An introduction to cooperative games

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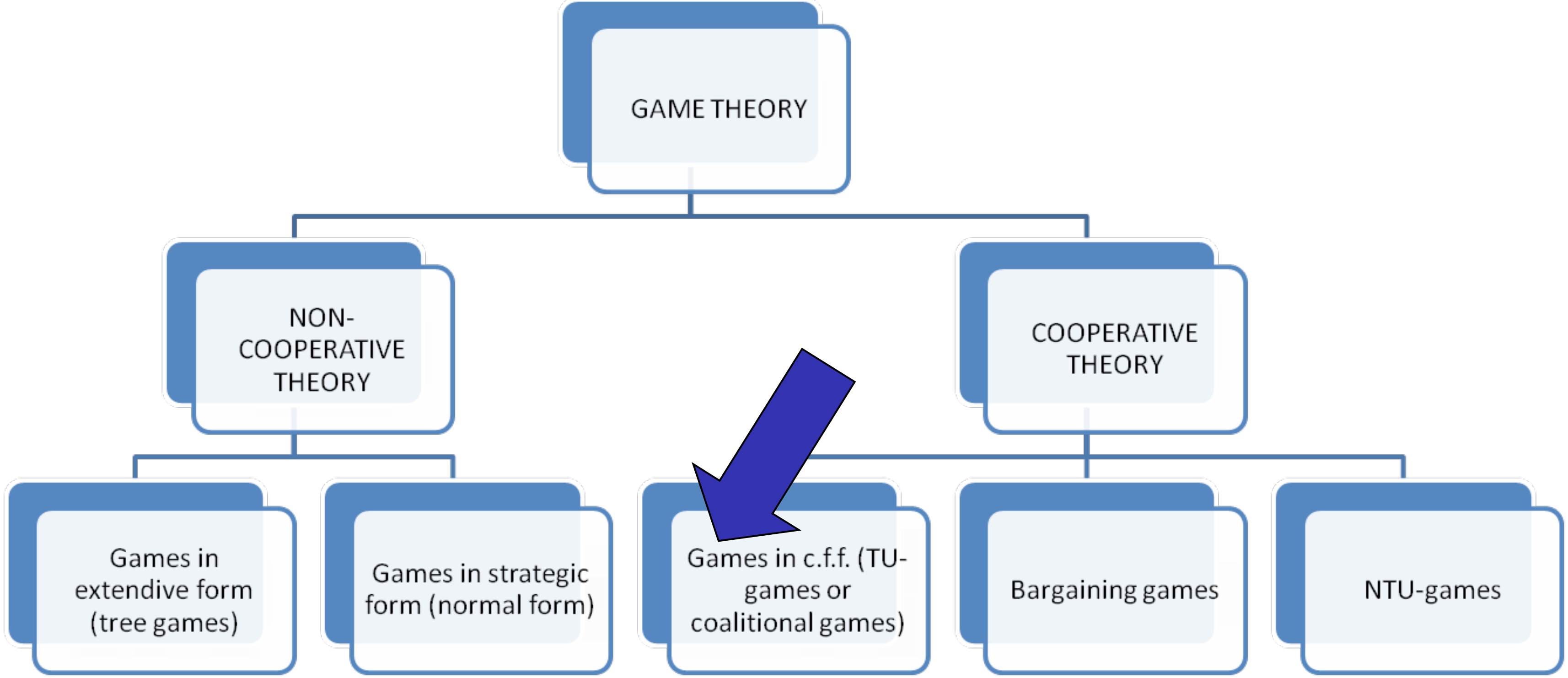

- Dominant strategies
- Nash eq. (NE)
- Subgame perfect NE
- NE & refinements

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- Core
- Shapley value
- Nucleolus
- τ-value
- PMAS

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- Nash sol.
- Kalai-Smorodinsky

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- CORE
- NTU-value
- Compromise value

...

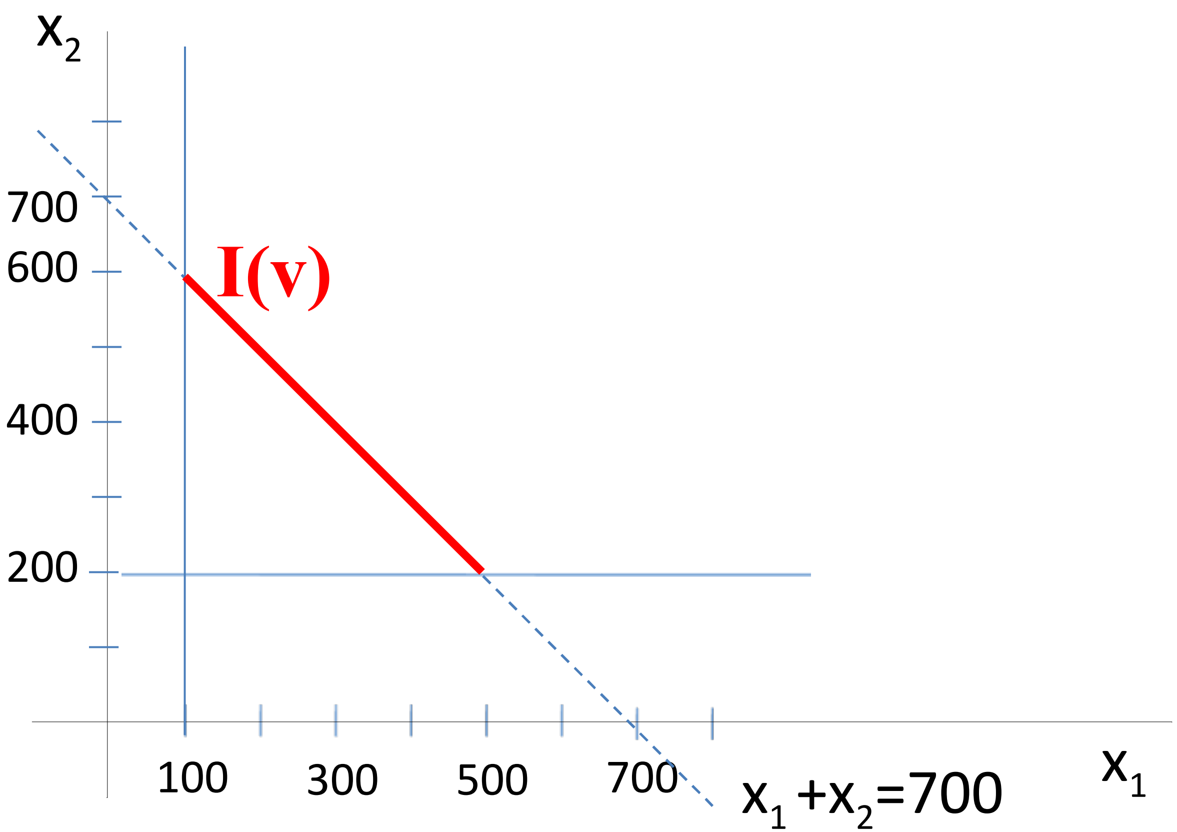
No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing

Cooperative games: a simple example

Alone, player 1 (singer) and 2 (pianist) can earn 100€ 200€ respect.
Together (duo) 700€

How to divide the (extra) earnings?



Imputation set: $I(v)=\{x \in IR^2 | x_1 \geq 100,\ x_2 \geq 200,\ x_1+x_2=700\}$

COOPERATIVE GAME THEORY

Games in coalitional form

TU-game: (N,v) or v

$N=\{1, 2, \ldots, n\}$	set of players
$S\subset N$	coalition
2^N	set of coalitions

DEF. v: $2^N \rightarrow IR$ with $v(\varnothing)=0$ is a Transferable Utility (TU)-game with player set N.

NB: $(N,v)\leftrightarrow v$

NB2: if $n=|N|$, it is also called *n-person TU-game*, *game in coalitional form*, *coalitional game*, *cooperative game with side payments*…

v(S) is the value (worth) of coalition S

COOPERATIVE GAME THEORY

Example

(Glove game) $N = L \cup R$, $L \cap R = \emptyset$

$i \in L$ ($i \in R$) possesses 1 left (right) hand glove

Value of a pair: 1€

$v(S) = \min\{|L \cap S|, |R \cap S|\}$ for each coalition $S \in 2^N \setminus \{\emptyset\}$.

Example

Glove game with $L=\{1,2\}$, $R=\{3\}$)

$v(1,3)=v(2,3)=v(1,2,3)=1$, $v(S)=0$ otherwise

Q.1: which coalitions form?

DEF. (N,v) is a superadditive game iff

$$v(S \cup T) \geq v(S) + v(T) \text{ for all } S,T \text{ with } S \cap T = \emptyset$$

Q.2: If the grand coalition N forms, how to divide v(N)?
(how to allocate costs?)

Many answers! (solution concepts)

One-point concepts:
- Shapley value (Shapley 1953)
- nucleolus (Schmeidler 1969)
- τ-value (Tijs, 1981)

…

Subset concepts:
- Core (Gillies, 1954)
- stable sets (von Neumann, Morgenstern, ’44)
- kernel (Davis, Maschler)
- bargaining set (Aumann, Maschler)

…..

Example

(Glove game) (N,v) such that $N=L\cup R$, $L\cap R=\emptyset$

$v(S)=\min\{|L\cap S|, |R\cap S|\}$ for all $S\in 2N\backslash\{\emptyset\}$

Claim: the glove game is superadditive.

Suppose $S,T\in 2^N\backslash\{\emptyset\}$ with $S\cap T=\emptyset$. Then

$v(S)+v(T)= \min\{|L\cap S|, |R\cap S|\} + \min\{|L\cap T|, |R\cap T|\}$

$=\min\{|L\cap S|+|L\cap T|,|L\cap S|+|R\cap T|,|R\cap S|+|L\cap T|,|R\cap S|+|R\cap T|\}$

$\leq\min\{|L\cap S|+|L\cap T|, |R\cap S|+|R\cap T|\}$

since $S\cap T=\emptyset$

$=\min\{|L\cap(S\cup T)|, |R\cap(S\cup T)|\}$

$=v(S\cup T)$.

The imputation set

DEF. Let (N,v) be a n-persons TU-game.
A vector $x=(x_1, x_2, \ldots, x_n) \in IR^N$ is called an imputation iff

(1) x is individual rational i.e.

$$x_i \geq v(i) \text{ for all } i \in N$$

(2) x is efficient

$$\Sigma_{i \in N} x_i = v(N)$$

[interpretation x_i: payoff to player i]

$$I(v) = \{x \in IR^N \mid \Sigma_{i \in N} x_i = v(N),\ x_i \geq v(i) \text{ for all } i \in N\}$$

Set of imputations

Example

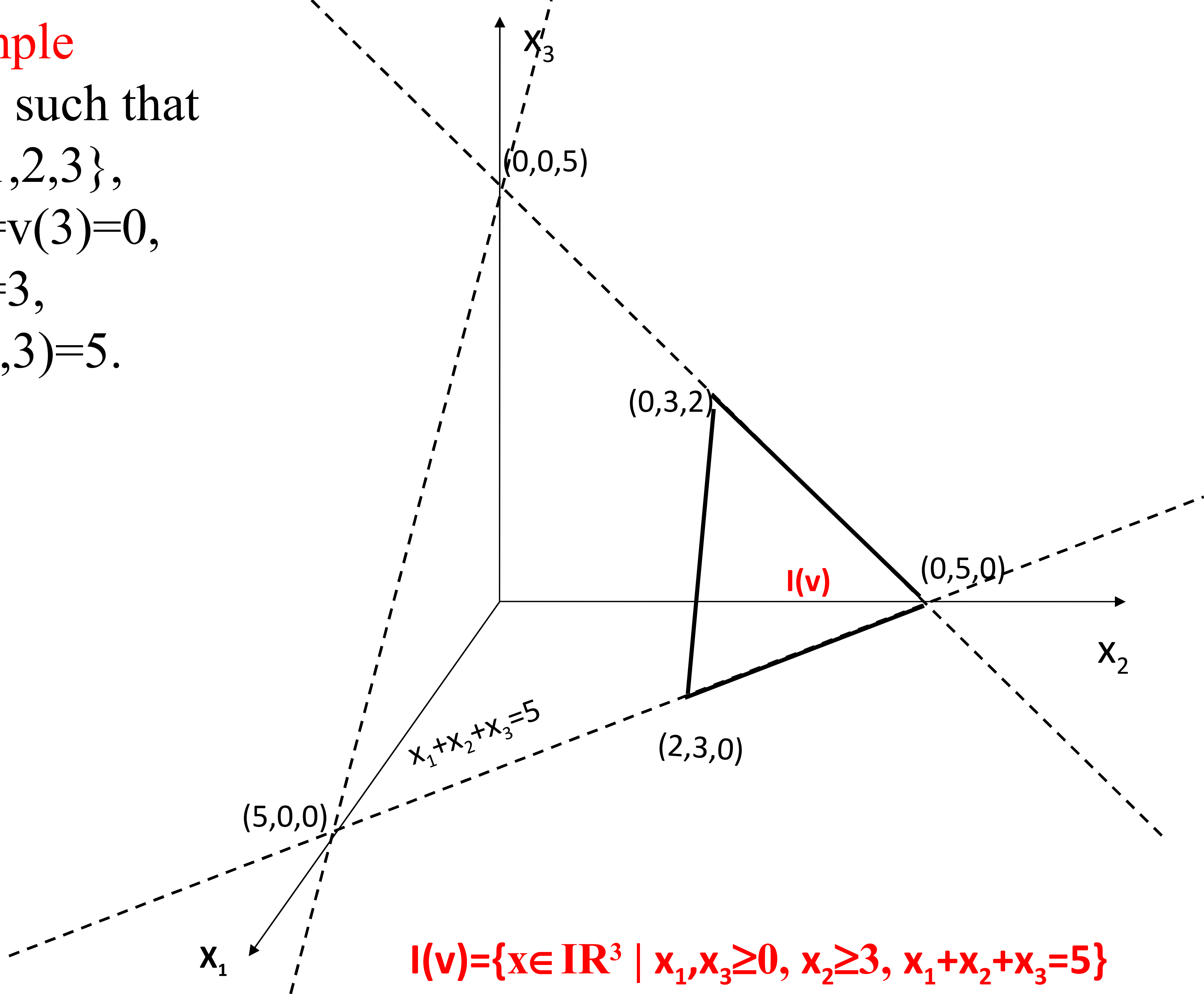
(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2,3)=5.



$I(v)=\{x\in \mathbb{R}^3 \mid x_1,x_3\geq 0,\ x_2\geq 3,\ x_1+x_2+x_3=5\}$

The core of a game

DEF. Let (N,v) be a TU-game. The core C(v) of (N,v) is the set

$C(v)=\{x\in I(v) \mid \Sigma_{i\in S}\, x_i \geq v(S) \text{ for all } S\in 2N\backslash\{\emptyset\}\}$

stability conditions

no coalition S has the incentive to split off if x is proposed

Note: $x \in C(v)$ iff

(1) $\Sigma_{i\in N}\, x_i = v(N)$ *efficiency*

(2) $\Sigma_{i\in S}\, x_i \geq v(S)$ for all $S\in 2N\backslash\{\emptyset\}$ *stability*

Bad news: C(v) can be empty

Good news: many interesting classes of games have a non-empty core.

Example

(N,v) such that

$N=\{1,2,3\}$,

$v(1)=v(3)=0$,

$v(2)=3$,

$v(1,2)=3$,

$v(1,3)=1$

$v(2,3)=4$

$v(1,2,3)=5$.

Core elements satisfy the following conditions:

$x_1,x_3\geq 0$, $x_2\geq 3$, $x_1+x_2+x_3=5$

$x_1+x_2\geq 3$, $x_1+x_3\geq 1$, $x_2+x_3\geq 4$

We have that

$5-x_3\geq 3 \Leftrightarrow x_3\leq 2$

$5-x_2\geq 1 \Leftrightarrow x_2\leq 4$

$5-x_1\geq 4 \Leftrightarrow x_1\leq 1$

$$C(v)=\{x\in IR^3 \mid 1\geq x_1\geq 0, 2\geq x_3\geq 0,\ 4\geq x_2\geq 3,\ x_1+x_2+x_3=5\}$$

Example

(N,v) such that

N={1,2,3},

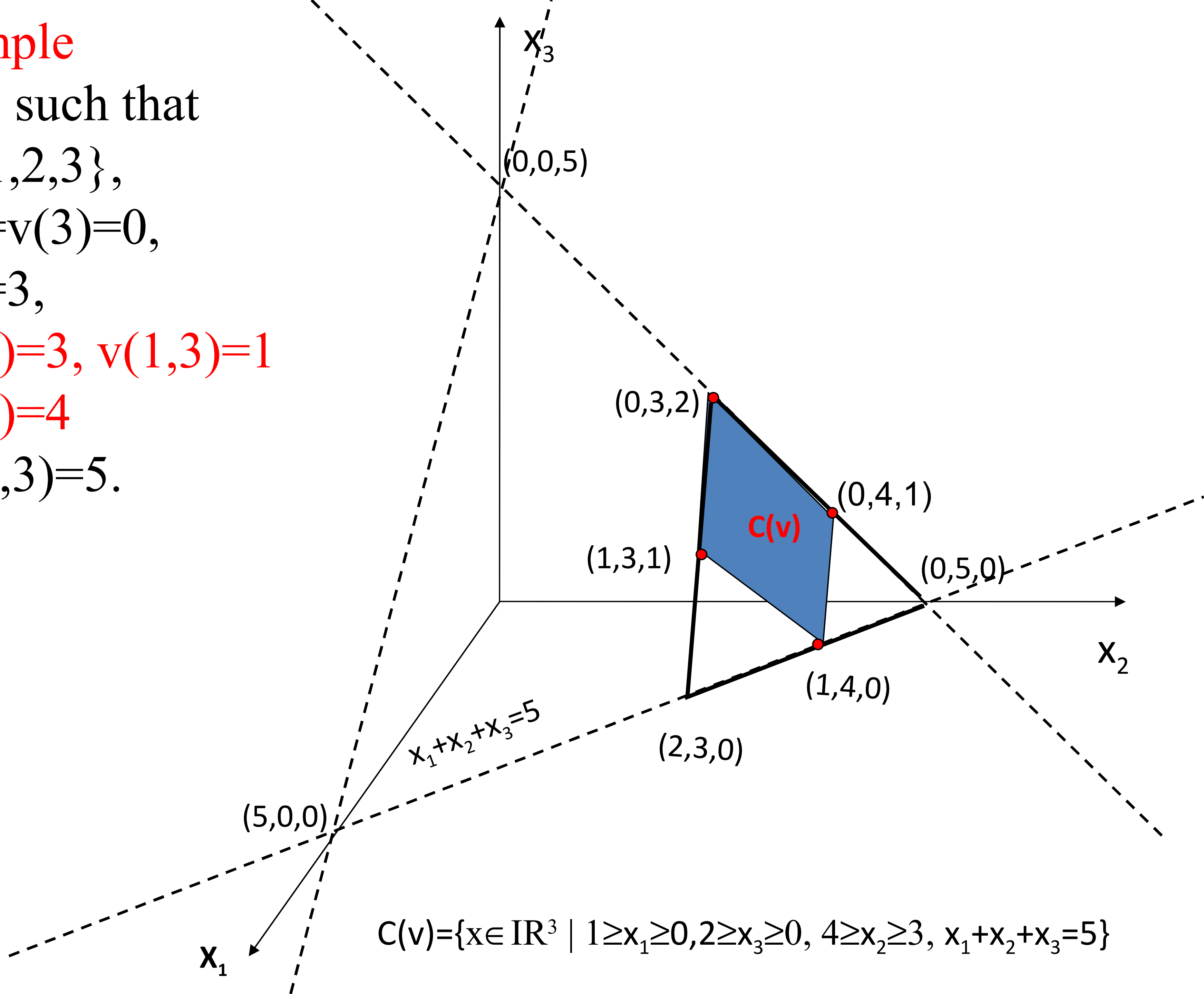
v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.



C(v)={x∈ IR³ | 1≥x₁≥0,2≥x₃≥0, 4≥x₂≥3, x₁+x₂+x₃=5}

Example (Game of pirates) Three pirates 1,2, and 3. On the other side of the river there is a treasure (10€). At least two pirates are needed to wade the river…

(N,v), $N=\{1,2,3\}$, $v(1)=v(2)=v(3)=0$,

$v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=10$

Suppose $(x_1, x_2, x_3)\in C(v)$. Then

efficiency $x_1+x_2+x_3=10$

stability
$$\begin{cases} x_1+x_2 \geq 10 \\ x_1+x_3 \geq 10 \\ x_2+x_3 \geq 10 \end{cases}$$

$20=2(x_1+x_2+x_3)\geq 30$ Impossible. So $C(v)=\varnothing$.

Note that (N,v) is superadditive.

Example

(Glove game with L={1,2}, R={3})

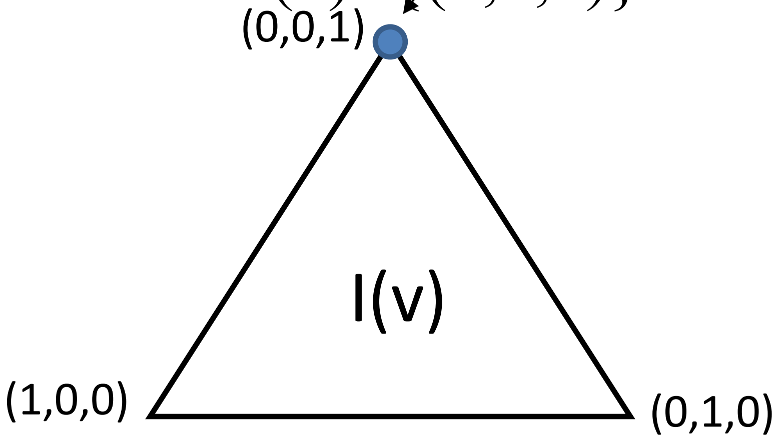
v(1,3)=v(2,3)=v(1,2,3)=1, v(S)=0 otherwise

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

$$\left.\begin{array}{l} x_1 + x_2 + x_3 = 1 \\ x_1 + x_3 \geq 1 \\ x_2 \geq 0 \end{array}\right\} \quad \begin{array}{l} x_2 = 0 \\ x_1 + x_3 = 1 \end{array}$$

$x_2 + x_3 \geq 1$ $\qquad x_1 = 0$ and $x_3 = 1$

So C(v)={(0,0,1)}.



How to share v(N)…

- The Core of a game can be used to exclude those allocations which are *not stable*.
- But the core of a game can be a bit “*extreme*” (see for instance the glove game)
- Sometimes the core is *empty* (pirates)
- And if it is not empty, there can be many allocations in the core (*which is the best*?)

An axiomatic approach (Shapley (1953)

- Similar to the approach of Nash in bargaining: *which properties an allocation method should satisfy in order to divide v(N) in a reasonable way*?
- Given a subset **C** of $\mathbf{G}^N$ (class of all TU-games with N as the set of players) a *(point map) solution* on **C** is a map $\Phi:\mathbf{C} \rightarrow \mathrm{IR}^N$.
- For a solution Φ we shall be interested in various properties…

Symmetry

PROPERTY 1(SYM) Let $v \in G^N$ be a TU-game.

Let $i, j \in N$. If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{N \setminus \{i,j\}}$, then $\Phi_i(v) = \Phi_j(v)$.

EXAMPLE

We have a TU-game $(\{1,2,3\}, v)$ s.t. $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(1, 3) = 4$, $v(2, 3) = 6$, $v(1, 2, 3) = 20$.

Players 2 and 3 are symmetric. In fact:

$v(\emptyset \cup \{2\}) = v(\emptyset \cup \{3\}) = 0$ and $v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4$

If Φ satisfies SYM, then $\Phi_2(v) = \Phi_3(v)$

Efficiency

PROPERTY 2 (EFF) Let $v \in G^N$ be a TU-game.

$\sum_{i \in N} \Phi_i(v) = v(N)$, i.e., $\Phi(v)$ is a *pre-imputation*.

Null Player Property

DEF. Given a game $v \in \mathbf{G}^N$, a player $i \in N$ s.t.

$v(S \cup i) = v(S)$ for all $S \in 2^N$ will be said to be a null player.

PROPERTY 3 (NPP) Let $v \in G^N$ be a TU-game. If $i \in N$ is a null player, then $\Phi_i(v) = 0$.

EXAMPLE We have a TU-game $(\{1,2,3\},v)$ such that $v(1) = 0$, $v(2) = v(3) = 2$, $v(1, 2) = v(1, 3) = 2$, $v(2, 3) = 6$, $v(1, 2, 3) = 6$. Player 1 is null. Then $\Phi_1(v) = 0$

EXAMPLE We have a TU-game ({1,2,3},v) such that v(1) =0, v(2) = v(3) = 2, v(1, 2) = v(1, 3) = 2, v(2, 3) = 6, v(1, 2, 3) = 6. On this particular example, if Φ satisfies NPP, SYM and EFF we have that

$\Phi_1(v) = 0$ by NPP

$\Phi_2(v) = \Phi_3(v)$ by SYM

$\Phi_1(v)+\Phi_2(v)+\Phi_3(v)=6$ by EFF

So $\Phi=(0,3,3)$

But our goal is to characterize Φ on $\mathbf{G}^N$. One more property is needed.

Additivity

PROPERTY 4 (ADD) Given $v,w \in \mathbf{G}^N$,

$$\Phi(v)+\Phi(w)=\Phi(v+w).$$

.EXAMPLE Two TU-games v and w on N={1,2,3}

v		w		v+w
v(1) =3		w(1) =1		v+w(1) =4
v(2) =4		w(2) =0		v+w(2) =4
v(3) = 1		w(3) = 1		v+w(3) = 2
v(1, 2) =8	+	w(1, 2) =2	=	v+w(1, 2) =10
v(1, 3) = 4		w(1, 3) = 2		v+w(1, 3) = 6
v(2, 3) = 6		w(2, 3) = 3		v+w(2, 3) = 9
v(1, 2, 3) = 10		w(1, 2, 3) = 4		v+w(1, 2, 3) = 14
Φ		Φ		Φ

Theorem 1 (Shapley 1953)

There is a unique map ϕ defined on $\mathbf{G}^N$ that satisfies EFF, SYM, NPP, ADD. Moreover, for any $i \in N$ we have that

$$\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_i^{\sigma}(v)$$

Here Π is the set of all permutations $\sigma: N \to N$ of N, while $m^{\sigma}_i(v)$ is the marginal contribution of player i according to the permutation σ, which is defined as:

$v(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\})$,

where j is the unique element of N s.t. $i = \sigma(j)$.

Probabilistic interpretation: (the "room parable")

- Players gather one by one in a room to create the "grand coalition", and each one who enters gets his marginal contribution.
- Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

Example

(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2)=3,

v(1,3)=1,

v(2,3)=4,

v(1,2,3)=5.

Permutation	1	2	3
1,2,3	**0**	**3**	**2**
1,3,2	**0**	**4**	**1**
2,1,3	**0**	**3**	**2**
2,3,1	**1**	**3**	**1**
3,2,1	**1**	**4**	**0**
3,1,2	**1**	**4**	**0**
Sum	**3**	**21**	**6**
$\phi(v)$	**3/6**	**21/6**	**6/6**

Example

(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.