DISCRETE-TIME

MARKOV CHAINS

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DISCRETE TIME MARKOV CHAIN (DTMC)

A sequence $(X_0,\ X_1,\ \ldots,\ X_n)$ of discrete r.v. taking values in the set S forms a Discrete Time - Discrete Parameter Markov Chain (DTMC) if:

$$Pr\{X_n = s_n \,|\, X_{n-1} = s_{n-1}\,;\ X_{n-2} = s_{n-2}\,;\ \ldots\,; X_0 = s_0\,\}$$

$$= Pr\{X_n = s_n \,|\, X_{n-1} = s_{n-1}\,\}$$

Let us introduce the unconditional probability:

$$p_j(n) = Pr\{X_n = j\,\}$$

and the conditional probability $P_{i\,j}(m,n)$ defined as the probability of finding the DTMC in state j at step n, given it was in state i at step m:

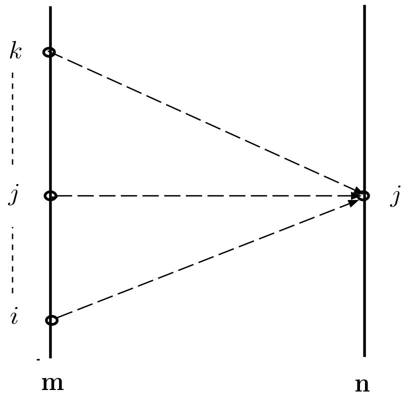
$$P_{i\,j}(m,n) = Pr\{X_n = j \,|\, X_m = i\}$$

$P_{i\,j}(m,n)$ is the transition probability function. From the above definitions:

$$\sum_{j \in S} p_j(n) = 1$$

$$\sum_{j \in S} P_{i\,j}(m,n) = 1$$

CHAPMAN KOLMOGOROV EQUATION



$$\begin{aligned} p_j(n) \;=\; & \ldots p_i(m)\, P_{ij}(m,n) \;+ \\ & \ldots p_j(m)\, P_{jj}(m,n) \;+ \\ & \ldots p_k(m)\, P_{kj}(m,n) \;+\; \ldots \end{aligned}$$

$$p_j(n) \;=\; \sum_{k \in S} p_k(m)\, P_{kj}(m,n)$$

From a similar reasoning (for $n > u > m$):

$$P_{ij}(m,n) \;=\; \sum_{k \in S} P_{ik}(m,u)\, P_{kj}(u,n)$$

The above is the Chapman Kolmogorov Equation for DTMC.

DTMC - MATRIX NOTATION

Define the probability vector:

$$\underline{\mathbf{p}}(n) \;=\; [\, p_1(n)\; p_2(n)\; \ldots]$$

and the transition probability matrix ($i,\, j \,\in\, S$):

$$\mathbf{P}(m,n) \;=\; [\, P_{ij}(m,n)\,]$$

In matrix notation:

$$\underline{\mathbf{p}}(n) \;=\; \underline{\mathbf{p}}(m)\, \mathbf{P}(m,n)$$

$$\mathbf{P}(m,n) \;=\; \mathbf{P}(m,u)\, \mathbf{P}(u,n)$$

Introducing the unity row vector $\underline{\mathbf{e}}$ with all the entries equal to 1 (being $\mathbf{e}^T$ the transpose unity column vector), the normalization conditions become:

$$\underline{\mathbf{p}}(n)\, \underline{\mathbf{e}}^T \;=\; 1$$

$$\mathbf{P}(m,n)\, \underline{\mathbf{e}}^T \;=\; \underline{\mathbf{e}}^T$$

The square matrix $\mathbf{P}(m,n)$ is called a stochastic matrix.

TIME HOMOGENEOUS DTMC

Time homogeneity: the transition probability matrix depends only on the difference $m - n$:

$$\mathbf{P}(m,n) = \mathbf{P}(n - m)$$

For homogeneous DTMC follows:

$$\mathbf{P}(n,n+1) = \mathbf{P}(m,m+1) = \mathbf{P}(1)$$

The one-step transition probability matrix $\mathbf{P}(1) = \mathbf{P}$, plays a special role:

$$\mathbf{P}(2) = \mathbf{P}\,\mathbf{P} = \mathbf{P}^2$$

$$\mathbf{P}(n) = \mathbf{P}(n-1)\,\mathbf{P} = \mathbf{P}^n$$

$$\mathbf{P}(n+m) = \mathbf{P}^n\,\mathbf{P}^m$$

The n-step transition probability matrix is the n-th power of $\mathbf{P}(1) = \mathbf{P}$.
$\mathbf{P}^n$ is a stochastic matrix:

$$\mathbf{P}^n\,\underline{\mathbf{e}}^T = \mathbf{P}^{n-1}\,\mathbf{P}\,\underline{\mathbf{e}}^T = \mathbf{P}^{n-1}\underline{\mathbf{e}}^T = \underline{\mathbf{e}}^T$$

THE n-step STATE PROBABILITY

Let $\underline{\mathbf{p}}_0 = \underline{\mathbf{p}}(0)$ be the vector of initial state occupation probabilities:

$$\underline{\mathbf{p}}(1) = \underline{\mathbf{p}}_0\,\mathbf{P}$$

$$\underline{\mathbf{p}}(n) = \underline{\mathbf{p}}(n-1)\,\mathbf{P} = \underline{\mathbf{p}}_0\,\mathbf{P}^n$$

The initial probability vector $\underline{\mathbf{p}}_0$ and the one-step transition probability matrix $\mathbf{P}$ univocally determine the state of the system at step n.

The state probability vector at step n, $\underline{\mathbf{p}}(n)$, is a probability vector:

$$\underline{\mathbf{p}}(n)\,\underline{\mathbf{e}}^T = \underline{\mathbf{p}}(n-1)\,\mathbf{P}\,\underline{\mathbf{e}}^T = \underline{\mathbf{p}}_0\underline{\mathbf{e}}^T = 1$$

since $\underline{\mathbf{p}}_0$ is a probability mass vector by assumption.

COMPUTING STATE PROBABILITY BY Z-TRANSFORM

The Z-transform $\underline{\mathbf{G}}(z)$ is defined as:

$$\underline{\mathbf{G}}(z) = \sum_{n=0}^{\infty} \underline{\mathbf{p}}(n)\, z^n$$

Let apply the Z-transform method to the basic equation:

$$\underline{\mathbf{p}}(n) = \underline{\mathbf{p}}(n-1)\,\mathbf{P} \qquad (n \geq 1)$$

$$\sum_{n=1}^{\infty} \underline{\mathbf{p}}(n)\, z^n = \sum_{n=1}^{\infty} \underline{\mathbf{p}}(n-1)\,\mathbf{P}\, z^n$$

$$\underline{\mathbf{G}}(z) - \underline{\mathbf{p}}_0 = z \left(\sum_{n=1}^{\infty} \underline{\mathbf{p}}(n-1)\, z^{n-1} \right) \mathbf{P}$$

$$\underline{\mathbf{G}}(z) - \underline{\mathbf{p}}_0 = z\, \underline{\mathbf{G}}(z)\, \mathbf{P}$$

$$\underline{\mathbf{G}}(z) = \underline{\mathbf{p}}_0 \left[\mathbf{I} - z\mathbf{P} \right]^{-1}$$

LIMITING DISTRIBUTION

From $\underline{\mathbf{p}}(n) = \underline{\mathbf{p}}(n-1)\,\mathbf{P}$, if a limiting distribution $\underline{\pi}$ exists for $n \to \infty$:

$$\underline{\pi} = \underline{\pi}\,\mathbf{P}$$

$$\underline{\pi}\,(\mathbf{I} - \mathbf{P}) = 0$$

with the normalization condition:

$$\underline{\pi}\, \underline{\mathbf{e}}^T = 1$$

If a limiting distribution $\underline{\pi}$ exists:

$$\lim_{n \to \infty} \mathbf{P}^n = \begin{pmatrix} \underline{\pi} \\ \underline{\pi} \\ \cdots \\ \underline{\pi} \end{pmatrix} = \underline{\mathbf{e}}^T\, \underline{\pi}$$

The limiting distribution has two major implications:

i) For any $\underline{\mathbf{p}}_0$

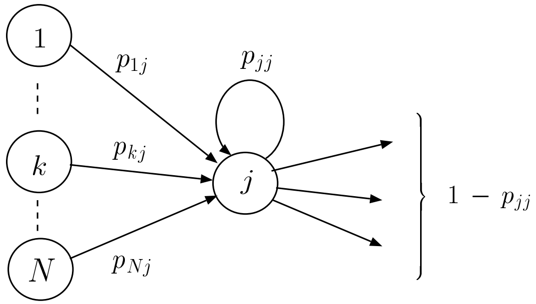
$$\lim_{n \to \infty} \underline{\mathbf{p}}(n) = \lim_{n \to \infty} \underline{\mathbf{p}}_0\, \mathbf{P}^n = \underline{\mathbf{p}}_0\, \underline{\mathbf{e}}^T\, \underline{\pi} = \underline{\pi}$$

ii) If $\underline{\mathbf{p}}_0 = \underline{\pi}$, then $\underline{\mathbf{p}}(n) = \underline{\pi}$ for all n.

LIMITING DISTRIBUTION AND BALANCE EQUATION

Writing the steady state equation explicitly, we get:

$$\pi_j \;=\; \sum_{k=1}^{N} \pi_k \cdot p_{kj}$$

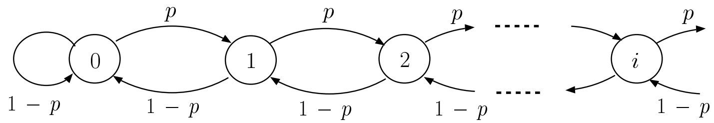


$$\pi_j \;=\; \pi_j\, p_{jj} + \sum_{k=1;k\neq j}^{N} \pi_k \cdot p_{kj}$$

$$\pi_j\,(1 - p_{jj}) \;=\; \sum_{k=1;k\neq j}^{N} \pi_k \cdot p_{kj}$$

In any state, the probability flow-in equals the probability flow-out

Random walk with reflecting state 0



The transition probability matrix is:

$$\mathbf{P} \;=\; \left|\begin{array}{cccccc} 1-p & p & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \cdots \\ 0 & 1-p & 0 & p & 0 & \cdots \\ 0 & 0 & 1-p & 0 & p & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|$$

Applying the balance equation:

$$p\,\pi_0 \;=\; (1-p)\,\pi_1 \qquad\qquad \pi_1 \;=\; \frac{p}{1-p}\,\pi_0$$

$$\pi_1 \;=\; p\,\pi_0 + (1-p)\,\pi_2 \qquad\qquad \pi_2 \;=\; \frac{p^2}{(1-p)^2}\,\pi_0$$

$$\cdots \qquad\qquad \cdots$$

$$\pi_i \;=\; p\,\pi_{i-1} + (1-p)\,\pi_{i+1} \qquad\qquad \pi_i \;=\; \frac{p^i}{(1-p)^i}\,\pi_0$$

Setting: $\alpha \;=\; \dfrac{p}{1-p}$ $(\alpha \le 1)$, and utilizing the normalization condition:

$$1 \;=\; \pi_0 \sum_{i=0}^{\infty} \alpha^i \;=\; \frac{\pi_0}{1 - \alpha}$$

we get: $\quad \pi_0 \;=\; 1 - \alpha \quad ; \quad \pi_i \;=\; (1 - \alpha)\,\alpha^i$

CLASSIFICATION OF STATES

Given a state j, if the ultimate return to that state is the certain event, the state is called *recurrent*, if the ultimate return has probability less than 1, the state is called *transient*.

Let $f_{jj}(n)$ be the probability that, given the DTMC is in state j, the next return to state j occurs in n steps.

$$f_{jj}(n) \;=\; Pr\{X_r \neq j,\, r = 1, 2, \ldots, n-1,\; X_n = j \,|\, X_0 = j\}$$

Similarly,

$$f_{jk}(n) \;=\; Pr\{X_r \neq k,\, r = 1, 2, \ldots, n-1,\; X_n = k \,|\, X_0 = j\}$$

From which we derive explicitly:

$$\begin{aligned} P_{jk} \;&=\; f_{jk}(1) \\ P_{jk}(2) \;&=\; f_{jk}(1) \cdot P_{kk} \,+\, f_{jk}(2) \\ \cdots \quad & \quad \cdots \\ P_{jk}(n) \;&=\; \sum_{r=1}^{n} f_{jk}(r) \cdot P_{kk}(n-r) \end{aligned}$$

CLASSIFICATION OF STATES

Define f_{jj} the probability that state j is eventually reentered and f_{jk} the probability that state k is entered from j.

Since the series $f_{jj(n)}$ and $f_{jk}(n)$ form mutually exclusive events as a function of n, we can write:

$$\begin{aligned} f_{jj} \;&=\; \sum_{n=1}^{\infty} f_{jj}(n) \\ f_{jk} \;&=\; \sum_{n=1}^{\infty} f_{jk}(n) \end{aligned}$$

If $f_{jj} = 1$ state j is recurrent, if $f_{jj} < 1$ state j is transient.

For a recurrent state $f_{jj}(n)$ is a probability mass function, and we can define the mean recurrence time:

$$\tau_j \;=\; \sum_{n=1}^{\infty} n\, f_{jj}(n)$$

If τ_j is finite, state j is positive-recurrent, if $\tau_i = \infty$, state j is null-recurrent.

CLASSIFICATION OF STATES

A state j communicates with k if $f_{jk} = 1$. If also k communicates with j, they are inter-communicating. Inter-communicating states must be of the same class (transient, positive or null recurrent).

As an equivalent definition, we can say that a state is recurrent if it is revisited infinitely often with probability 1, while it is transient if it is revisited finitely often with probability 1.

Suppose that when the chain starts in j, subsequent returns to j can occur at instants r, $2\,r$, $3\,r$ etc.. where $r > 1$ is an integer. State j is called periodic with period r and $p_{jj}(n)$ vanishes except when n is an integral multiple of r.

A closed set $C \subset S$ is a subset of S in which all the states are inter-communicating. A closed set with a single state is an absorbing state (for an absorbing state $p_{jj} = 1$).

The state space S of a DTMC, can be partitioned into a transient set and closed sets of recurrent states (possibly absorbing).

A DTMC is *irriducible* if all the states belong to a single closed set.

Classification of states and limiting distributions

For an irreducible DTMC the limiting distribution always exists.

Two properties of the limiting distribution are presented without proof.

◇ - The equilibrium probability equals the inverse of the mean recurrence time:

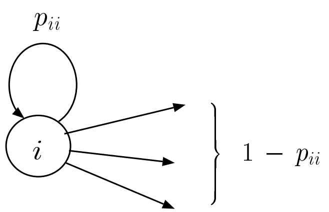
$$\lim_{n\to\infty} p_j(n) = \pi_j = \frac{1}{\tau_j}$$

◇ - The equilibrium probability equals the proportion of time spent in a given state in an indefinitely long realization of the process.
Given an interval of time $[0 - n]$, let $R_j(n)$ be the number of times state j is occupied:

$$\lim_{n\to\infty} \frac{R_j(n)}{n} = \pi_j = \frac{1}{\tau_j}$$

Expected sojourn time in state i

Let T_i be the sojourn time in state i:



The probability $p_n^{(i)}$ of remaining n steps in state i is:

$$\begin{array}{lcl} p_1^{(i)} & = & 1 - p_{ii} \\ p_2^{(i)} & = & p_{ii}\,(1 - p_{ii}) \\ p_3^{(i)} & = & p_{ii}^2\,(1 - p_{ii}) \\ \cdots & & \cdots \\ p_n^{(i)} & = & p_{ii}^{(n-1)}\,(1 - p_{ii}) \end{array}$$

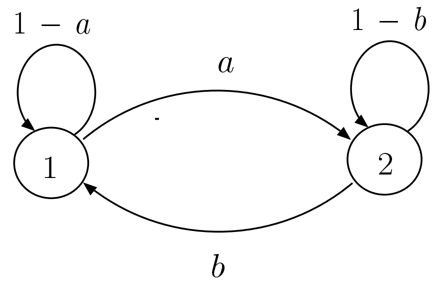
The expected value of the sojourn time becomes:

$$\begin{aligned} E[T_i] &= \sum_{n=1}^{\infty} n\, p_n^{(i)} = (1 - p_{ii}) \sum_{n=1}^{\infty} n\, p_{ii}^{n-1} \\ &= (1 - p_{ii}) \sum_{n=0}^{\infty} \frac{d\,p_{ii}^n}{d\,p_{ii}} = (1 - p_{ii}) \frac{1}{(1 - p_{ii})^2} = \frac{1}{1 - p_{ii}} \end{aligned}$$

We have made use of the following series identity:

$$\sum_{n=1}^{\infty} n\, x^{n-1} = \sum_{n=0}^{\infty} \frac{d\,x^n}{d\,x} = \frac{d}{d\,x} \sum_{n=0}^{\infty} x^n = \frac{d}{d\,x}\left(\frac{1}{1-x}\right) = \frac{1}{1-x^2}$$

EXAMPLE OF SOJOURN TIME



$$E[T_1] = \frac{1}{1 - p_{11}} = \frac{1}{1 - (1-a)} = \frac{1}{a}$$

$$E[T_2] = \frac{1}{1 - p_{22}} = \frac{1}{b}$$

The balance equation applied to this example provides:

$$\left\{ \begin{array}{rcl} a\,\pi_1 & = & b\,\pi_2 \\ \pi_1 + \pi_2 & = & 1 \end{array} \right. \qquad \left\{ \begin{array}{rcl} \frac{a}{b}\,\pi_1 & = & \pi_2 \\ \pi_1 + \frac{a}{b}\,\pi_1 & = & 1 \end{array} \right.$$

$$\pi_1 = \frac{b}{a + b} \qquad \pi_2 = \frac{a}{a + b}$$

DTMC WITH ABSORBING STATES

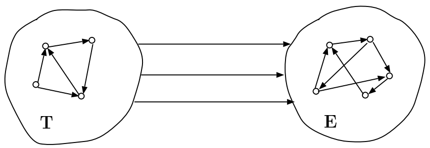
If a DTMC consists in set of transient states and a closed set of recurrent states, the one-step probability matrix can be partitioned in the following form:

$$\mathbf{P} = \begin{vmatrix} \mathbf{T} & \mathbf{B} \\ \mathbf{0} & \mathbf{E} \end{vmatrix}$$

$\mathbf{T}$ is a square sub-stochastic matrix (with at least one row-sum less than 1) describing the connections among the transient states.

$\mathbf{E}$ is a square stochastic matrix describing the connections among the recurrent states

$\mathbf{B}$ is rectangular matrix (a column vector if the recurrent subsets is a single absorbing state) describing the connections from the transient to the recurrent states.

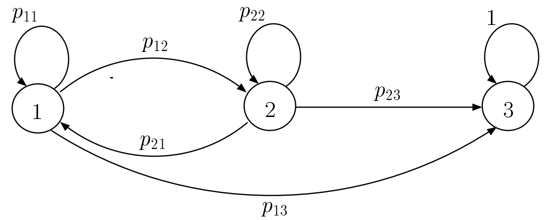


The steady state probability vector is equal to 0 in the states $s \in \mathbf{T}$ and sums to 1 in the states $s \in \mathbf{E}$.

Since return to $\mathbf{T}$ is not certain, a measure of interest is the time to leave $\mathbf{T}$ (conicident with the first passage time in $\mathbf{E}$).

DTMC WITH ABSORBING STATES

Consider the following DTMC in which states 1 and 2 are transient and 3 absorbing.



The transition probability matrix can be partitioned in:

$$\mathbf{P} = \left|\begin{array}{cc|c} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ \hline 0 & 0 & 1 \end{array}\right|$$

Let $f_{j3}(n)$ be the probability of absorption in state 3 starting from state j in n steps. We can write (for $j =\in T$ and $k \in T$):

$$\begin{aligned} f_{j3}(1) &= p_{j3} \\ f_{j3}(2) &= \sum_{k=1,2} p_{jk}\, f_{k3}(1) \\ \ldots & \quad \ldots \\ f_{j3}(n) &= \sum_{k=1,2} p_{jk}\, f_{k3}(n-1) \\ \ldots & \quad \ldots \end{aligned}$$

The ultimate probability of absorption from state j to state 3 is:

$$f^a_{j3} = \sum_{k=1}^{\infty} f_{j3}(k)$$

DTMC WITH ABSORBING STATES

Expliciting the above equations for $j = 1$ we get:

$$\begin{aligned} f_{13}(1) &= p_{13} \\ f_{13}(2) &= p_{11}\, f_{13}(1) + p_{12}\, f_{23}(1) \;=\; p_{11}\, p_{13} + p_{12}\, p_{23} \\ f_{13}(3) &= p_{11}\, f_{13}(2) + p_{12}\, f_{23}(2) \\ &= p_{11}\,(p_{11}\, p_{13} + p_{12}\, p_{23}) \;+\; p_{12}\,(p_{11}\, p_{13} + p_{12}\, p_{23}) \\ \ldots & \quad \ldots \end{aligned}$$

Expliciting the above equations for $j = 2$ we get:

$$\begin{aligned} f_{23}(1) &= p_{23} \\ f_{23}(2) &= p_{21}\, f_{13}(1) + p_{22}\, f_{23}(1) \;=\; p_{21}\, p_{13} + p_{22}\, p_{23} \\ f_{23}(3) &= p_{21}\, f_{13}(2) + p_{22}\, f_{23}(2) \\ &= p_{21}\,(p_{21}\, p_{13} + p_{22}\, p_{23}) \;+\; p_{22}\,(p_{21}\, p_{13} + p_{22}\, p_{23}) \\ \ldots & \quad \ldots \end{aligned}$$

Matrix Solution of DTMC with Absorbing States

Given the partitioned matrix:

$$\mathbf{P} \;=\; \begin{vmatrix} \mathbf{T} & \mathbf{B} \\ \mathbf{0} & \mathbf{E} \end{vmatrix}$$

let $\mathbf{f}(k)$ the $(T \times E)$ rectangular matrix that contains the probability of absorption from any state in T to any state in E. In matrix form we have:

$$\begin{aligned} \mathbf{f}(1) &= \mathbf{B} \\ \mathbf{f}(2) &= \mathbf{T}\,\mathbf{B} \\ \mathbf{f}(3) &= \mathbf{T}^2\,\mathbf{B} \\ \ldots & \quad \ldots \\ \mathbf{f}(k) &= \mathbf{T}^{k-1}\,\mathbf{B} \\ \ldots & \quad \ldots \end{aligned}$$

The ultimate probability of absorption from any state T to any state E is:

$$\mathbf{f}^a \;=\; \sum_{k=1}^{\infty} \mathbf{f}(k) \;=\; (\mathbf{I}+\mathbf{T}+\mathbf{T}^2+\ldots+\mathbf{T}^k+\ldots)\,\mathbf{B} \;=\; (\mathbf{I}-\mathbf{T})^{-1}\,\mathbf{B}$$

Matrix Solution of DTMC with Absorbing States

In the simpler case in which there exists a single absorbing state ($E = 1$), we prove that absorption is the certain events from any state T.

Since $\mathbf{P}$ is a stochastic matrix (row sum equal to 1), we can write:

$$\mathbf{T}\,\mathbf{e}^T = \mathbf{e}^T - \mathbf{B} \qquad ; \qquad \mathbf{B} = (\mathbf{I} - \mathbf{T})\,\mathbf{e}^T$$

Using this identity the absorption probability vector assumes the form:

$$\mathbf{f}^a = (\mathbf{I} - \mathbf{T})^{-1}\,\mathbf{B} = \mathbf{e}^T$$

Each entry of the vector $\mathbf{f}^a$ results to be equal to 1.

Mean Absorption Time

Suppose the DTMC has a single absorbing state a, the mean absorption time starting from state $j \in T$ can be calculated as:

$$m_j^a = \sum_{k=1}^{\infty} k\, f_{ja}(k)$$

Let $\mathbf{m}^a$ be the T dimensional vector containing the mean absorption times from any state in t to the absorbing state a.

In vector form:

$$\mathbf{m}^a = \sum_{k=1}^{\infty} k\,\mathbf{f}(k) = \sum_{k=1}^{\infty} k\,\mathbf{T}^{k-1}\,\mathbf{B}$$

$$= (\mathbf{I} - \mathbf{T})^{-2}\,\mathbf{B} = (\mathbf{I} - \mathbf{T})^{-1}\,\mathbf{e}^T$$

In deriving the previous expression we have made use of the fact that:

$$\mathbf{B} = (\mathbf{I} - \mathbf{T})^{-1}\,\mathbf{e}^T$$

$$\sum_{k=1}^{\infty} k\,x^{k-1} = \frac{1}{(1 - x)^2}$$