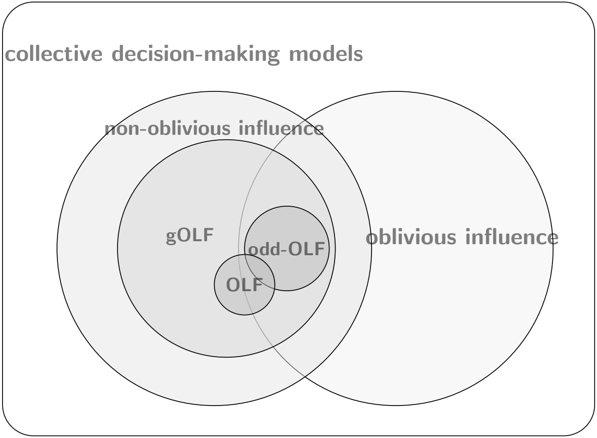
- Opinion leader-follower model (OLF) (van den Brink et al. 2011)
  n must be odd and q is set to $(n+1)/2$
  Decision follows single majority rule on final decision.
- odd-opinion leader-follower model (odd-OLF)
  $r = 1/2$ and for all $i \in \mathtt{F}$, $\delta^{-}(i)$ is odd.
  Change of decision follows single majority rule on an odd set.

Oblivious and non-oblivious influence systems

Let (G, f, q, N) be an influence game and $x \in \{0,1\}^n$.
Compute $F(x)$ and let $y_i = 1$ iff $i \in F(x)$

- non-oblivious influence model
  The final decision vector is y and a collective decision is taken with quota q.
- non-oblivious influence model
  The final decision vector considers y and if a follower $(V \setminus N)$ detects a tie on yes-no retracts to its initial decision.

Relationship among decision systems



Satisfaction in influence decision systems

Introduced for OLF in (van den Brink et. al., 2011)

- Let $\mathcal{M}$ be a collective decision-making model over a set of n actors.
- For an initial decision vector $x \in \{0,1\}^n$, an actor i is *satisfied* when $C_{\mathcal{M}}(x) = x_i$.
- The Satisfaction Measure of the actor i corresponds to the number of initial decision vectors for which the actor is satisfied, i.e.,

$$\textsc{Sat}_{\mathcal{M}}(i) = |\{x \in \{0,1\}^n \mid C(x) = x_i\}|.$$

Satisfaction and power indices

Let $\Gamma = (N, \mathcal{W})$ be a simple game.

- The Banzhaf value of player $i \in N$ is $\mathrm{Bz}_\Gamma(i) = |\{X \in \mathcal{W} \mid X \setminus \{i\} \notin \mathcal{W}\}|$.
- The Rae index of player $i \in N$ is $\mathrm{Rae}_\Gamma(i) = |\{X \in \mathcal{W} \mid i \in X\}| + |\{X \notin \mathcal{W} \mid i \notin X\}|$.

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- $\mathrm{RAE}_\Gamma(i) = 2^{n-1} + \mathrm{Bz}_\Gamma(i)$. (Dubey and Shapley 1979)

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Let $\mathcal{M}$ be a monotonic decision-making model on a set of actors V, for $i \in V$, $\mathrm{Sat}_\mathcal{M}(i) = \mathrm{Rae}_{\Gamma_\mathcal{M}}(i)$.

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Lemma

Oblivious and non-oblivious decision models are monotonic.

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Theorem

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Corollary

Computing the Satisfaction measure, the Banzhaf value or the Rae index if #P-hard for two layered bipartite oblivious and non-oblivious influence models.

Strong Hierarchical digraphs: Graph operations

- Disjoint union
  Given two graphs H_1 and H_2 with $V(H_1) \cap V(H_2) = \emptyset$,
  $H_1 + H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$.
- One layer extension
  Given a graph H, an a set $V' \neq \emptyset$ with $V(H) \cap V' = \emptyset$,
  $H \otimes V' = (V(H) \cup V', E(H) \cup \{(u, v) \mid u \in \mathtt{FI}(H), v \in V'\}$.

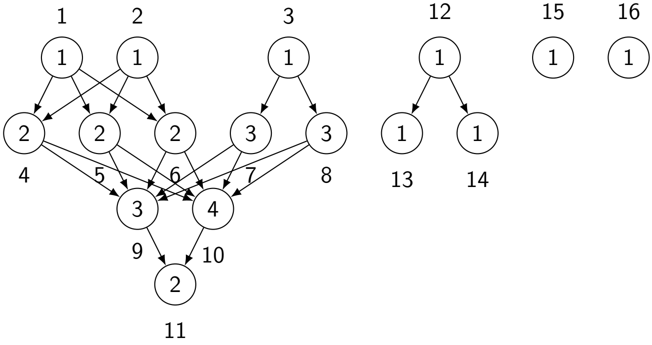
$\mathtt{FI}(H)$ is formed by all nodes with out degree 0.
$\mathtt{LI}(H)$ is formed by all nodes with in degree 0.
In addition to leaders and follower we have mediators

Strong Hierarchical digraphs

The family of strong hierarchical graphs is defined recursively as follows.

- The graph I_a, for $a > 0$, is a strong hierarchical graph.
- If H_1 and H_2 are disjoint strong hierarchical graphs, their disjoint union $H_1 + H_2$ is a strong hierarchical graph.
- If H is a strong hierarchical graph and $V' \neq \emptyset$ is a set of vertices with $V(H) \cap V' = \emptyset$, the graph $H \otimes V'$ is a strong hierarchical graph.

Strong Hierarchical digraphs



Strong hierarchical influence models

- A strong hierarchical influence graph is an influence graph (G, f) where G is a strong hierarchical graph.
- A strong hierarchical influence game is an influence game (G, f, q, N) where G is a strong hierarchical graph and $N = \mathtt{L}(G) \cup \mathtt{I}(G)$.

Computing satisfaction

Let (G, f, q, N) be an influence game. Let $|F_k(N, G, f)|$ be the number of $X \subseteq N$ with $|F(X)| = k$.

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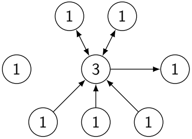
Theorem

The SATISFACTION *problem, for oblivious and non-oblivious models corresponding to strong hierarchical influence games, is polynomial time solvable.*

Star influence systems

- A star influence graph is an influence graph (G, f), where $V(G) = \mathtt{L} \cup \mathtt{I} \cup \mathtt{R} \cup \{c\} \cup \mathtt{F}$ and $E(G) = \{(u, c) \mid u \in \mathtt{L} \cup \mathtt{R}\} \cup \{(c, v) \mid v \in \mathtt{R} \cup \mathtt{F}\}$.



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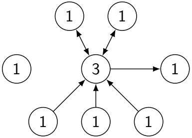

- A star influence game is a game $\Gamma = (G, f, q, N)$, where $N = \mathtt{L} \cup \mathtt{R} \cup \mathtt{I}$ and (G, f) is a star influence graph.

Computing the satisfaction measure

An extended star influence graph is obtained from a star influence graph (G', f'), by selecting one vertex $u \in \mathtt{R}(G')$ and adding a set of vertices F_u with label 1 and the set of edges $\{(u, v) \mid v \in F_u\}$.

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Theorem

The SATISFACTION *problem, for oblivious and non-oblivious models corresponding to star influence games, is polynomial time solvable.*

Conclusions

- We have analyzed simple, weighted and influence games.
- We have concentrated on the study of four computational problems.
- Each of the problems requires different tools for the analysis
- We had a glimpse to decision systems and models inspired by process in social networks.
- This later aspects provide an interesting area for further research

References

Contents taken from a subset of the results in

- J. Freixas, X. Molinero, M. Olsen, M. Serna: On the complexity of problems on simple games. RAIRO - Operations Research 45(4): 295-314 (2011)
- X. Molinero, F. Riquelme, M. Serna: Cooperation through social influence. European Journal of Operational Research 242(3): 960-974 (2015)
- X. Molinero, F. Riquelme, M. Serna: Measuring satisfaction in societies with opinion leaders and mediators. Submitted.

References

Further suggested reading

- H. Aziz: Algorithmic and complexity aspects of simple coalitional games
  PhD. Thesis, CS Dept. University of Warwick
- F. Riquelme: Structural and computational aspects of simple and influence games
  PhD. Thesis, CS Dept, UPC

Thanks!

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