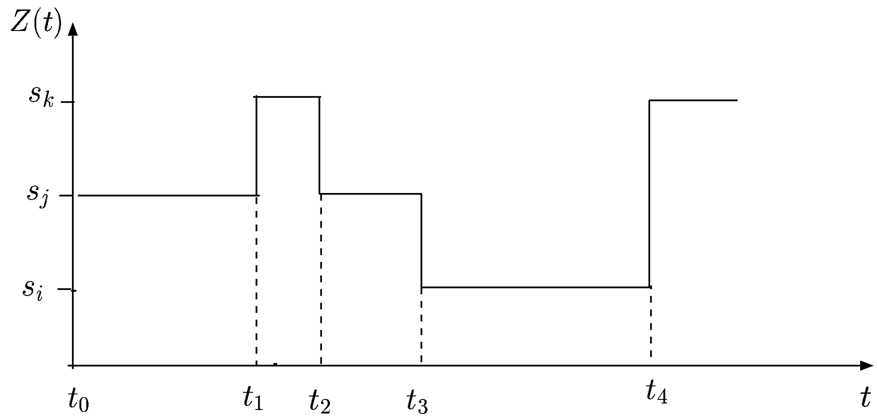
CONTINUOUS-TIME MARKOV CHAINS

Andrea Bobbio
Last revision: Anno Accademico 2002-2003

Continuous Time Markov Process

A stochastic process $Z(t)$ defined over a discrete state space S of cardinality N is a continuous-time discrete-state Markov process (or Markov chain - CTMC) if:



for any sequence

$$(0 < t_1 < t_2 < \ldots < t_{m-1} < t_m)$$

The following property holds:

$$Pr\{ Z(t_m) = s_{j_m} | Z(t_{m-1}) = s_{j_{m-1}}, \ldots, Z(t_1) = s_{j_1} \}$$

$$= Pr\{ Z(t_m) = s_{j_m} | Z(t_{m-1}) = s_{j_{m-1}} \}$$

Transition Probability Matrix

Let us introduce the following notation:

$$p_{ij}(u,x) \;=\; Pr\,\{\,Z(x) \;=\; s_j\,|Z(u) \;=\; s_i\,\} \qquad\qquad (u \leq x)$$

With:

$$p_{ii}(x,x) \;=\; 1 \qquad ; \qquad p_{ij}(x,x) \;=\; 0$$

$p_{ij}(u,x)$ is the conditional probability of transition in state s_j at time x given the process was in state s_i at time u.

$$p_i(x) \;=\; Pr\,\{\,Z(x) \;=\; s_i\,\}$$

$p_i(x)$ is the probability that the process is in state s_i at time x and is called the occupancy state probability or simply *state probability*.

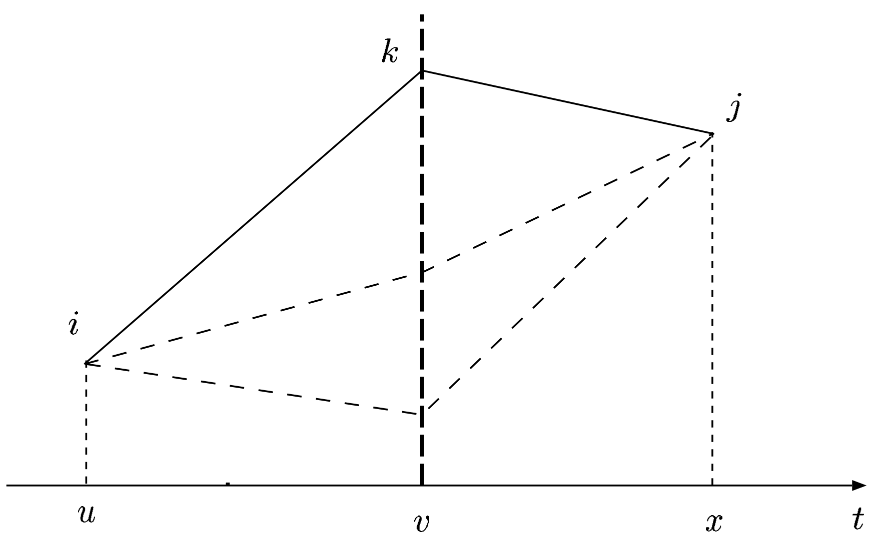
From the above definitions:

$$\sum_{j=1}^{N} \; p_{ij}(u,x) \;=\; 1 \qquad ; \qquad \sum_{i=1}^{N} \; p_i(x) \;=\; 1$$

Chapman-Kolmogorov Equations

The Markov property implies the following Chapman-Kolmogorov equations:

$$
\begin{aligned}
p_j(x) &= \sum_i p_i(u) \cdot p_{ij}(u,x) \\
p_{ij}(u,x) &= \sum_k p_{ik}(u,v) \cdot p_{kj}(v,x) \qquad \text{for } u \le v \le x
\end{aligned}
$$



Let $\mathbf{P}(u,x) = [p_{ij}(u,x)]$ be the $(N \times N)$ square transition probability matrix.

Let $\mathbf{p}(x) = [p_i(x)]$ be the (N)-dimensional row vector of the state probabilities.

Using matrix algebra, the C-K equations can be written as:

$$
\begin{aligned}
\mathbf{p}(x) &= \mathbf{p}(u) \cdot \mathbf{P}(u,x) \\
\mathbf{P}(u,x) &= \mathbf{P}(u,v) \cdot \mathbf{P}(v,x) \\
\mathbf{P}(x,x) &= \mathbf{I}
\end{aligned}
$$

where $\mathbf{I}$ is the identity matrix.

Time-Homogeneous CTMC

A CTMC is said to be time-homogeneous (or simply homogeneous), when the transition probability matrix $\mathbf{P}(u, x)$ depends only on the difference $(x - u)$.

Substituting: $x - v = t$ e $v - u = \theta$ the C-K equations become:

$$\mathbf{P}(t + \theta) = \mathbf{P}(t) \cdot \mathbf{P}(\theta) \qquad ; \qquad \mathbf{P}(0) = \mathbf{I}$$

Define (for $i \neq j$ and for $\Delta t \geq 0$):

$$q_{ij} = \left. \frac{d\, p_{ij}(t)}{d\, t} \right|_{t=0} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t) - p_{ij}(0)}{\Delta\, t} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta\, t}$$

From the above we get:

$$q_{ij} \geq 0$$

$$p_{ij}(\Delta\, t) = Pr\{\, Z(t + \Delta\, t)\} = j \,|\, Z(t) = i\} = q_{ij}\, \Delta t + O(\Delta\, t)$$

Define (for $i = j$ and for $\Delta\, t \geq 0$):

$$q_{ii} = \left. \frac{d\, p_{ii}(t)}{d\, t} \right|_{t=0} = \lim_{\Delta t \to 0} \frac{p_{ii}(\Delta t) - p_{ii}(0)}{\Delta\, t} = - \lim_{\Delta t \to 0} \frac{1 - p_{ii}(\Delta t)}{\Delta\, t}$$

From the above we get:

$$q_{ii} < 0$$

$$p_{ii}(\Delta\, t) = Pr\{\, Z(t + \Delta\, t)\} = i \,|\, Z(t) = i\} = 1 + q_{ii}\, \Delta t + O(\Delta\, t)$$

CK Equations for Time-Homogeneous CTMC

q_{ij} are the transition rates, whose physical interpretation is:

$$p_{ij}(\Delta t) \;=\; Pr\{\, Z(t+\Delta t)\} \;=\; j \,|\, Z(t) \;=\; i\} \;=\; q_{ij}\,\Delta t + O(\Delta t)$$

$$p_{ii}(\Delta t) \;=\; Pr\{\, Z(t+\Delta t)\} \;=\; i \,|\, Z(t) \;=\; i\} \;=\; 1 + q_{ii}\,\Delta t + O(\Delta t)$$

The C-K equation can be written as:

$$\begin{aligned} p_{ij}(t+\Delta t) &= \sum_k p_{ik}(t)\, p_{kj}(\Delta t) \\ &= p_{ij}(t)\, p_{jj}(\Delta t) + \sum_{k:k\neq j} p_{ik}(t)\, p_{kj}(\Delta t) \end{aligned}$$

$$p_{ij}(t+\Delta t) \;=\; p_{ij}(t)(1 + q_{jj}\,\Delta t) + \sum_{k:k\neq j} p_{ik}(t)\, q_{kj}(\Delta t) + O(\Delta t)$$

From the above:

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} \;=\; p_{ij}(t)\, q_{jj} + \sum_{k:k\neq j} p_{ik}(t)\, q_{kj} + \frac{O(\Delta t)}{\Delta t}$$

Taking the limit as $\Delta t \to 0$,

$$\frac{d\,p_{ij}(t)}{d\,t} = \sum_k p_{ik}(t)\, q_{kj} \quad \text{with initial condition} \quad p_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The Transition Rate Matrix

Since the transition to a state from time t to $(t+\Delta t)$ is the certain event:

$$1 \;=\; \sum_{j} \; p_{ij}(\Delta t) \;=\; 1 \;+\; q_{ii}\,\Delta t \;+\; \sum_{j:\, j \neq i} \; q_{ij}\,\Delta t$$

$$q_{ii} \;=\; -\sum_{j:\, j \neq i} \; q_{ij}$$

Define the transition rate matrix (infinitesimal generator) $\mathbf{Q}$ of the process as:

$$\mathbf{Q} \;=\; [\, q_{ij} \,] \quad \text{where} \quad \begin{array}{ll} q_{ij} \geq 0 & i \;\neq\; j \\ q_{ii} < 0 & q_{ii} \;=\; -\displaystyle\sum_{j:\, j \neq i} \; q_{ij} \end{array}$$

The row sum of the transition rate matrix $\mathbf{Q}$ is equal to 0.

In matrix form, the C-K equations become:

$$\mathbf{P}'(t) \;=\; \mathbf{P}(t) \,\cdot\, \mathbf{Q} \qquad ; \qquad \mathbf{P}(0) = \mathbf{I}$$

The State Probability Vector

Let $\mathbf{p}(t)$ be the state probability vector in a homogeneous CTMC, and let $\mathbf{p}(0)$ be the initial state probability vector (the initial condition). We have:

$$\mathbf{p}(t) \;=\; \mathbf{p}(0) \cdot \mathbf{P}(t)$$

Differentiating both sides:

$$\mathbf{p}'(t) \;=\; \mathbf{p}(0) \cdot \mathbf{P}'(t) \;=\; \mathbf{p}(0) \cdot \mathbf{P}(t) \cdot \mathbf{Q}$$

From which we derive the state probability equation:

$$\mathbf{p}'(t) \;=\; \mathbf{p}(t) \cdot \mathbf{Q}$$

$$\text{with initial condition} \quad \mathbf{p}(0)$$

The state probability equation has formal solution:

$$\mathbf{p}(t) \;=\; \mathbf{p}(0) \cdot e^{\mathbf{Q}\,t}$$

where:

$$e^{\mathbf{Q}\,t} \;=\; \mathbf{I} + \mathbf{Q}\,t + \frac{1}{2}\,(\mathbf{Q}\,t)^2 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i\,!}\,(\mathbf{Q}\,t)^i$$

The Laplace Transform of the State Probability Equation

The state probability equation can be written explicitly:

$$\begin{cases} p_1'(t) \;=\; p_1(t)\,q_{11} \;+\; p_2(t)\,q_{21} \;+\; \ldots \;+\; p_N(t)\,q_{N1} \\ p_2'(t) \;=\; p_1(t)\,q_{12} \;+\; p_2(t)\,q_{22} \;+\; \ldots \;+\; p_N(t)\,q_{N2} \\ \ldots \qquad \ldots \end{cases}$$

Denoting the Laplace transform: $\mathcal{L}[p_i(t)] \;=\; p_i^*(s)$, the Laplace transform of the state probability equation becomes:

$$\begin{cases} s\,p_1^*(s) \;-\; p_1(0) \;=\; p_1^*(s)\,q_{11} + p_2^*(s)\,q_{21} + \ldots + p_N^*(s)\,q_{N1} \\ s\,p_2^*(s) \;-\; p_2(0) \;=\; p_1^*(s)\,q_{12} + p_2^*(s)\,q_{22} + \ldots + p_N^*(s)\,q_{N2} \\ \ldots \qquad \ldots \end{cases}$$

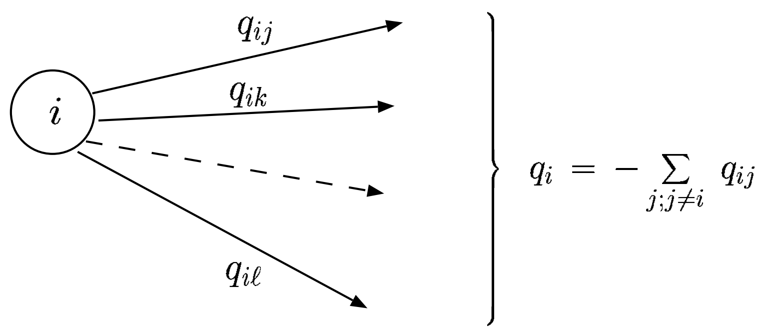
$$\begin{cases} p_1^*(s)\,(s \;-\; q_{11}) - p_2^*(s)\,q_{21} - \ldots - p_N^*(s)\,q_{N1} \;=\; p_1(0) \\ -\,p_1^*(s)\,q_{12} + p_2^*(s)\,(s \;-\; q_{22}) - \ldots - p_N^*(s)\,q_{N2} \;=\; p_2(0) \\ \ldots \qquad \ldots \end{cases}$$

In matrix form, the above equations become:

$$\mathbf{p}^*(s)\,(s\,\mathbf{I} \;-\; \mathbf{Q}) \;=\; \mathbf{p}_0 \quad \Rightarrow \quad \mathbf{p}^*(s) \;=\; \mathbf{p}_0\,(s\,\mathbf{I} \;-\; \mathbf{Q})^{-1}$$

Sojourn time in state i

We isolate state i by deleting transitions entering state i. We have:



$$\frac{d\,p_i(t)}{d\,t} = -\,p_i(t)\;q_i \qquad\qquad p_i(0) = 1$$

Where: $q_i = -\,q_{ii} = \sum_{j:\,j\neq i} q_{ij}$ is a negative constant equal to the sum of the rates out of state i.

Solution of the above equation is:

$$p_i(t) = 1 - e^{-\,q_i\,t}$$

The sojourn time in each state is exponentially distributed with a rate equal to the sum of the exit rates.

The probability that the sojourn time in state i terminates by a transition toward state j, is given by:

$$p_{ij}(t) = \frac{q_{ij}}{q_i}\,e^{-\,q_i\,t}$$

Expected State Occupancy in $(0 - t)$

Let $\theta_i(t)$ be the random variable representing the time spent by the CTMC $Z(t)$ in state s_i in the interval $(0 - t)$.

To evaluate the expected value of $\theta_i(t)$, let us introduce an indicator process $y(t)$ defined as follows:

$$\begin{cases} y(t) = 1 & \text{if} \quad Z(t) = i \\ y(t) = 0 & \text{if} \quad Z(t) \neq i \end{cases}$$

By construction:

$$\theta_i(t) = \int_0^t y(u)\, du$$

with initial condition $\theta_i(0) = 0$.
Hence:

$$\begin{aligned} E[\,\theta_i(t)\,] &= E[\int_0^t y(u)\, du] = \int_0^t E[\,y(u)\, du\,] \\ &= \int_0^t 0 \cdot Pr\{y(t) = 0\} + 1 \cdot Pr\{y(t) = 1\} \\ &= \int_0^t p_i(u)\, du \end{aligned}$$

Introducing the vector $\boldsymbol{\theta}(t)$ whose entries are the $E[\,\theta_i(t)\,]$, we can write:

$$\boldsymbol{\theta}(t) - \boldsymbol{\theta}(0) = \int_0^t \mathbf{p}(u)\, du$$

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \mathbf{p}_0\,(\mathbf{I}\,t + \frac{t^2}{2}\,\mathbf{Q} + \ldots + \frac{t^{i+1}}{(i+1)\,!}\,\mathbf{Q}^i + \ldots)$$

Classification of states and stationary distribution

The classification of states for a CTMC is similar to the DTMC case. Given a state i, if the ultimate return to that state is the certain event, the state is called *recurrent*, if the ultimate return has probability less than 1, the state is called *transient*.

An absorbing state is a state with no outgoing arcs: state i is absorbing if $q_{ij} = 0$ for any $j \neq i$.

A state j is *reachable* from i for some $t > 0$, if $p_{ij}(t) > 0$.

The state space of a CTMC can be partitioned into a set of transient states and closed sets of recurrent states.

A CTMC is *irreducible* if every state is reachable from every other state.

An irreducible CTMC reaches a steady-state condition as $t \to \infty$ independently of the initial condition.

$$\lim_{t \to \infty} p_i(t) \;=\; \pi_i$$

If the limit exists then: $\displaystyle\lim_{t \to \infty} \frac{d\, p_i(t)}{d\, t} \;=\; 0$
and the steady state matrix equation becomes:

$$\boldsymbol{\pi} \cdot \mathbf{Q} \;=\; \mathbf{0} \qquad \text{with} \qquad \sum_{i=1}^{N} \pi_i \;=\; 1$$

Properties of the steady-state distribution

Similarly to the discrete case, the equilibrium distribution for an irreducible CTMC has the following properties:

◇ for all initial conditions, the occupancy state probability $p_i(t)$ tends to a constant value π_i as $t \to \infty$, and the π_i's form a probability distribution.

◇ if the initial probability is π_i, then $p_i(t) = \pi_i$ for all t;

◇ the proportion of time spent in state i in the interval $(0 - t)$ tends to π_i as $t \to \infty$:

$$\lim_{t \to \infty} \frac{E[\theta_i]}{t} = \pi_i$$

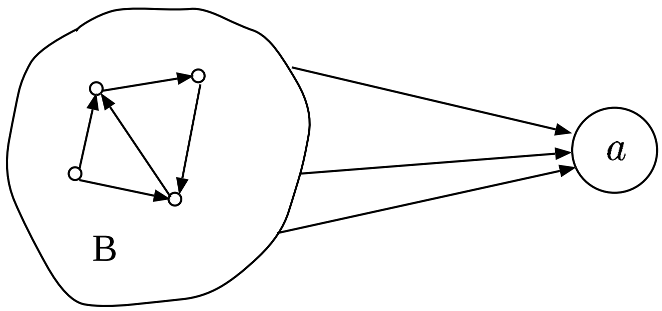
◇ the steady-state probabilities satisfy a system of ordinary linear equations.

$$\boldsymbol{\pi} \cdot \mathbf{Q} = \mathbf{0} \qquad \text{with} \qquad \boldsymbol{\pi} \cdot \mathbf{e}^T = 1$$

◇ the steady state equation can be interpreted as a probability balance equation (for every state the probability flow-in equals the probability flow-out)

CTMC with Absorbing States

Let state a be an absorbing state. We can partition the Markov equation as follows (being the row corresponding to the absorbing state equal to 0).



$$[\,\mathbf{p}'(t)\;\; p'_a(t)\,] \;=\; [\,\mathbf{p}(t)\;\; p_a(t)\,] \left| \begin{array}{c|c} \mathbf{B} & \mathbf{A} \\ - & - \\ \mathbf{0} & 0 \end{array} \right|$$

where the square matrix $\mathbf{B}$ groups the transition rates inside the transient states and the column vector $\mathbf{A} = -\mathbf{B}\,\mathbf{e}^T$ the rates from the transient states to the absorbing state.

Solving the above equations in partitioned form, we get:

$$\begin{cases} \mathbf{p}'(t) &= \mathbf{p}(t)\,\mathbf{B} \\ p'_a(t) &= \mathbf{p}(t)\,\mathbf{A} \end{cases}$$

$$\begin{cases} \mathbf{p}(t) &= \mathbf{p_0}\, e^{\mathbf{B}\,t} \\ p'_a(t) &= \mathbf{p_0}\, e^{\mathbf{B}\,t}\,\mathbf{A} \end{cases}$$

Given τ_a is the time to reach the absorbing state from $t = 0$ (time to absorption), we have:

$$\begin{aligned} F_a(t) &= Pr\{\tau_a \le t\} = Pr\{Z(t) = a\} = p_a(t) \\ &= 1 - \mathbf{p}(t)\,\mathbf{e}^T = 1 - \mathbf{p_0}\, e^{\mathbf{B}\,t}\,\mathbf{e}^T \end{aligned}$$

CTMC with Absorbing States in Laplace transform

$$[\, s\,\mathbf{p}^*(s) - \mathbf{p_0} \quad s\,p_a^*(s)\,] = [\,\mathbf{p}^*(s) \quad p_a^*(s)\,] \left| \begin{array}{c|c} \mathbf{B} & \mathbf{A} \\ - & - \\ \mathbf{0} & 0 \end{array} \right|$$

$$\begin{cases} s\,\mathbf{p}^*(s) - \mathbf{p_0} \;=\; \mathbf{p}^*(s)\,\mathbf{B} \\ s\,p_a^*(s) \;=\; \mathbf{p}^*(s)\,\mathbf{A} \end{cases}$$

$$\begin{cases} \mathbf{p}^*(s) \;=\; \mathbf{p_0}(s\,\mathbf{I} - \mathbf{B})^{-1} \\ \\ p_a^*(s) \;=\; \frac{1}{s}\,\mathbf{p}^*(s)\,\mathbf{A} = \frac{1}{s}\,\mathbf{p_0}(s\,\mathbf{I} - \mathbf{B})^{-1}\,\mathbf{A} \end{cases}$$

It turns out that:

$$F_a^*(s) \;=\; \frac{1}{s}\,\mathbf{p_0}(s\,\mathbf{I} - \mathbf{B})^{-1}\,\mathbf{A}$$

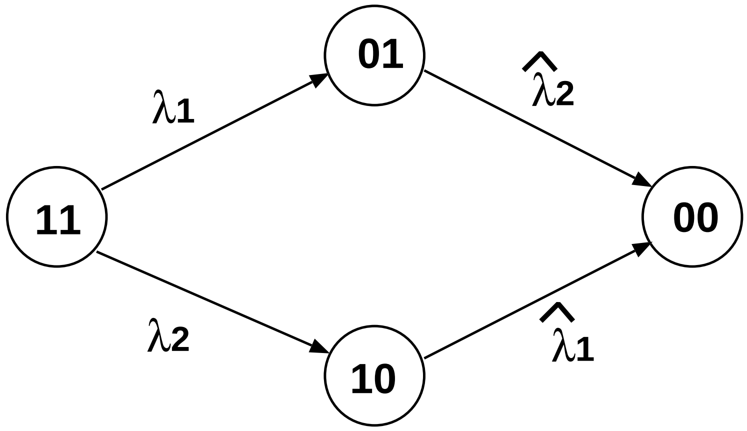
$$f_a^*(s) \;=\; s\,F_a^*(s) \;=\; \mathbf{p_0}(s\,\mathbf{I} - \mathbf{B})^{-1}\,\mathbf{A}$$

Resorting to the moment theorem for Laplace transforms, we can evaluate the mean time to absorption (with $\mathbf{A} = -\mathbf{B}\,\mathbf{e}^T$):

$$\begin{aligned} E[\,\tau_a\,] &= (-1)\,\frac{d\,f_a^*(s)}{d\,s}\bigg|_{s=0} \\ &= (-1)\,\mathbf{p_0}\,(s\,\mathbf{I} - \mathbf{B})^{-2}\,\mathbf{A}\Big|_{s=0} \\ &= \mathbf{p_0}\,(-\mathbf{B})^{-1}\,\mathbf{e}^T \end{aligned}$$

System of two Dependent Components

Markov Analysis - 1



The transition rate matrix **Q** assumes the form:

$$
\mathbf{Q} \;=\; \begin{array}{c|cccc|}
 & (1{,}1) & (0{,}1) & (1{,}0) & (0{,}0) \\
(1{,}1) & -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\
(0{,}1) & 0 & -\hat{\lambda}_2 & 0 & \hat{\lambda}_2 \\
(1{,}0) & 0 & 0 & -\hat{\lambda}_1 & \hat{\lambda}_1 \\
(0{,}0) & 0 & 0 & 0 & 0
\end{array}
$$

System of two Dependent Components - 2

The solution equation for the state probabilities is:

$$\left| \, p_1'(t) \;\; p_2'(t) \;\; p_3'(t) \;\; p_4'(t) \, \right| =$$

$$\left| \, p_1(t) \;\; p_2(t) \;\; p_3(t) \;\; p_4(t) \, \right| \cdot \left| \begin{array}{cccc} -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\ 0 & -\hat{\lambda}_2 & 0 & \hat{\lambda}_2 \\ 0 & 0 & -\hat{\lambda}_1 & \hat{\lambda}_1 \\ 0 & 0 & 0 & 0 \end{array} \right|$$

From the matrix equation the following set of linear differential equations is obtained:

$$\frac{d\,p_1(t)}{d\,t} = -\,(\lambda_1 + \lambda_2)\,p_1(t)$$

$$\frac{d\,p_2(t)}{d\,t} = \lambda_1\,p_1(t) - \hat{\lambda}_2\,p_2(t)$$

$$\frac{d\,p_3(t)}{d\,t} = \lambda_2\,p_1(t) - \hat{\lambda}_1\,p_3(t)$$

$$\frac{d\,p_4(t)}{d\,t} = \hat{\lambda}_2\,p_2(t) + \hat{\lambda}_1\,p_3(t)$$

System of two Dependent Components - 3

Let us assume, as initial condition, that the system is in the good state $s_1 = \{0,0\}$ at time $t = 0$ with probability 1.

$$p_1(0) = 1\,, \quad p_2(0) = 0\,, \quad p_3(0) = 0\,, \quad p_4(0) = 0$$

The solution of the set of linear differential equations can be obtained by resorting to the Laplace transform method. Let $p_i^*(s)$ denote the Laplace transform of $p_i(t)$. We get:

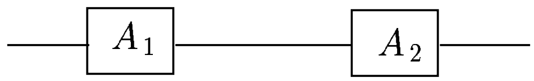
$$\begin{aligned}
s\,p_1^*(s) - 1 &= -(\lambda_1 + \lambda_2)\,p_1^*(s) \\
s\,p_2^*(s) &= \lambda_1\,p_1^*(s) - \hat{\lambda}_2\,p_2^*(s) \\
s\,p_3^*(s) &= \lambda_2\,p_1^*(s) - \hat{\lambda}_1\,p_3^*(s) \\
s\,p_4^*(s) &= \hat{\lambda}_2\,p_2^*(s) + \hat{\lambda}_1\,p_3^*(s)
\end{aligned}$$

The state probabilities in the time domain are:

$$\begin{aligned}
p_1(t) &= e^{-(\lambda_1 + \lambda_2)\,t} \\
p_2(t) &= \frac{\lambda_1}{\lambda_1 + \lambda_2 - \hat{\lambda}_2}\,(e^{-\hat{\lambda}_2\,t} - e^{-(\lambda_1 + \lambda_2)\,t}) \\
p_3(t) &= \frac{\lambda_2}{\lambda_1 + \lambda_2 - \hat{\lambda}_1}\,(e^{-\hat{\lambda}_1\,t} - e^{-(\lambda_1 + \lambda_2)\,t}) \\
p_4(t) &= 1 - p_1(t) - p_2(t) - p_3(t)
\end{aligned}$$

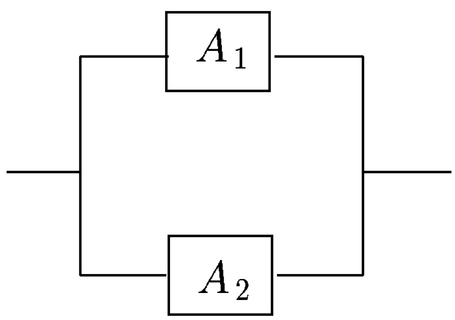
System of two Dependent Components - 4

Series System:



$$R_{ser}(t) = p_1(t) = e^{-(\lambda_1 + \lambda_2)t}$$

Parallel System:

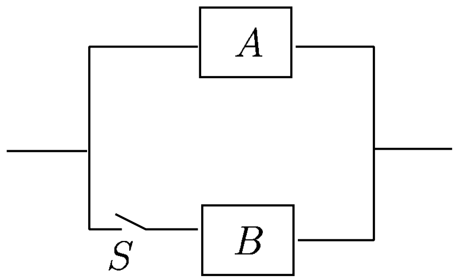


$$R_{par}(t) = p_1(t) + p_2(t) + p_3(t) =$$

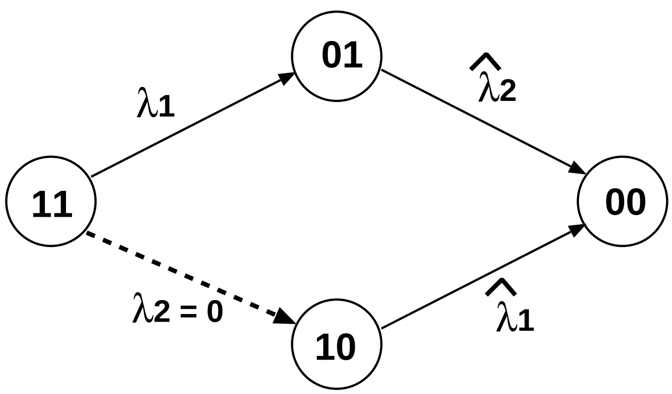
$$= e^{-(\lambda_1 + \lambda_2)t}$$
$$+ \frac{\lambda_1}{\lambda_1 + \lambda_2 - \hat{\lambda}_2} \left(e^{-\hat{\lambda}_2 t} - e^{-(\lambda_1 + \lambda_2)t} \right)$$
$$+ \frac{\lambda_2}{\lambda_1 + \lambda_2 - \hat{\lambda}_1} \left(e^{-\hat{\lambda}_1 t} - e^{-(\lambda_1 + \lambda_2)t} \right)$$

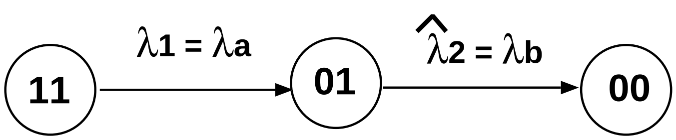
System of two Dependent Components - 5

Stand-by redundancy:



Since component 2 is not operating in state $s_1 = \{1, 1\}$, we have $\lambda_2 = 0$. The Markov graph becomes in this case:



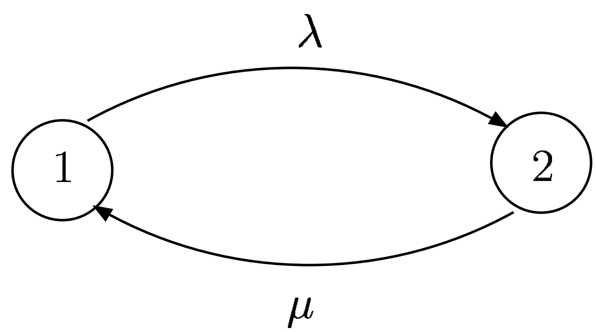


$$R_{standby}(t) = \frac{\lambda_a}{\lambda_a - \lambda_b} e^{-\lambda_b t} - \frac{\lambda_b}{\lambda_a - \lambda_b} e^{-\lambda_a t}$$

If the two components are identical ($\lambda_a = \lambda_b = \lambda$), the previous equation becomes:

$$R_{standby}(t) = (1 + \lambda t) e^{-\lambda t}$$

Repairable System - Availability



$$\left| \, p_1'(t) \;\; p_2'(t) \, \right| \; = \; \left| \, p_1(t) \;\; p_2(t) \, \right| \; \cdot \; \left| \begin{matrix} -\lambda & \lambda \\ \mu & -\mu \end{matrix} \right|$$

Expliciting the matrix equation we obtain:

$$\frac{d\,p_1(t)}{d\,t} \;=\; -\,\lambda\,p_1(t) \;+\; \mu\,p_2(t)$$

$$\frac{d\,p_2(t)}{d\,t} \;=\; \lambda\,p_1(t) -\; \mu\,p_2(t)$$

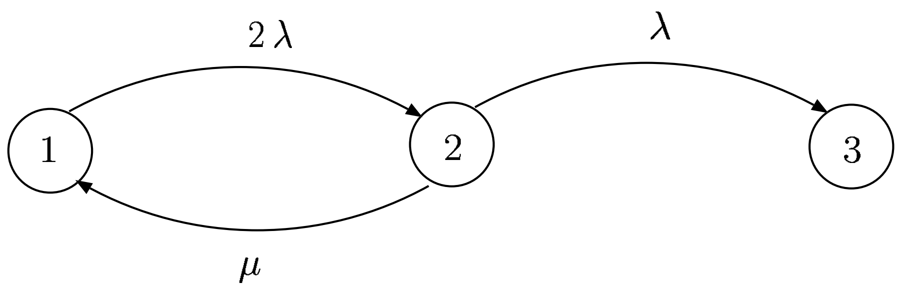
Assuming as initial condition [$p_1(0) \;=\; 1$, $p_2(0) \;=\; 0$], the state probabilities become:

$$p_1(t) \;=\; \frac{\mu}{\lambda \,+\, \mu} \;+\; \frac{\lambda}{\lambda \,+\, \mu}\; e^{-(\lambda + \mu)\,t}$$

$$p_2(t) \;=\; \frac{\lambda}{\lambda \,+\, \mu} \;-\; \frac{\lambda}{\lambda \,+\, \mu}\; e^{-(\lambda + \mu)\,t}$$

Two Component System

Identical and Repairable Components - 1



$$\left| \begin{array}{ccc} p'_1(t) & p'_2(t) & p'_3(t) \end{array} \right| = \left| \begin{array}{ccc} p_1(t) & p_2(t) & p_3(t) \end{array} \right| \cdot \left| \begin{array}{ccc} -2\,\lambda & 2\,\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 \end{array} \right|$$

Expliciting the matrix equation, we obtain the following set of linear differential equations:

$$\frac{d\,p_1(t)}{d\,t} = -\,2\,\lambda\,p_1(t) + \mu\,p_2(t)$$

$$\frac{d\,p_2(t)}{d\,t} = 2\,\lambda\,p_1(t) - (\,\lambda + \mu\,)\,p_2(t)$$

$$\frac{d\,p_3(t)}{d\,t} = \lambda\,p_2(t)$$

Two Component System

Identical and Repairable Components - 2

We resort to the Laplace transform method.

$$\begin{aligned} s\,p_1^*(s) \,-\, 1 \;&=\; -\,2\,\lambda\,p_1^*(s) \,+\, \mu\,p_2^*(s) \\ s\,p_2^*(s) \;&=\; 2\,\lambda\,p_1^*(s) - (\,\lambda \,+\, \mu\,)\,p_2^*(s) \\ s\,p_3^*(s) \;&=\; \lambda\,p_2^*(s) \end{aligned}$$

Solving the algebraic set of equations:

$$p_3^*(s) \;=\; \frac{2\,\lambda^2}{s\,[\,s^2 \,+\, (3\,\lambda \,+\, \mu)\,s \,+\, 2\,\lambda^2\,]}$$

Inverting by means of the partial fraction expansion:

$$R(t) \;=\; 1 \,-\, p_3(t) \;=\; \frac{\alpha_2}{\alpha_1 \,-\, \alpha_2}\,e^{-\,\alpha_1\,t} \;-\; \frac{\alpha_1}{\alpha_1 \,-\, \alpha_2}\,e^{-\,\alpha_2\,t}$$

where α_1 and α_2 are roots of the equation:

$$s^2 \,+\, (3\,\lambda \,+\, \mu)\,s \,+\, 2\,\lambda^2 \;=\; (\,s \,+\, \alpha_1\,)\,(\,s \,+\, \alpha_2\,)$$