RENEWAL PROCESSES

AND

POISSON PROCESSES

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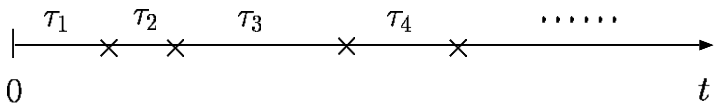
Renewal Processes



A **renewal process** is a point process characterized by the fact that the successive inter-arrival times τ_1, τ_2, τ_3 etc... are distributed with the same cdf $F(t)$ with density $f(t)$ and expected value $E[\tau] = m_1$:

$$F(t) = Pr\{\tau \le t\} \quad ; \quad f(t) = \frac{d\,F(t)}{d\,t}$$

$$E[\tau] = m_1$$

Let us define the corresponding Laplace transforms:

$$f^*(s) = \mathcal{L}[f(t)] \quad ; \quad F^*(s) = \mathcal{L}[F(t)]$$

The τ_i form a sequence of independent identically distributed random variables.
Let s_k denote the time up to the k arrival:

$$s_k = \sum_{i=1}^{k} \tau_i \tag{1}$$

Let $F_k(t)$ denote the cdf of s_k, and $f_k(t)$ its density.

$$F_k(t) = Pr\{s_k \le t\} \quad ; \quad f_k(t) = \frac{d\,F_k(t)}{d\,t}$$

In terms of Laplace transforms, we obtain:

$$F_k^*(s) = \frac{1}{s}\,[f^*(s)]^k \quad ; \quad f_k^*(s) = [f^*(s)]^k$$

Renewal Processes - number of arrivals

Let $N(t)$ denote the number of arrivals in the interval $0 - t$.

$$N(t) < k \quad \text{if and only if} \quad s_k > t$$

from which:

$$Pr\{N(t) < k\} = Pr\{s_k > t\} = 1 - F_k(t) \tag{2}$$

$$\begin{aligned} Pr\{N(t) = k\} &= Pr\{N(t) < k + 1\} - Pr\{N(t) < k\} \\ &= F_k(t) - F_{k+1}(t) \end{aligned} \tag{3}$$

Expected number of arrivals in $(0 - t)$, $H(t)$:

$$\begin{aligned} H(t) = E[N(t)] &= \sum_{k=0}^{\infty} k \, Pr\{N(t) = k\} \\ &= \sum_{k=0}^{\infty} k \, [F_k(t) - F_{k+1}] = \sum_{k=1}^{\infty} F_k(t) \end{aligned} \tag{4}$$

In Laplace transforms:

$$H^*(s) = \sum_{k=1}^{\infty} F_k^*(s) = \frac{1}{s} \sum_{k=1}^{\infty} f^{*\,k}(s)$$

The Renewal Equation

The renewal density $h(t)$ is defined as:

$$h(t) \;=\; \lim_{\Delta t \to 0} \; \frac{Pr\{one\ or\ more\ events\ occur\ in\ (t - t + \Delta t)\}}{\Delta t}$$

The probability that the k-th event occurs in $(t - t + \Delta t)$ is:

$$Pr\{k - th\ event\ occurs\ in\ (t - t + \Delta t)\} \;=\; f_k(t) \;+\; O(\Delta t)$$

Hence:

$$h(t) \;=\; \sum_{k=1}^{\infty} \; f_k(t) \;=\; \frac{d\,H(t)}{d\,t}$$

In Laplace transform:

$$\begin{aligned} h^*(s) \;&=\; \sum_{k=1}^{\infty} \; f_k^*(s) \\ &=\; f^*(s) \;+\; [f^*(s)]^2 \;+\; \ldots \;+\; [f^*(s)]^k \;+ \ldots \\ &=\; \frac{f^*(s)}{1 \;-\; f^*(s)} \end{aligned}$$

From which we obtain the **renewal equation** in LT and time domain:

$$h^*(s) \;=\; f^*(s) \;+\; h^*(s) \;\cdot\; f^*(s)$$

$$h(t) \;=\; f(t) \;+\; \int_0^t \; h(t - u) \;\cdot\; f(u)\, du$$

The Fundamental Renewal Equation

In terms of the renewal function $H(t)$, the renewal equation can be derived in the following way.

In Laplace transform:

$$H^*(s) = \frac{h^*(s)}{s} = \frac{1}{s}\frac{f^*(s)}{1 - f^*(s)}$$

In a similar way, we can write:

$$\begin{aligned} H^*(s) &= \sum_{k=1}^{\infty} F_k^*(s) = \frac{1}{s}\sum_{k=1}^{\infty} f^{*\,k}(s) \\ &= \frac{1}{s}\{f^*(s) + f^{*\,2}(s) + \ldots + f^{*\,k}(s) + \ldots\} = \frac{1}{s}\frac{f^*(s)}{1 - f^*(s)} \end{aligned}$$

From the above we derive:

$$H^*(s) = \frac{f^*(s)}{s} + H^*(s)\, f^*(s)$$

In time domain:

$$H(t) = F(t) + \int_0^t H(t-u) \cdot f(u)\, du$$

This is known as the fundamental **renewal equation**.

Poisson Process

A **Poisson process** is a renewal process in which the inter-arrival times are exponentially distributed with parameter λ.

$$f(t) = \lambda\, e^{-\lambda t} \quad \Longrightarrow \quad f^*(s) = \frac{\lambda}{s + \lambda}$$

The cdf and density of the time up to the k-th arrival s_k are in LT:

$$f_k^*(s) = \left(\frac{\lambda}{s + \lambda}\right)^k \quad ; \quad F_k^*(s) = \frac{\lambda^k}{s\,(s + \lambda)^k}$$

$$f_1(t) = \mathcal{L}^{-1}\left[\frac{\lambda}{s + \lambda}\right] = \lambda\, e^{-\lambda t}$$

$$f_2(t) = \mathcal{L}^{-1}\left[\frac{\lambda^2}{(s + \lambda)^2}\right] = \lambda^2\, t\, e^{-\lambda t}$$

$$\cdots \quad \cdots \quad \cdots$$

$$f_k(t) = \mathcal{L}^{-1}\left[\frac{\lambda^k}{(s + \lambda)^k}\right] = \frac{\lambda\,(\lambda\, t)^{k-1}}{(k - 1)!}\, e^{-\lambda t}$$

$$F_k(t) = \int_0^t f_k(u)\, du = 1 - \sum_{i=0}^{k-1} \frac{(\lambda\, t)^i}{i\,!}\, e^{-\lambda t}$$

Poisson Process

Let us define:

$$P_k(t) \;=\; Pr\{N(t) \;=\; k\} \;=\; F_k(t) \;-\; F_{k+1}(t)$$

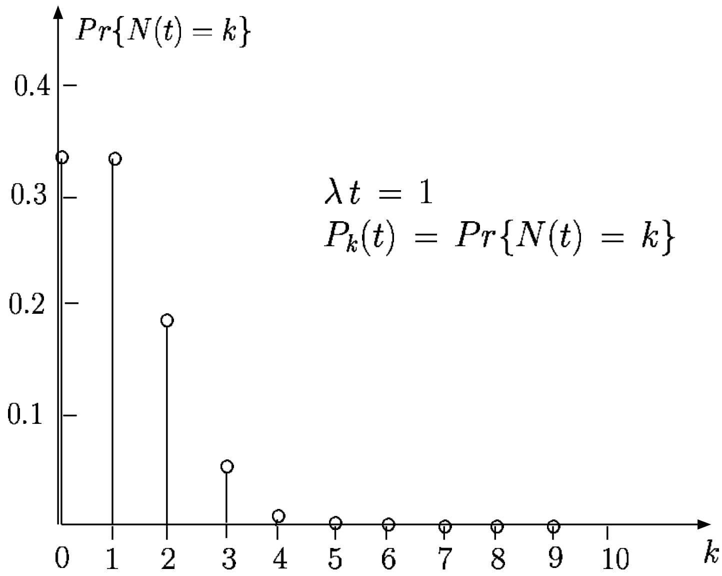
Taking Laplace transforms:

$$P_k^*(s) \;=\; \frac{\lambda^k}{s\,(s\,+\,\lambda)^k} \;-\; \frac{\lambda^{k+1}}{s\,(s\,+\,\lambda)^{k+1}} \;=\; \frac{\lambda^k}{(s\,+\,\lambda)^{k+1}}$$

Inverting again in the time domain, we obtain the **Poisson distribution**:

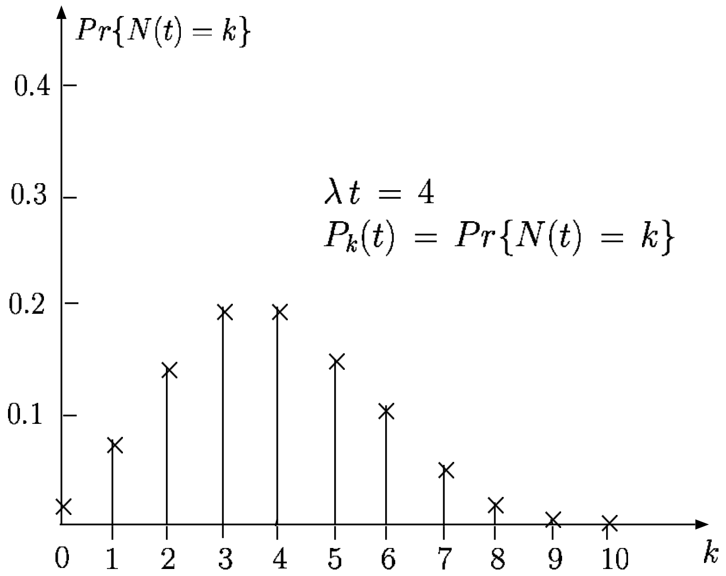
$$P_k(t) \;=\; Pr\{N(t) \;=\; k\} \;=\; \frac{(\lambda\, t)^{\,k}}{k\,!}\; e^{-\lambda\, t}$$

Poisson Distribution



k	P_k
0	0.368
1	0.368
2	0.184
3	0.061
4	0.015
5	0.003
6	$5.1 \cdot 10^{-4}$
...	...
10	$1.01 \cdot 10^{-7}$

Poisson Distribution



k	P_k
0	0.0183
1	0.073
2	0.146
3	0.195
4	0.195
5	0.156
6	0.104
7	0.059
8	0.029
9	0.013
10	0.005

Poisson Process
Expected number of events

$$H(t) = E[N(t)] = \sum_{k=0}^{\infty} k\, Pr\{N(t) = k\} = \sum_{k=0}^{\infty} k\, P_k(t)$$

$$= e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}$$

$$= \lambda t \cdot e^{-\lambda t} \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots\right) = \lambda t$$

An alternative derivation in Laplace transforms:

$$H^*(s) = E^*[N(s)] = \frac{1}{s} \frac{f^*(s)}{1 - f^*(s)}$$

Since in the Poisson process:

$$f^*(s) = \frac{\lambda}{s + \lambda} \quad \Longrightarrow \quad E^*[N(s)] = \frac{\lambda}{s^2}$$
$$\Longrightarrow \quad h^*(s) = \frac{\lambda}{s} \tag{5}$$

We obtain:

$$H(t) = E[N(t)] = \lambda t$$

$$h(t) = \lambda$$

Buffer design in a Poisson Process

Jobs arrive according to a Poisson process of rate λ and must be stored in a buffer during an interval T.

The design problem consists in evaluating the number of slots in a buffer such that the probability of refusing an incoming job in the interval T is less than a prescribed (small) risk $\alpha \quad (0 \leq \alpha \leq 1)$.

Let $N(T)$ be the number of arrivals in the interval T and let K be the number of slots to be determined.

The design problem can be formulated as:

$$Pr\{N(T) > K\} \leq \alpha$$

$$Pr\{N(T) \leq K\} > 1 - \alpha$$

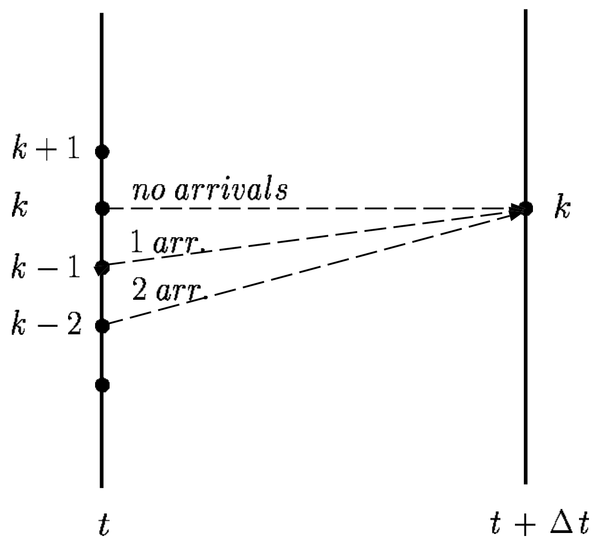
$$Pr\{N(T) \leq K\} = \sum_{j=0}^{K} \frac{(\lambda T)^j}{j!} e^{-\lambda T} > 1 - \alpha$$

The value of K is the smallest integer that satisfies the above equation.

Poisson Process
An alternative derivation

A counting process $N(t)$ is a stochastic point process identified by a sequence of random points in time. The **state** of the process at time t is defined by the number of arrivals $N(t)$ in the interval $(0 - t)$.

Let $E_i(t)$ be the event $N(t) = i$, i.e. i arrivals in $(0 - t)$.



According to the figure, the event $E_i(t + \Delta\, t)$ can be decomposed into the sequence of independent and mutually exclusive events:

$$\begin{aligned} E_i(t + \Delta\, t) &= \{E_i(t),\ \text{no arrivals in } \Delta\, t\} \\ &+ \{E_{i-1}(t),\ 1 \text{ arrival in } \Delta\, t\} \\ &+ \{E_{i-2}(t),\ 2 \text{ arrivals in } \Delta\, t\} \\ &+ \cdots \end{aligned}$$

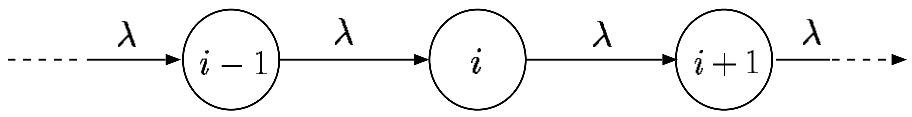
Poisson Process
An alternative derivation

By the theorem of the total probability, we can write:

$$
\begin{aligned}
Pr\{N(t+\Delta t)=i\} \;=\;& Pr\{\text{no arrivals in } \Delta t \,|\, N(t)=i\} \cdot \\
& \cdot Pr\{N(t)=i\} \\
+\;& Pr\{\text{1 arrival in } \Delta t \,|\, N(t)=i-1\} \cdot \\
& \cdot Pr\{N(t)=i-1\} \\
+\;& Pr\{\text{2 arrivals in } \Delta t \,|\, N(t)=i-2\} \cdot \\
& \cdot Pr\{N(t)=i-2\} \\
+\;& \cdots
\end{aligned}
$$

Poisson Process
An alternative derivation

A stochastic point process $N(t)$ is a Poisson process, if the probability of having one event in any interval $d\,t$ is constant and equal to λ.



By the theorem of the total probability, we can write for $i > 0$:

$$Pr\{N(t+\Delta\,t) = i\,|N(t) = i\} \;=\; 1 \,-\, \lambda\,\Delta\,t \,+\, O(\Delta\,t)$$

$$Pr\{N(t+\Delta\,t) = i\,|N(t) = i-1\} \;=\; \lambda\,\Delta\,t \,+\, O(\Delta\,t)$$

Where:

$$\lim_{\Delta\,t \to 0} \frac{O(\Delta\,t)}{\Delta\,t} \;=\; 0$$

For $i = 0$, we can write:

$$Pr\{N(t+\Delta\,t) = 0\,|N(t) = 0\} \;=\; 1 \,-\, \lambda\,\Delta\,t \,+\, O(\Delta\,t)$$

Poisson Process
An alternative derivation

Let us define:

$$P_i(t) \;=\; Pr\{N(t) \;=\; i\}$$

According to the above relations we can write:

$$\begin{cases} P_0(t+\Delta\, t) \;=\; (1 \,-\, \lambda\,\Delta\, t)\, P_0(t) & i=0 \\[2ex] P_i(t+\Delta\, t) \;=\; (1 \,-\, \lambda\,\Delta\, t)\, P_i(t) \,+\, \lambda\,\Delta\, t P_{i-1}(t) & i>0 \end{cases}$$

$$\begin{cases} \dfrac{P_0(t+\Delta\, t) \,-\, P_0(t)}{\Delta\, t} \;=\; -\,\lambda\, P_0(t) & i=0 \\[2ex] \dfrac{P_i(t+\Delta\, t) \,-\, P_i(t)}{\Delta\, t} \;=\; -\,\lambda\, P_i(t) \,+\, \lambda\, P_{i-1}(t) & i>0 \end{cases}$$

Taking the limit $\Delta\, t \longrightarrow 0$, a Poisson process is characterized by the following set of linear differential equations:

$$\begin{cases} \dfrac{d\, P_0(t)}{d\, t} \;=\; -\,\lambda\, P_0(t) & i=0 \\[2ex] \dfrac{d\, P_i(t)}{d\, t} \;=\; -\,\lambda\, P_i(t) \,+\, \lambda\, P_{i-1}(t) & i>0 \end{cases}$$

with initial conditions:

$$\begin{cases} P_0(0) \;=\; 1 & i=0 \\[2ex] P_i(0) \;=\; 0 & i>0 \end{cases}$$

Poisson Process
An alternative derivation

Taking Laplace transforms of the time domain differential equations:

$$\begin{cases} s\,P_0^*(s) - 1 = -\lambda\,P_0^*(s) & i = 0 \\ s\,P_i^*(s) = -\lambda\,P_i^*(s) + \lambda\,P_{i-1}^*(s) & i > 0 \end{cases}$$

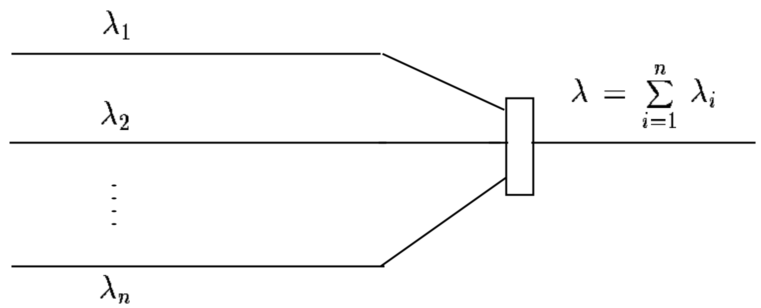
$$\begin{aligned} P_0^*(s) &= \frac{1}{s + \lambda} \\ P_1^*(s) &= \frac{\lambda}{s + \lambda} \cdot P_0^*(s) = \frac{\lambda}{(s + \lambda)^2} \\ \cdots & \quad \cdots \\ P_i^*(s) &= \frac{\lambda}{s + \lambda} \cdot P_{i-1}^*(s) = \frac{\lambda^i}{(s + \lambda)^{i+1}} \\ \cdots & \quad \cdots \end{aligned}$$

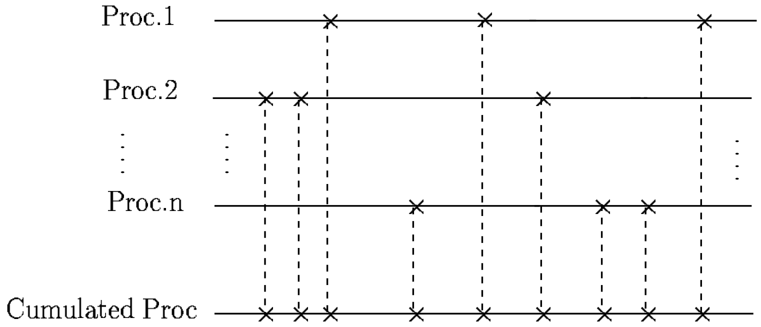
From the above equations we derive again the Poisson distribution:

$$P_k(t) = Pr\{N(t) = k\} = \frac{(\lambda\,t)^{\,k}}{k\,!}\,e^{-\lambda\,t}$$

Superposition of Poisson Processes

A superposition of Poisson processes is obtained by cumulating the occurrences of n independent sources of Poisson processes with parameters λ_1, λ_2, $\ldots$, λ_n, respectively.





The cumulated process is still a Poisson process:

$$Prob\{\, N(t + \Delta\, t) = k + 1 \,|\, N(t) = k\} \;=$$

$$\lambda_1 \,\Delta\, t \;+\; \lambda_2 \,\Delta\, t \;+\; \ldots \;+\; \lambda_n \,\Delta\, t \;+\; O(t)$$

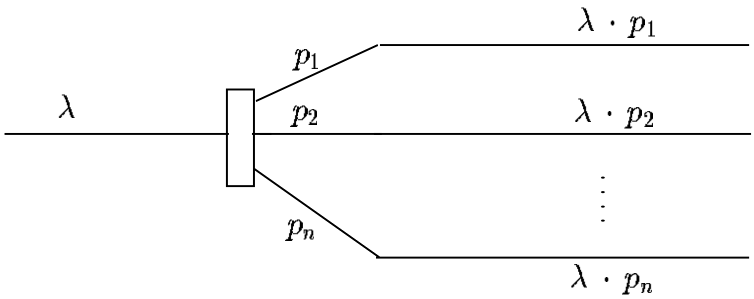
The parameter of the cumulated Poisson process is given by:

$$\lambda \;=\; \sum_{i=1}^{n} \;\lambda_i$$

Decomposition of Poisson Processes

A Poisson process with parameter λ is split into n indipendent branches according to a probability distribution $p_1,\ p_2,\ \ldots,\ p_n$ with:

$$\sum_{i=1}^{n} p_i \ = \ 1$$

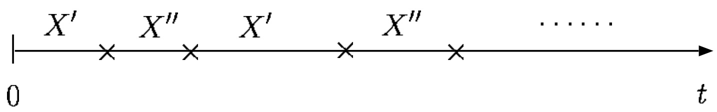


By decomposing the original Poisson process of parameter λ, n Poisson processes are generated: $N_1(t),\ N_2(t),\ \ldots,\ N_n(t)$, with parameters: $\lambda\, p_1,\ \lambda\, p_2,\ \ldots,\ \lambda\, p_n$.

$$Pr\{N_i(t+\Delta\, t) = k+1 \,|\, N_i(t) = k\} \ = \ p_i\, \lambda\, \Delta\, t \ + \ O(\Delta\, t)$$

Alternating Renewal Processes

The process is constituted by a sequence of *Type I* variables X' with density $f_1(x)$ followed by a *Type II* variables X'' with density $f_2(x)$. The process starts with probability 1 with a *Type I* variable.



If we look at the sequence formed by the occurrence of the *Type II* variables, the process is an ordinary renewal process with inter-arrival time $(X' + X'')$.

The mean number of *Type II* occurrences satisfies (in LT):

$$H_2^*(s) = \frac{f_1^*(s) \cdot f_2^*(s)}{s\,\{1 - f_1^*(s) \cdot f_2^*(s)\}}$$

For the *Type I* occurrences we have a modified renewal process, for which the expected number of renewals is:

$$H_1^*(s) = \frac{f_1^*(s)}{s\,\{1 - f_1^*(s) \cdot f_2^*(s)\}}$$

In both cases:

$$h_i^*(s) = s\,H_i^*(s) \qquad\qquad i = 1, 2$$

Alternating Renewal Processes

Let be:

$\pi_1(t)$ **-** Probability *Type I* variable occurs at time t

$\pi_2(t)$ **-** Probability *Type II* variable occurs at time t

Type I is in use at time t if:

a) - No *Type I* event occurs in $(0 - t)$;

b) - A *Type II* event occurs in $u - u + du$ $(u < t)$, and no *Type I* events occur in $(t - u)$:

$$\pi_1(t) = [1 - F_1(t)] + \int_0^t h_2(u)\,[1 - F_1(t - u)]\,du$$

In Laplace transform:

$$\pi_1^*(s) = \frac{1 - f_1^*(s)}{s} + \frac{f_1^*(s) \cdot f_2^*(s)}{1 - f_1^*(s) \cdot f_2^*(s)} \frac{1 - f_1^*(s)}{s}$$

$$\pi_1^*(s) = \frac{1 - f_1^*(s)}{s\,\{\,1 - f_1^*(s) \cdot f_2^*(s)\,\}}$$

Note also that:

$$\pi_1^*(s) = H_2^*(s) - H_1^*(s) + \frac{1}{s}$$

$$\pi_1(t) = H_2(t) - H_1(t) + 1$$

Alternating Poisson Process

Type I variable is exponential with rate λ;
Type II variable is exponential with rate μ.

$$\begin{aligned}\pi_1^*(s) &= \frac{s+\mu}{s\,(s+\lambda+\mu)} \\ &= \frac{\mu}{\lambda+\mu}\cdot\frac{1}{s}+\frac{\lambda}{\lambda+\mu}\cdot\frac{1}{s+\lambda+\mu}\end{aligned}$$

$$\pi_1(t) = \frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}\,e^{-(\lambda+\mu)\,t}$$

$$\pi_2^*(s) = \frac{\lambda}{s\,(s+\lambda+\mu)}$$

$$\pi_2(t) = \frac{\lambda}{\lambda+\mu}-\frac{\lambda}{\lambda+\mu}\,e^{-(\lambda+\mu)\,t}$$

$$\lim_{t\to\infty}\pi_1(t) = \frac{\mu}{\lambda+\mu} \qquad ; \qquad \lim_{t\to\infty}\pi_2(t) = \frac{\lambda}{\lambda+\mu}$$