Analytic Tableaux for KLM Preferential and Cumulative Logics*

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Abstract. We present tableau calculi for some logics of default reasoning, as defined by Kraus, Lehmann and Magidor. We give a tableau proof procedure for preferential and cumulative logics. Our calculi are obtained by introducing suitable modalities to interpret conditional assertions. Moreover, they give a decision procedure for the respective logics and can be used to establish their complexity.

1 Introduction

In the early 90' [11] Kraus, Lehmann and Magidor (from now on KLM) proposed a formalization of non-monotonic reasoning that was early recognized as a landmark. Their work stemmed from two sources: the theory of nonmonotonic consequence relations initiated by Gabbay [6] and the preferential semantics proposed by Shoham [13] as a generalization of Circumscription. Their works lead to a classification of nonmonotonic consequence relations, determining a hierarchy of stronger and stronger systems.

According to the KLM framework, defeasible knowledge is represented by a (finite) set of nonmonotonic conditionals or assertions of the form $A \mid\sim B$ whose reading is *normally (or typically) the A's are B's*. The operator "$\mid\sim$" is nonmonotonic, in the sense that $A \mid\sim B$ does not imply $A \wedge C \mid\sim B$. For instance, a knowledge base K may contain the following set of conditionals: $adult \mid\sim work, adult \mid\sim taxpayer, student \mid\sim adult, student \mid\sim \neg work, student \mid\sim \neg taxpayer, retired \mid\sim adult, retired \mid\sim \neg work$, whose meaning is that adults typically work, adults typically pay taxes, students are typically adults, but they typically do not work, nor do they pay taxes, and so on. Observe that if $\mid\sim$

* This research has been partially supported by the project MIUR PRIN 2003 "Logic-based development and verication of multi-agent systems".

G. Sutcliffe and A. Voronkov (Eds.): LPAR 2005, LNAI 3835, pp. 666–681, 2005.

were interpreted as classical (or intuitionistic) implication, we simply would get *student* $\mid\sim \bot$, *retired* $\mid\sim \bot$, i.e. typically there are not students, nor retired people, thereby obtaining a trivial knowledge base. One can derive new conditional assertions from the knowledge base by means of a set of inference rules.

In KLM framework, the set of adopted inference rules defines some fundamental types of inference systems, namely, from the weakest to the strongest: Cumulative (**C**) , Loop-Cumulative (**CL**), Preferential (**P**) and Rational logic (**R**). All these systems allow one to infer new assertions from K without incurring in the trivialising conclusions of classical logic: regarding our example, in none of them, one can infer *student* $\mid\sim$ *work* or *retired* $\mid\sim$ *work*. In cumulative logics (both **C** and **CL**) one can infer *adult* $\wedge$ *student* $\mid\sim \neg$*work* (giving preference to more specific information), in Preferential logic **P** one can also infer that *adult* $\mid\sim \neg$*retired* (i.e. typical adults are not retired). In the rational case **R**, if one further knows that *adult* $\not\mid\sim \neg$*married* (i.e. it is not the case the adults are typically unmarried), one can also infer that *adult* $\wedge$ *married* $\mid\sim$ *work*.

From a semantic point of view, to the each logic (**C**, **CL**, **P**, **R**) corresponds one kind of models, namely, possible-world structures equipped with a preference relation among worlds or states. More precisely, for **P** we have models with a preference relation (an irreflexive and transitive relation) on worlds. For the stronger **R** the preference relation is further assumed to be *modular*. For the weaker logic **CL**, the preference relation is defined on *states*, where a state can be identified, intuitively, with a set of worlds. In the weakest case of **C**, the preference relation is on states, as for **CL**, but it is no longer assumed to be transitive. In all cases, the meaning of a conditional assertion $A \mid\sim B$ is that B holds in the *most preferred* worlds/states where A holds.

In KLM framework the operator "$\mid\sim$" is considered as a meta-language operator, rather than as a connective in the object language. However, it has been readily observed that KLM systems **P** and **R** coincide to a large extent with the flat (i.e. unnested) fragments of well-known conditional logics, once we interpret the operator "$\mid\sim$" as a binary connective [3], [2], [10].

A recent result by Halpern and Friedman [4] has shown that preferential and rational logic are quite natural and general systems: surprisingly enough, the axiom system of preferential (likewise of rational logic) is complete with respect to a wide spectrum of semantics, from ranked models, to parametrized probabilistic structures, ϵ-semantics and possibilistic structures. The reason is that all these structures are examples of *plausibility structures* and the truth in them is captured by the axioms of preferential (or rational) logic. These results, and their extensions to the first order setting [5] are the source of a renewed interest in KLM framework.

Even if KLM was born as an inferential approach to nonmonotonic reasoning, curiously enough, there has not been much investigation on deductive mechanisms for these logics. In short, the state of the art is as follows:

- Lehmann and Magidor [12] have proved that validity in **P** is **coNP**-complete. Their decision procedure for **P** is more a theoretical tool than a practical algorithm, as it requires to guess sets of indexes and propositional evaluations.

They have also provided another procedure for **P** that exploits its reduction to **R**. However, the reduction of **P** to **R** breaks down if boolean combinations of conditionals are allowed, indeed it is exactly when such combinations are allowed that the difference between **P** and **R** arises.
- A tableau proof procedure for **C** has been given in [1]. Their tableau procedure is fairly complicated; it uses labels and it contains a cut-rule. Moreover, it is not clear how it can be adapted to **CL** and **P**.
- In [7] it is defined a labelled tableau calculus for the conditional logic **CE** whose flat fragment (i.e. without nested conditionals) corresponds to **P**. That calculus needs a fairly complicated loop-checking mechanism to ensure termination. It is not clear if it matches complexity bounds and if it can be adapted in a simple way to **CL**.
- Finally, decidability of **P** and **R** has also been obtained by interpreting them into standard modal logics, as it is done by Boutilier [2]. However, his mapping is not very direct and natural, as we discuss below.
- To the best of our knowledge, for **CL** no decision procedure and complexity bound was known before the present work.

In this work we begin our investigation of tableau procedures for KLM logics, by considering the cases of **P** and **CL**. The investigation of tableau calculi for the weakest **C** and the strongest **R** is left for future work. Our approach is based on a novel interpretation of **P** into modal logics. As a difference with previous approaches (e.g. Lamarre [3] and Boutillier [2]), that take S4 as the modal counterpart of **P**, we consider here modal logic G. Our tableau method provides a sort of run-time translation of **P** into modal logic G.

The idea is simply to interpret the preference relation as an accessibility relation: a conditional $A \mid\sim B$ holds in a model if B is true in all A-worlds w that are minimal. An A-world is minimal if all smaller worlds are not A-worlds. The relation with modal logic G is motivated by the fact that we assume, following KLM, the so-called *smoothness condition*, which is related to the well-known *limit assumption*. This condition ensures indeed that A-minimal worlds exist, by preventing an infinitely descending chain of worlds. This condition is therefore ensured by the finite-chain condition on the accessibility relation (as in modal logic G). Therefore, our interpretation of conditionals is different from the one proposed by Boutilier, who rejects the smoothness condition and then gives a less natural (and more complicated) interpretation of **P** into modal logic S4.

However, we do not give a formal translation of **P** into G, we appeal to the correspondence as far as it is needed to derive the tableau rules for **P**. For deductive purposes, we believe that our approach is more direct, intuitive, and efficient than translating **P** into G and then using a calculus for G.

We are able to extend our approach to the case of **CL** by using a second modality which takes care of states. More precisely, we show that we can map **CL**-models into **P**-models with an additional modality. The very fact that one can interpret **CL** into **P** by means of an additional modality does not seem to be previously known and might be of independent interest. In both cases, **P** and **CL**, we can define a decision procedure and obtain also a complexity bound for these logics, namely that they are both **coNP**-complete. In case of **CL** this bound is new, to the best of our knowledge.

2 KLM Logics

We briefly recall the axiomatizations and semantics of the two KLM systems we consider: **P** and **CL**. For a complete picture of KLM systems, see [11].

2.1 Preferential Logic P

The language of KLM logics consists just of conditional assertions $A \mid\sim B$. We consider a richer language allowing boolean combinations of assertions and propositional formulas. Our language $\mathcal{L}$ is defined from a set of propositional variables ATM, the boolean connectives and the conditional operator $\mid\sim$. We use $A, B, C, ...$ to denote propositional formulas, whereas $F, G, ...$ are used to denote all formulas (even conditionals); $\Gamma, \Delta, ...$ represent sets of formulas, whereas $X, Y, ...$ denote sets of sets of formulas. The formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\sim B \in \mathcal{L}$; if F is a boolean combination of formulas of $\mathcal{L}$, $F \in \mathcal{L}$.

The axiomatization of **P** consists of all axioms and rules of propositional calculus together with the following axioms and rules (notice that $\vdash$ denotes provability in the propositional calculus):

- REF. $A \mid\sim A$ (reflexivity)
- LLE. If $\vdash A \leftrightarrow B$, then $(A \mid\sim C) \rightarrow (B \mid\sim C)$ (left logical equivalence)
- RW. If $\vdash A \rightarrow B$, then $(C \mid\sim A) \rightarrow (C \mid\sim B)$ (right weakening)
- CM. $((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \wedge B \mid\sim C)$ (cautious monotonicity)
- AND. $((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \mid\sim B \wedge C)$
- OR. $((A \mid\sim C) \wedge (B \mid\sim C)) \rightarrow (A \vee B \mid\sim C)$

The semantics of **P** is defined by considering possible world structures with a preference relation (a strict partial order) $w < w^{'}$ whose meaning is that w is preferred to $w^{'}$. We have that $A \mid\sim B$ holds in a model $\mathcal{M}$ if B holds in all *minimal worlds* (with respect to the relation $<$) where A holds. This definition makes sense provided minimal worlds for A exist whenever there are A-worlds. This is ensured by the *smoothness condition* in the next definition.

Definition 1 (Semantics of P, Definition 16 in [11]). *A preferential model is a triple $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ where: $\mathcal{W}$ is a non-empty set of items called worlds; $<$ is an irreflexive and transitive relation on $\mathcal{W}$; V is a function $V : \mathcal{W} \longmapsto pow(ATM)$, which assigns to every world w the set of atoms holding in that world. We define the truth conditions for a formula F as follows:*

- *If F is a boolean combination of formulas, $\mathcal{M}, w \models F$ is defined as for propositional logic;*
- *Let A be a propositional formula; we define $Min_<(A) = \{w \in W \mid \mathcal{M}, w \models A$ and $\forall w^{'}.w^{'} < w$ implies $\mathcal{M}, w^{'} \not\models A\}$;*
- *$\mathcal{M}, w \models A \mid\sim B$ if for all $w^{'} \in Min_<(A)$ we have $\mathcal{M}, w^{'} \models B$.*

The relation $<$ satisfies the following condition, called smoothness*: if $\mathcal{M}, w \models A$ then $w \in Min_<(A)$ or $\exists w' \in Min_<(A)$ such that $w' < w$.*
We say that a formula F is valid in a model $\mathcal{M}$*, denoted with $\mathcal{M} \models F$, if $\mathcal{M}, w \models F$ for every $w \in \mathcal{W}$. A formula is* valid *if it is valid in every model $\mathcal{M}$.*

Notice that the truth conditions for conditional formulas are given with respect to single possible worlds for uniformity sake. Since the truth value of a conditional only depends on global properties of $\mathcal{M}$, we have that: $\mathcal{M}, w \models A \mid\sim B$ iff $\mathcal{M} \models A \mid\sim B$.

Now we introduce the language $\mathcal{L}_P$ of the calculus introduced in the next section. $\mathcal{L}_P$ extends $\mathcal{L}$ by formulas of the form $\Box A$, where A is propositional, whose intuitive meaning is as follows: $\Box A$ holds in a world w if A holds in all the worlds w' such that $w' < w$:

Definition 2 (Truth condition of modality $\Box$). *We define the truth condition of a boxed formula as follows:*

$$\mathcal{M}, w \models \Box A \textit{ if for every } w' \in \mathcal{W} \textit{ if } w' < w \textit{ then } \mathcal{M}, w' \models A$$

It is easy to see that $\Box$ has the properties of the modal system G: the accessibility relation (defined as xRy if $y < x$) is transitive and does not have infinite ascending chains. From definition of $Min_<(A)$ in Definition 1 above, and Definition 2, it follows that for any formula A, $w \in Min_<(A)$ iff $\mathcal{M}, w \models A \land \Box\neg A$.

2.2 Loop Cumulative Logic CL

The next KLM logic we consider is **CL**, weaker than **P**. The axiomatization of **CL** can be obtained from the axiomatization of **P** by removing the axiom OR and by adding the following infinite set of axioms LOOP:

$$(LOOP)\ (A_0 \mid\sim A_1) \land (A_1 \mid\sim A_2)...(A_{n-1} \mid\sim A_n) \land (A_n \mid\sim A_0) \rightarrow (A_0 \mid\sim A_n)$$

Notice that these axioms are derivable in **P**.

Definition 3 (Loop-cumulative models, Definition 13 in [11]). *A loop-cumulative model is a tuple $\mathcal{M} = \langle S, l, <, V\rangle$. S is a set, whose elements are called states. Given a set $\mathcal{U}$ of possible worlds, $l : S \mapsto 2^{\mathcal{U}}$ is a function that labels every state with a nonempty set of worlds. $<$ is an irreflexive and transitive relation on S. V is a valuation function $V : \mathcal{U} \longmapsto pow(ATM)$, which assigns to every world w the atoms holding in that world. For $s \in S$ and A propositional, we let $s \models\!\!\equiv A$ if $\forall w \in l(s)$, $w \models A$. Let $Min_<(A)$ be the set of minimal states s such that $s \models\!\!\equiv A$. We define $\mathcal{M}, s \models\!\!\equiv A \mid\sim B$ if $\forall s' \in Min_<(A)$, $s' \models\!\!\equiv B$. We assume that $<$ satisfies the smoothness condition.*

Here again, we define satisfiability of conditionals with respect to states rather than to models for uniformity reasons. Indeed, a conditional is satisfied by a state of a model only if it is satisfied by all the states of that model, hence by the whole model. We show that we can map loop-cumulative models into preferential models extended with an additional accessibility relation R. We call these preferential models *CL-preferential structures*. The idea is to represent states as sets of possible worlds related by R in such a way that a formula is satisfied in a state s just in case it is satisfied in all possible worlds $w^{'}$ accessible from its corresponding w. The syntactic counterpart of the extra accessibility relation R is a modality L. Given a loop-cumulative model $\mathcal{M}$ and the corresponding CL-structure $\mathcal{M}^{'}$, $\mathcal{M}, s \models\!\equiv A$ iff for its corresponding w, $\mathcal{M}^{'}, w \models LA$.

As we will see, this mapping enables us to use a variant of the tableau calculus for **P** to deal with system **CL**. As for **P**, the tableau calculus for **CL** will use boxed formulas. Thus, the formulas that appear in the tableau for **CL** belong to the language $\mathcal{L}_L$ obtained from $\mathcal{L}$ as follows: (i) if A is propositional, then $A \in \mathcal{L}_L$; $LA \in \mathcal{L}_L$; $\Box\neg LA \in \mathcal{L}_L$; ($ii$) if A, B are propositional, then $A \mid\!\sim B \in \mathcal{L}_L$; ($iii$) if F is a boolean combination of formulas of $\mathcal{L}_L$, then $F \in \mathcal{L}_L$. Observe that the only allowed combination of $\Box$ and L is in formulas of the form $\Box\neg LA$ where A is propositional.

We can map loop-cumulative models into preferential structures with an additional accessibility relation as defined below:

Definition 4 (CL-preferential structures). *A model has the form* $\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$ *where:* $\mathcal{W}$, $<$, *and* V *are defined as in Definition 1, and* R *is a serial accessibility relation. We add to the truth conditions for preferential models in Definition 1 the following clause:*

$$\mathcal{M}, w \models LA \textit{ if for all } w^{'} \; wRw^{'} \textit{ implies } \mathcal{M}, w^{'} \models A$$

Moreover, we need to change the truth condition for conditional formulas as follows: $\mathcal{M}, w \models A \mid\!\sim B$ *if for all* $w^{'} \in Min_{<}(LA)$ *we have* $\mathcal{M}, w^{'} \models LB$.

We can prove the following proposition:

Proposition 1. *A set of conditional formulas* $\{(\neg)A_1 \mid\!\sim B_1, \ldots, (\neg)A_n \mid\!\sim B_n\}$ *is satisfiable in a loop-cumulative model* $\langle S, l, <, V \rangle$ *iff it is satisfiable in a CL-preferential model* $\langle W, R, <, V \rangle$.

3 The Tableau Calculus for Preferential Logic P

In this section we present a tableau calculus for **P** called $\mathcal{T}$**P**, then we analyze it in order to obtain a decision procedure for this logic. We also give an explicit complexity bound for **P**.

Definition 5 (The calculus $\mathcal{T}$P). *The rules of the calculus manipulate sets of formulas* Γ. *We write the shorthand* Γ, F *to denote* $\Gamma \cup \{F\}$. *Moreover, given* Γ *we define the following notation:*

- $\Gamma^{\Box} = \{\Box A \mid \Box A \in \Gamma\}$ – $\Gamma^{\Box^{\downarrow}} = \{A \mid \Box A \in \Gamma\}$ – $\Gamma^{\mid\sim^{+}} = \{A \mid\!\sim B \mid A \mid\!\sim B \in \Gamma\}$
- $\Gamma^{\mid\sim^{-}} = \{\neg(A \mid\!\sim B) \mid \neg(A \mid\!\sim B) \in \Gamma\}$ – $\Gamma^{\mid\sim^{\pm}} = \Gamma^{\mid\sim^{+}} \cup \Gamma^{\mid\sim^{-}}$

The tableau rules are given in Figure 1. Due to space limitations, we only give propositional rules for $\neg$ *and* $\wedge$. *We say that a tableau is closed if all its leaves contain both* F *and* $\neg F$, *for a formula* $F \in \mathcal{L}_P$.

$$(\mathbf{AX})\ \Gamma, F, \neg F \qquad\qquad (\neg)\ \frac{\Gamma, \neg\neg F}{\Gamma, F}$$

$$(\mid\!\sim^{+})\ \frac{\Gamma, A \mid\!\sim B}{\Gamma, \neg A, A \mid\!\sim B \qquad \Gamma, \neg\Box\neg A, A \mid\!\sim B \qquad \Gamma, B, A \mid\!\sim B}$$

$$(\mid\!\sim^{-})\ \frac{\Gamma, \neg(A \mid\!\sim B)}{A, \Box\neg A, \neg B, \Gamma^{\mid\sim\pm}} \qquad\qquad (\Box^{-})\ \frac{\Gamma, \neg\Box\neg A}{\Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\sim\pm}, A, \Box\neg A}$$

$$(\wedge^{+})\ \frac{\Gamma, F \wedge G}{\Gamma, F, G} \qquad\qquad (\wedge^{-})\ \frac{\Gamma, \neg(F \wedge G)}{\Gamma, \neg F \qquad \Gamma, \neg G}$$

Fig. 1. Tableau system $\mathcal{TP}$

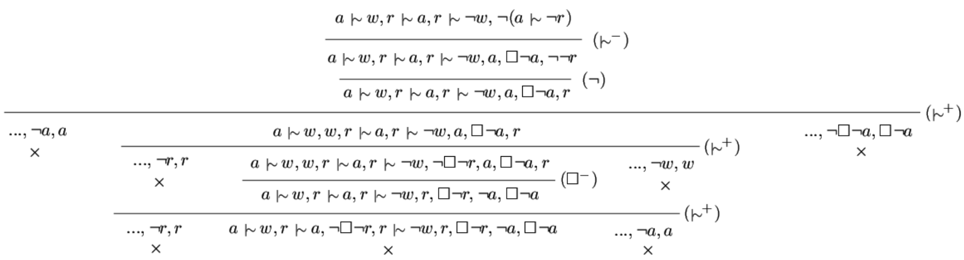

Fig. 2. A derivation of $((adult \mid\!\sim work) \wedge (retired \mid\!\sim adult) \wedge (retired \mid\!\sim \neg work)) \rightarrow (adult \mid\!\sim \neg retired)$. For readability, we use a to denote *adult*, r for *retired*, and so on.

Our tableau calculus $\mathcal{TP}$ is based on a runtime translation of conditional assertions into modal logic G. As we have seen this allows a characterization of the minimal worlds satisfying a formula A (i.e., the worlds in $Min_{<}(A)$) as the worlds w satisfying the formula $A \wedge \Box\neg A$. It is tempting to provide a full translation of the conditionals in the logic G, and then to use the standard tableau calculus for G. To this purpose, we can exploit the transitivity properties of G frames to capture the fact that conditionals are global to all worlds by the formula $\Box(A \wedge \Box\neg A \rightarrow B)$. Hence, the overall translation of a conditional formula $A \mid\!\sim B$ could be the following one: $(A \wedge \Box\neg A \rightarrow B) \wedge \Box(A \wedge \Box\neg A \rightarrow B)$. However, there are significant differences between the calculus resulting from the translation and our calculus.

Using the standard tableau rules for G on the translation, we get the rule $(\mid\!\sim^{+})$ as a derived rule. Instead, the rule for dealing with negated conditionals (which are translated in G into a disjunction of two formulas, namely $(A \wedge \Box\neg A \wedge \neg B) \vee \Diamond(A \wedge \Box\neg A \wedge \neg B))$, is rather different.

Let us first observe that the rule ($\mid\!\sim^-$) we have introduced precisely captures the intuition that: (1) conditionals are global (all conditionals are kept in the conclusion of the rule) and (2) when moving to a new minimal world, all the boxed formulas (positive and negated) are removed. Conversely, when the tableau rules for G are applied to the translation of the negated conditionals, we get two branches (due to the disjunction). None of the branches can be eliminated. In both branches all the boxed formulas are kept, while negated conditionals are erased. This is quite different from our rule ($\mid\!\sim^-$), and it is not that obvious that the calculus obtained by the translation of **P** conditionals in G is equivalent to $\mathcal{T}$**P**.

Also observe that, from the semantic point of view, the model extracted from an open tableau has the structure of a forest, while the model constructed by applying the tableau for G to the translation of conditionals has the structure of a tree. This difference is due to the fact that the above translation of **P** in G uses the same modality $\Box$ both for capturing the minimality condition and for modelling the fact that conditionals are global. For this reason, a translation to G as the one proposed above for **P**, would not be applicable to the cumulative logic **C**, as the relation $<$ is not transitive in **C**. Moreover, the treatment of both the logics **C** and **CL** would anyhow require the addition to the language of a new modality to deal with states. The advantage of the runtime translation we have adopted is that of providing a uniform approach to deal with the different logics.

The system $\mathcal{T}$**P** is sound and complete with respect to the semantics.

Theorem 1 (Soundness of $\mathcal{T}$P). *The system $\mathcal{T}$**P** is sound with respect to the semantics, i.e. if there is a closed tableau for a set Γ, then Γ is unsatisfiable.*

To prove the completeness of $\mathcal{T}$**P** we have to show that if F is unsatisfiable, then there is a closed tableau starting with F. We prove the contrapositive, that is: if there is no closed tableau for F, then there is a model satisfying F. This proof is inspired by [8]. First of all, we distinguish *static* and *dynamic* rules. The rules ($\mid\!\sim^-$) and ($\Box^-$) are called *dynamic*, since their conclusion represents another world with respect to the premise; the other rules are called *static*, since the world represented by premise and conclusion(s) is the same. Moreover, we have to introduce the *saturation* of a set of formulas Γ. Given a set of formulas Γ, we say that it is saturated if all the static rules have been applied.

Definition 6 (Saturated sets). *A set of formulas Γ is saturated with respect to the static rules if the following conditions hold:*

- *if* $F \wedge G \in \Gamma$ *then* $F, G \in \Gamma$;
- *if* $\neg(F \wedge G) \in \Gamma$ *then* $\neg F \in \Gamma$ *or* $\neg G \in \Gamma$;
- *if* $\neg\neg F \in \Gamma$ *then* $F \in \Gamma$;
- *if* $A \mid\!\sim B \in \Gamma$ *then* $\neg A \in \Gamma$ *or* $\neg\Box\neg A \in \Gamma$ *or* $B \in \Gamma$.

Lemma 1. *Given a consistent finite set of formulas Γ, there is a consistent, finite, and saturated set $\Gamma^{'} \supseteq \Gamma$.*

By Lemma 1, we can think of having a function which, given a consistent set Γ, returns one fixed consistent saturated set, denoted by $\texttt{SAT}(\Gamma)$. Moreover, we denote by $\texttt{APPLY}(\Gamma, F)$ the result of applying to Γ the rule for the principal connective in F. In case the rule for F has more conclusions (the case of a branching), we suppose that the function APPLY chooses one consistent conclusion in an arbitrary but fixed manner.

Theorem 2 (Completeness of $\mathcal{T}$P). *$\mathcal{T}$**P** is complete with respect to the semantics.*

Proof. As mentioned above, we assume that no tableau for Γ_0 is closed, then we construct a model for Γ_0. We build X, the set of worlds of the model, as follows:

1. initialize $X = \{\texttt{SAT}(\Gamma_0)\}$;
while X contains unresolved nodes do
  2. choose an unresolved Γ from X;
  3. **for** each formula $\neg(A \mid\!\sim B) \in \Gamma$
    3a. let $\Gamma_{\neg(A \mid\sim B)} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg(A \mid\!\sim B)))$;
    3b. **if** $\Gamma_{\neg(A \mid\sim B)} \notin X$ **then** $X = X \cup \{\Gamma_{\neg(A \mid\sim B)}\}$;
  4. **for** each formula $\neg\Box\neg A \in \Gamma$, let $\Gamma_{\neg\Box\neg A} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg\Box\neg A))$;
    4a. add the relation $\Gamma_{\neg\Box\neg A} < \Gamma$;
    4b. **if** $\Gamma_{\neg\Box\neg A} \notin X$ **then** $X = X \cup \{\Gamma_{\neg\Box\neg A}\}$.
  5. mark Γ as resolved;
endWhile;

This procedure terminates, since the number of possible sets of formulas that can be obtained by applying $\mathcal{T}$**P**'s rules to an initial finite set Γ is finite. We construct the model $\mathcal{M} = \langle X, <_X, V \rangle$ for Γ as follows:

- $<_X$ is the transitive closure of the relation $<$;
- $V(\Gamma) = \{P \mid P \in \Gamma \cap ATM\}$

In order to show that $\mathcal{M}$ is a preferential model for Γ, we prove the following:

Fact 1. *The relation $<_X$ is acyclic.*

Fact 2. *For all formulas F and for all sets $\Gamma \in X$ we have that:
(i) if $F \in \Gamma$ then $\mathcal{M}, \Gamma \models F$; (ii) if $\neg F \in \Gamma$ then $\mathcal{M}, \Gamma \not\models F$.*

By the above Facts the proof of the completeness of $\mathcal{T}$**P** is over, since $\mathcal{M}$ is a model for the initial set Γ_0. □

A relevant property of the calculus that will be useful to estimate the complexity of logic **P** is the so-called *disjunction property* of conditional formulas:

Proposition 2 (Disjunction property). *If there is a closed tableau for $\Gamma, \neg(A \mid\!\sim B), \neg(C \mid\!\sim D)$, then there is a closed tableau either for $\Gamma, \neg(A \mid\!\sim B)$ or for $\Gamma, \neg(C \mid\!\sim D)$.*

The reason why this property holds is that the $(\mid\!\sim^-)$ rule discards all the other formulas that could have been introduced by its previous application.

3.1 Decision Procedure for P

In general, non-termination in tableau calculi can be caused by two different reasons: 1. some rules copy their principal formula in the conclusion, thus can be reapplied over the same formula without any control; 2. dynamic rules can generate infinitely-many worlds, creating infinite branches.

Concerning the second source of non-termination (point 2.) we show that the generation of infinite branches due to the interplay between rules ($\mid\!\sim^+$) and ($\Box^-$) cannot occur. Indeed, as we will see, the application of ($\Box^-$) to a formula $\neg\Box\neg A$ (introduced by ($\mid\!\sim^+$)) adds the formula $\Box\neg A$ to the conclusion, so that ($\mid\!\sim^+$) can no longer consistently introduce $\neg\Box\neg A$. This is due to the properties of $\Box$ in G, which do not hold in other systems as K4. Furthermore, the ($\mid\!\sim^-$) rule can be applied only once to a given negated conditional on a branch, thus infinitely-many worlds cannot be generated on a branch.

Concerning point 1. the above calculus $\mathcal{T}\mathbf{P}$ does not ensure a terminating proof search due to ($\mid\!\sim^+$), which can be applied without any control. We ensure the termination by putting some constraints on $\mathcal{T}\mathbf{P}$. The intuition is as follows: one does not need to apply ($\mid\!\sim^+$) on the same conditional formula $A \mid\!\sim B$ *more than once in the same world*, therefore we keep track of positive conditionals already used by moving them in an additional set Σ in the conclusions of ($\mid\!\sim^+$), and restrict the application of this rule to unused conditionals only. The dynamic rules re-introduce formulas from Σ in order to allow further applications of ($\mid\!\sim^+$) in the other worlds. This machinery is standard.

Theorem 4 below shows that no additional machinery is needed to ensure termination. Notice that this would not work in other systems (for instance, in K4 one needs a more sophisticated loop-checking as described in [9]).

The terminating calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ is presented in Figure 3. The calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ is sound and complete with respect to the semantics: the soundness is immediate, and the completeness easily follows from the fact that two successive applications of ($\mid\!\sim^+$) to the same conditional in the same world are useless.

Theorem 3 (Soundness and completeness of $\mathcal{T}\mathbf{P}^{\mathbf{T}}$). *The calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ is sound and complete w.r.t. the semantics.*

In order to prove that $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ ensures a terminating proof search, we define a complexity measure on a set of formulas Γ and the corresponding set of pos-

$$(\mid\!\sim^+)\ \frac{\Gamma, A \mid\!\sim B; \Sigma}{\Gamma, \neg A; \Sigma, A \mid\!\sim B \quad \Gamma, \neg\Box\neg A; \Sigma, A \mid\!\sim B \quad \Gamma, B; \Sigma, A \mid\!\sim B}$$

$$(\mid\!\sim^-)\ \frac{\Gamma, \neg(A \mid\!\sim B); \Sigma}{\Sigma, A, \Box\neg A, \neg B, \Gamma^{\mid\sim^{\pm}}; \emptyset} \qquad (\Box^-)\ \frac{\Gamma, \neg\Box\neg A; \Sigma}{\Sigma, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\sim^{\pm}}, A, \Box\neg A; \emptyset}$$

Fig. 3. The calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$. Propositional rules are as in Figure 1 addicting Σ.

itive conditionals already used Σ, denoted by $m(\Gamma;\Sigma)$, which consists of four measures c_1, c_2, c_3 and c_4 in a lexicographic order. We write $A \mid\!\sim B \in_+ \Gamma$ (resp. $A \mid\!\sim B \in_- \Gamma$) if $A \mid\!\sim B$ occurs positively (resp. negatively) in Γ, where positive and negative occurrences are defined in the standard way. We also denote by $cp(F)$ the complexity of a formula F.

Definition 7 (Lexicographic order). *We define* $m(\Gamma;\Sigma) = \langle c_1, c_2, c_3, c_4\rangle$ *where:* $c_1 =\mid \{A \mid\!\sim B \in_- \Gamma\} \mid$, $c_2 =\mid \{A \mid\!\sim B \in_+ \Gamma \cup \Sigma \mid \Box\neg A \notin \Gamma\} \mid$, $c_3 =\mid \{A \mid\!\sim B \in_+ \Gamma\} \mid$, *and* $c_4 = \sum_{F\in\Gamma} cp(F)$. *We consider the lexicographic order given by* $m(\Gamma;\Sigma)$.

Intuitively, c_2 represents the number of positive conditionals *which can still create a new world*. The application of $(\Box^-)$ reduces c_2: indeed, if $(\mid\!\sim^+)$ is applied to $A \mid\!\sim B$, this application introduces a branch containing $\neg\Box\neg A$; when a new world is generated by an application of $(\Box^-)$ on $\neg\Box\neg A$, it contains A and $\Box\neg A$. If $(\mid\!\sim^+)$ is applied to $A \mid\!\sim B$ once again, then the conclusion where $\neg\Box\neg A$ is introduced is closed, by the presence of $\Box\neg A$ in that branch. c_3 is the number of conditionals not yet considered in that branch.

Theorem 4 (Termination of $\mathcal{T}\mathbf{P^T}$). $\mathcal{T}\mathbf{P^T}$ *ensures a terminating proof search.*

Proof sketch. Let $\Gamma^{'};\Sigma^{'}$ be obtained by an application of a rule of $\mathcal{T}\mathbf{P^T}$ to a premise $\Gamma;\Sigma$. It can be easily proved that $m(\Gamma^{'};\Sigma^{'}) < m(\Gamma;\Sigma)$. □

We conclude this section with a complexity analysis of $\mathcal{T}\mathbf{P^T}$, in order to prove that validity in **P** is **coNP**-complete. First of all, notice that we could take advantage of the disjunction property (Proposition 2). By this property we can reformulate the $(\mid\!\sim^-)$ rule as follows:

$$\frac{\Gamma, \neg(A \mid\!\sim B);\Sigma}{\Sigma, A, \Box\neg A, \neg B, \Gamma^{\mid\sim^+};\emptyset}\ (\mid\!\sim^-)$$

This rule reduces the length of a branch at the price of making the proof search more non-deterministic.

We give a non-deterministic algorithm for testing satisfiability in **P** that: (i) takes a set of formulas Γ as input; (ii) returns `SAT` iff Γ is satisfiable.

By the disjunction property, we can consider a negated conditional at a time. Indeed, for $\Gamma, \neg(A \mid\!\sim B), \neg(C \mid\!\sim D)$ to be satisfiable, it is sufficient that both $\Gamma, \neg(A \mid\!\sim B)$ and $\Gamma, \neg(C \mid\!\sim D)$, separately considered, are satisfiable. For each negated conditional, the algorithm `GENERAL-CHECK` applies the rule $(\mid\!\sim^-)$ to it, and calls the algorithm `CHECK` on the resulting set of formulas. `CHECK` is a non-deterministic algorithm that tests satisfiability in **P** of a set of formulas not containing negated conditionals. One can see that, when a negated conditional at a time is considered, a set of formulas is satisfiable in a preferential model if and only if it is satisfiable in a linearly ordered model (this can be proven directly, by transforming our canonical model in Theorem 2 into a linearly ordered model, and has also been proved in [12]). The algorithm `CHECK` verifies if there is a *linearly ordered* model satisfying the initial set of formulas. To this purpose, it makes use of a stronger version of the rule ($\Box^-$) in which, roughly speaking, each branch coming from the conclusion represents a possible linear model of the premise. The strengthened version of ($\Box^-$) is the following (we use $\Gamma^{\Box^-}_{-i}$ to denote $\{\neg\Box\neg A_j \lor A_j \mid \neg\Box\neg A_j \in \Gamma \land j \neq i\}$):

$$\frac{\Gamma, \neg\Box\neg A_1, \neg\Box\neg A_2, ..., \neg\Box\neg A_n}{\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, A_1, \Box\neg A_1, \Gamma^{\Box^-}_{-1} \mid ... \mid \Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, A_n, \Box\neg A_n, \Gamma^{\Box^-}_{-n}} \;(\Box^-)$$

An important feature of this reformulation with respect to the original ($\Box^-$) rule is that no backtracking on the choice of the formula $\neg\Box\neg A_i$ is needed as all alternatives are kept in the conclusion.

We call $L\mathcal{T}\mathbf{P^T}$ the calculus obtained by replacing in $\mathcal{T}\mathbf{P^T}$ the initial rules ($\mid\sim^-$) and ($\Box^-$) with the ones reformulated above. We can prove that $L\mathcal{T}\mathbf{P^T}$ is sound and complete w.r.t. the preferential models by proving the following proposition:

Proposition 3. *There is a closed tableau for Γ in $\mathcal{T}\mathbf{P^T}$ iff there is a closed tableau for Γ in $L\mathcal{T}\mathbf{P^T}$.*

Let `EXPAND`(Γ) be a procedure that returns one saturated expansion of Γ w.r.t. all static rules. In case of a branching rule, `EXPAND` nondeterministically selects (guesses) and applies one conclusion of the rule. The algorithm is defined below; in brackets we give the complexity of each operation, considering that $n = \mid \Gamma \mid$.

`CHECK`(Γ)
1. $\Gamma \longleftarrow$ `EXPAND`(Γ); ($O(n)$)
2. **if** Γ contains an axiom **then return** `UNSAT`; ($O(n^2)$)
3. **if** $\{\neg\Box\neg A \mid \neg\Box\neg A \in \Gamma\} = \emptyset$ **then return** `SAT`;
4. **else if** ($\{\neg\Box\neg A \mid \neg\Box\neg A \in \Gamma\} \neq \emptyset$) **then**
   4a. let $\{\neg\Box\neg A_1, \ldots, \neg\Box\neg A_k\}$ be all the negated boxed conditionals in Γ;
   4b. choose $i = 1, \ldots, k$;
   4c. `CHECK(APPLY(`$\Gamma, \neg\Box\neg A_i$`))`;

The above procedure allows to decide the satisfiability of a set of formulas (not containing negated conditionals). To see that the decision problem is in **NP**, observe that: (1) the complexity of each call to the procedure `EXPAND` is polynomial. Indeed, as the number of different subformulas is at most $O(n)$, `EXPAND` makes at most $O(n)$ applications of the static rules. (2) The test that a set Γ (of size $O(n)$) of formulas contains an axiom has at most complexity $O(n^2)$. (3) The number of recursive calls to the procedure `CHECK` is at most $O(n)$, since in a branch the rule ($\Box^-$) can be applied only once to each formula $\neg\Box\neg A_i$, and the number of different negated box formulas is at most $O(n)$.

Let us now define a procedure to decide whether an arbitrary set of formulas Γ (possibly containing negated conditionals) is satisfiable:

`GENERAL-CHECK`(Γ)
1. $\Gamma \longleftarrow$ `EXPAND`(Γ); ($O(n)$)
2. let $\neg(A_1 \mid\sim B_1), \ldots, \neg(A_k \mid\sim B_k)$ be all negated conditionals in Γ;

2.1. **for all** $i = 1, \ldots, k$ result[i] $\longleftarrow$ `CHECK(APPLY(`$\Gamma, \neg(A_i \mid\!\sim B_i)$`))` ;
3. **if for all** $i = 1, \ldots, n$ result[i]==`SAT` **then return** `SAT`;
else return `UNSAT`;

By the subformula property, the number of negated conditionals which can occur in Γ is at most $O(n)$. Hence, the procedure `GENERAL-CHECK` calls to the algorithm `CHECK` at most $O(n)$ times.

Theorem 5 (Complexity of P). *The problem of deciding validity for preferential logic* **P** *is* **coNP***-complete.*

Proof. The procedure `GENERAL-CHECK` allows the satisfiability of a set of formulas of logic **P** to be decided in nondeterministic polynomial time. The validity problem for **P** is therefore in **coNP**. As **coNP**-hardness is immediate (this logic includes classical propositional logic), we conclude that the validity problem for logic **P** is **coNP**-complete. □

This result matches with the known complexity results for logic **P** [12]. Due to the **coNP** lower bound, the above method provides a computationally optimal reasoning procedure for logic **P**.

4 The Tableau Calculus for Loop Cumulative Logic CL

In this section we develop a tableau calculus $\mathcal{T}$**CL** for **CL**, and we show that it can be turned into a terminating calculus. This provides a decision procedure for **CL** and a **coNP**-membership upper bound for validity in **CL**.

The calculus $\mathcal{T}$**CL** can be obtained from the calculus $\mathcal{T}$**P** for preferential logics, by adding a suitable rule for dealing with the modality L. We define $\Gamma^{L^{\downarrow}} = \{A \mid LA \in \Gamma\}$. Our tableau system $\mathcal{T}$**CL** for **CL** is shown in Figure 4 and is obtained by introducing the new modality L in the rules of $\mathcal{T}$**P** and by adding the new rule (L^-). Observe that rules $(\mid\!\sim^+)$ and $(\mid\!\sim^-)$ have been changed as they introduce the modality L in front of the propositional formulas A and B in their conclusions. The new rule (L^-) is a dynamic rule.

$$(\mid\!\sim^+)\ \frac{\Gamma, A \mid\!\sim B}{\Gamma, \neg LA, A \mid\!\sim B \quad \Gamma, \neg\Box\neg LA, A \mid\!\sim B \quad \Gamma, LB, A \mid\!\sim B} \qquad (\mid\!\sim^-)\ \frac{\Gamma, \neg(A \mid\!\sim B)}{LA, \Box\neg LA, \neg LB, \Gamma^{\mid\sim^{\pm}}}$$

$$(L^-)\ \frac{\Gamma, \neg LA}{\Gamma^{L^{\downarrow}}, \neg A}\ \text{where either } \{\neg LA\} \neq \emptyset \text{ or } \Gamma^{L^{\downarrow}} \neq \emptyset \qquad (\Box^-)\ \frac{\Gamma, \neg\Box\neg LA}{\Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\sim^{\pm}}, LA, \Box\neg LA}$$

Fig. 4. Tableau system $\mathcal{T}$**CL**. If $\neg LA$ is not in the premise of (L^-) (i.e. $\{\neg LA\} = \emptyset$) the rule allows to step from Γ to $\Gamma^{L^{\downarrow}}$. The boolean rules are omitted.

The proof of the completeness of the calculus can be done as for the preferential case, provided we suitably modify the procedure for constructing a model for a finite consistent set of formulas Γ of $\mathcal{L}_L$. First of all, we modify the definition of saturated sets as follows:

- if $A \mid\sim B \in \Gamma$ then $\neg LA \in \Gamma$ or $\neg\Box\neg LA \in \Gamma$ or $LB \in \Gamma$

For this notion of saturated set of formulas we can still prove Lemma 1 for language $\mathcal{L}_L$.

Theorem 6 (Completeness of $\mathcal{T}$CL). *$\mathcal{T}$**CL** is complete with respect to the semantics.*

Proof. We define a procedure for constructing a model satisfying a set of formulas $\Gamma_0 \in \mathcal{L}_L$ by modifying the procedure for the preferential logic **P**. We add to the procedure two new steps 4' and 4", between step 4 and step 5 as follows:

4'. **if** $\{\neg LA \mid \neg LA \in \Gamma\} \neq \emptyset$ **then**
  for each $\neg LA \in \Gamma$, let $\Gamma_{\neg LA}$ =SAT(APPLY($\Gamma, \neg LA$));
    4' a. add the relation $\Gamma\ R\ \Gamma_{\neg LA}$;
    4' b. **if** $\Gamma_{\neg LA} \notin X$ **then** $X = X \cup \{\Gamma_{\neg LA}\}$;
4". **else if** $\Gamma^{L^{\downarrow}} \neq \emptyset$ **then**, let Γ' =SAT(APPLY(Γ, L^-));
  4" a. add the relation $\Gamma\ R\ \Gamma'$;
  4" b. **if** $\Gamma' \notin X$ **then** $X = X \cup \{\Gamma'\}$;

This procedure terminates. We construct the model $\mathcal{M} = \langle X, R_X, <_X, V\rangle$ by defining $<_X$ and V as in the case of **P** and by letting R_X the relation obtained from R above augmented with all the pairs (Γ, Γ) such that $\Gamma \in X$ and Γ has no R-successor. It is easy to show that the following properties hold for $\mathcal{M}$:

- for all $\Gamma, \Gamma' \in X$, if $(\Gamma, \Gamma') \in R_X$ and $LA \in \Gamma$ then $A \in \Gamma'$;
- for all formulas F and for all sets $\Gamma \in X$ we have that: (i) if $F \in \Gamma$ then $\mathcal{M}, \Gamma \models F$; (ii) if $\neg F \in \Gamma$ then $\mathcal{M}, \Gamma \not\models F$. □

4.1 Decision Procedure for CL

Let us now analyze the calculus $\mathcal{T}$**CL** in order to obtain a decision procedure for **CL** logic. First of all, we reformulate the calculus as we made for **P**, obtaining a system called $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$: we reformulate the $(\mid\sim^+)$ rule so that it applies only once to each conditional in each world, by adding of an extra set Σ. We reformulate the other rules accordingly. Notice that the rule (L^-) does not need to be further reformulated since it can only be applied a finite number of times. Exactly as we made for **P**, we consider a lexicographic order given by $m(\Gamma; \Sigma) =< c_1, c_2, c_3, c_4 >$, and easily prove that each application of the rules of $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$ reduces this measure. Thus, $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$ ensures termination. Furthermore, the decision algorithm for **P** described in section 3 can be adapted to **CL**. The procedure CHECK has to be modified by introducing the following steps 4' and 4" between steps 2. and 3.:

4'. **else if** $\{\neg LA \mid \neg LA \in \Gamma\} \neq \emptyset$ **then**
4'a. **for all** $\neg LA_i \in \Gamma$ **do** `CHECK(APPLY(`$\Gamma, \neg LA_i$`));`
4". **else if** $\{LA \mid LA \in \Gamma\} \neq \emptyset$ **then**
4"a. `CHECK(APPLY` (Γ, L^-));

Observe that the two recursive calls of `CHECK` in 4'a and 4"a do not generate further recursive calls. By this reason, one obtains the following result:

Theorem 7 (Complexity of CL). *The problem of deciding validity for* **CL** *is* **coNP***-complete.*

5 Conclusions

In this paper, we have presented tableau calculi for some of the KLM logical systems for default reasoning. We have given a tableau calculus for preferential logic **P** and for loop-cumulative logic **CL**. The calculi presented give a decision procedure for the respective logics, whose complexity is **coNP** for both **P** and **CL**. We will make a detailed comparison with existing works ([1], [7], [12]) in a full paper.

We plan to extend our calculi to the other KLM systems, namely to the weaker **C** and to the stronger **R**. For **C** we conjecture that a complete calculus is given by a variant of $\mathcal{T}$**CL** in which the $(\Box^-)$ rule is weakened so that it does not enforce the transitivity of the preferential relation $<$. Another development of our work will be the extension to the first order case. The starting point will be the analysis of first order preferential and rational logics by Friedman, Halpern and Koller in [5].

Acknowledgements. We are grateful to the anonymous referees for their very helpful comments.

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