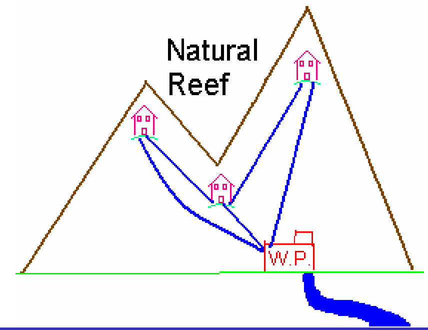
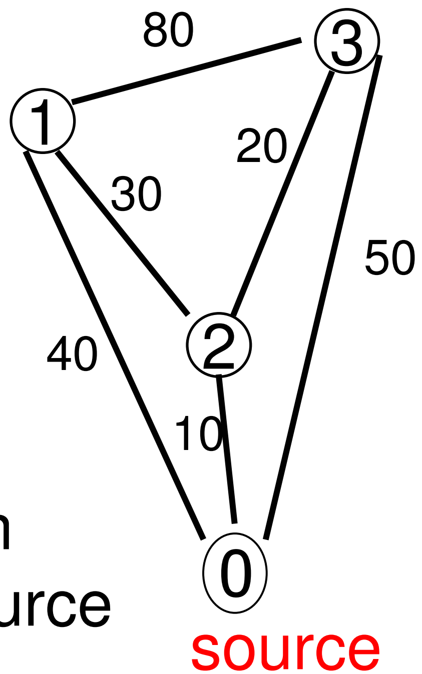


Section 2. Connection situations

- A connection situation takes place in the presence of a group of agents N={1,2, …,n}, each of which needs to be connected directly or via other agents to a source.
- If connections among agents are costly, then each agent will evaluate the opportunity of cooperating with other agents in order to reduce costs.
- If a group of agents decides to cooperate, a configuration of links which minimizes the total cost of connection is provided by a minimum cost spanning tree (mcst).
- The problem of finding a mcst may be easily solved thanks to different algorithms proposed in literature (Boruvka (1926), Kruskal (1956), Prim (1957), Dijkstra (1959))

Minimum Cost Spanning Tree Situation

Consider a complete weighted graph



– whose vertices represent agents

– vertex 0 is the source

– edges represent connections between agents or between an agent and the source

– numbers close to edges are connection costs

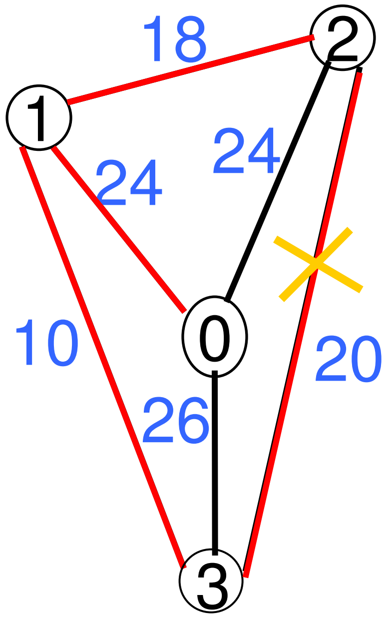
Minimum cost spanning tree (mcst) problem

Optimization problem:

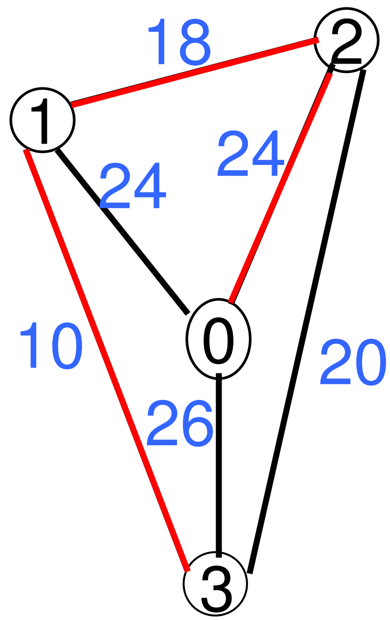
How to connect each node to the source 0 in such a way that the cost of construction of a spanning network (which connects every node directly or indirectly to the source 0) is minimum?

Example $N=\{1,2,3\}, E_{N'}=\{\{1,0\},\{2,0\},\{2,1\},\{3,0\},\{3,1\},\{3,2\}\}$
cost function shown on graphs

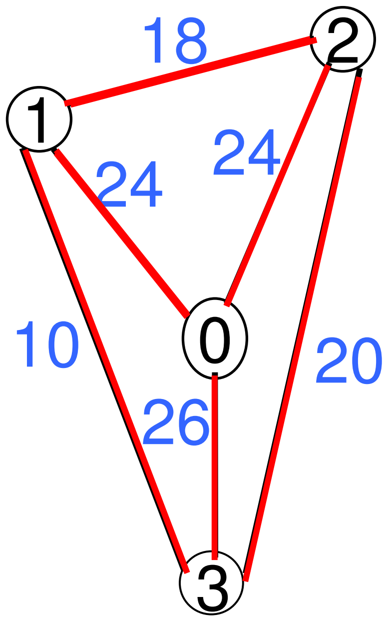
Kruskal algorithm



Prim algorithm



Example: mcst cost game ({1,2,3},c) defined on the following connection situation:



c(1)=24

c(2)=24

c(3)=26

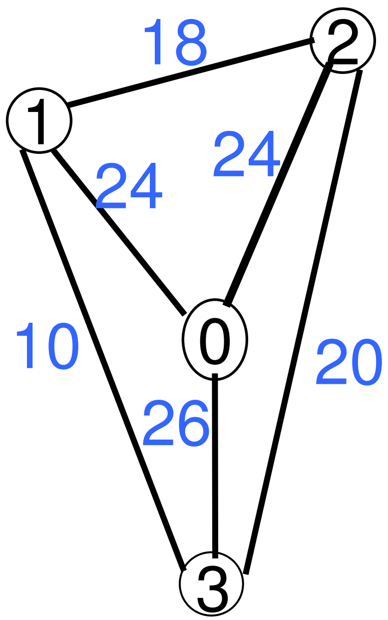
c(1,3)=34

c(1,2)=42

c(2,3)=44

c(1,2,3)=52

Example: The cost game ({1,2,3},c) is defined on the following connection situation:



c(1)=24

c(2)=24

c(3)=26

c(1,3)=34

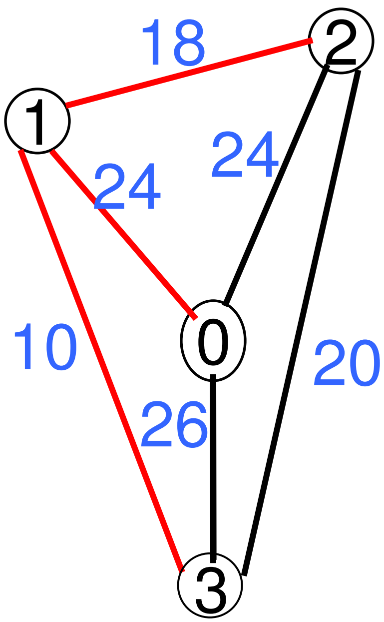
c(1,2)=42

c(2,3)=44

c(1,2,3)=52

The game ({1,2,3}, c) is said mcst game (Bird (1976))

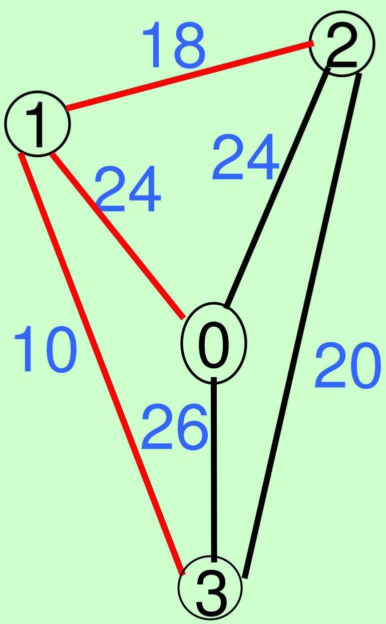
How to divide the total cost? (Bird 1976)



- The predecessor of 1 is 0: the Bird allocation gives to player 1 the cost of {0,1}.
- The predecessor of 2 is 1: the Bird allocation gives to player 2 the cost of {1,2};
- The predecessor of 3 is 1: the Bird allocation gives to player 3 the cost of {1,3}.

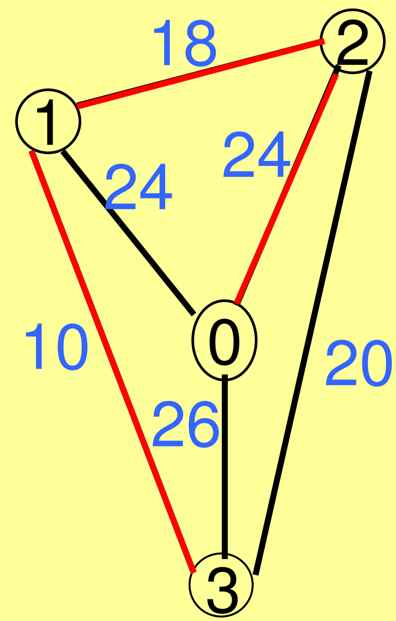
$w(\Gamma)=52$

Bird allocation w.r.t. to Γ, $(x_1, x_2, x_3)=(24, 18, 10)$ is in the core of ({1,2,3},c).



The Bird allocation w.r.t .this mcst is

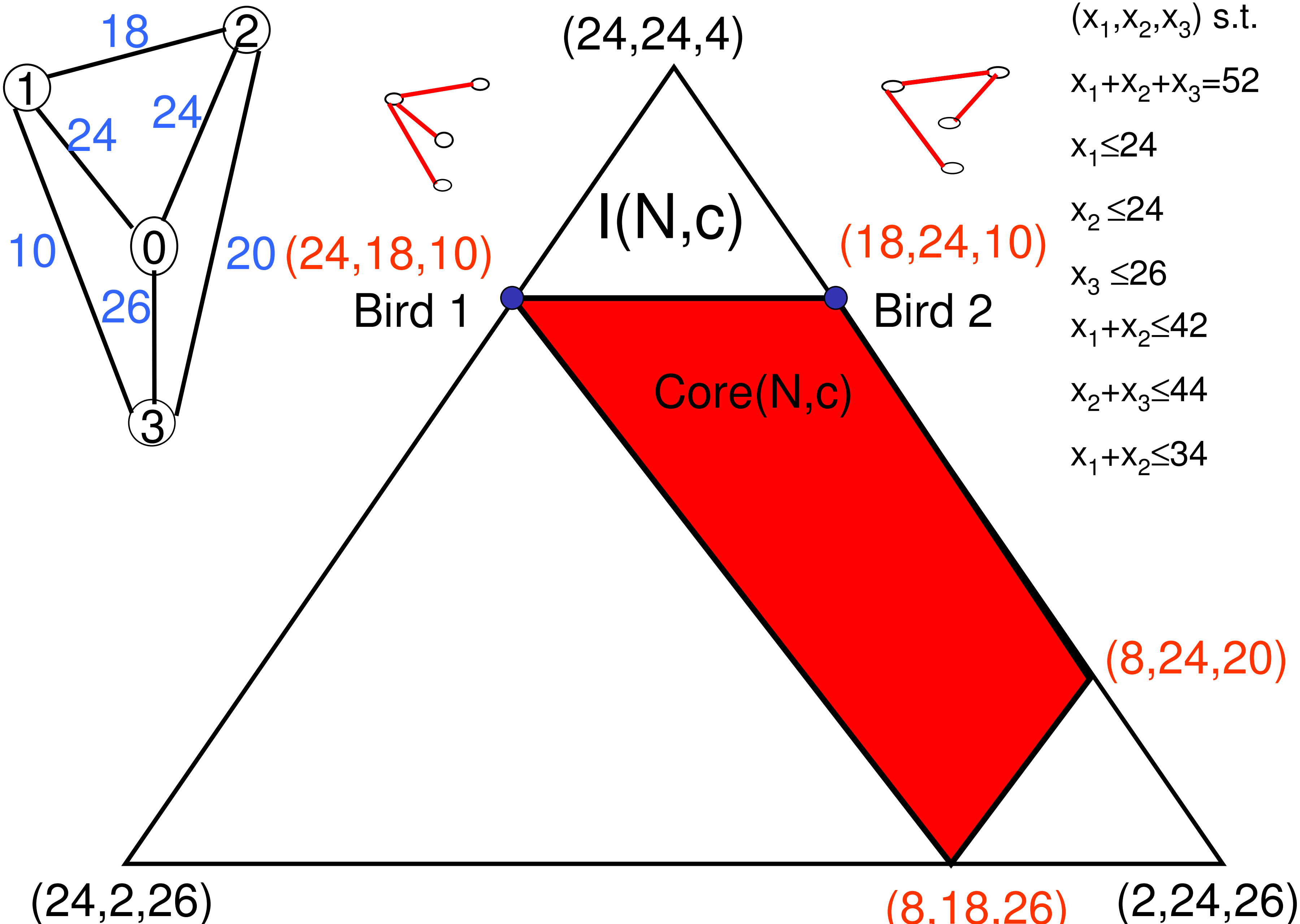
(x_1, x_2, x_3)=(24, 18 ,10)



The Bird allocation w.r.t. this mcst is

(x_1, x_2, x_3)=(18, 24 ,10)

Both allocations belong to the core of the mcst game (and also their convex combination).



1
2
0
3
18
24
24
10
26
20
(24,24,4)
I(N,c)
(24,18,10)
(18,24,10)
Bird 1
Bird 2
Core(N,c)
(8,24,20)
(24,2,26)
(8,18,26)
(2,24,26)
(x1,x2,x3) s.t.
x1+x2+x3=52
x1≤24
x2 ≤24
x3 ≤26
x1+x2≤42
x2+x3≤44
x1+x2≤34

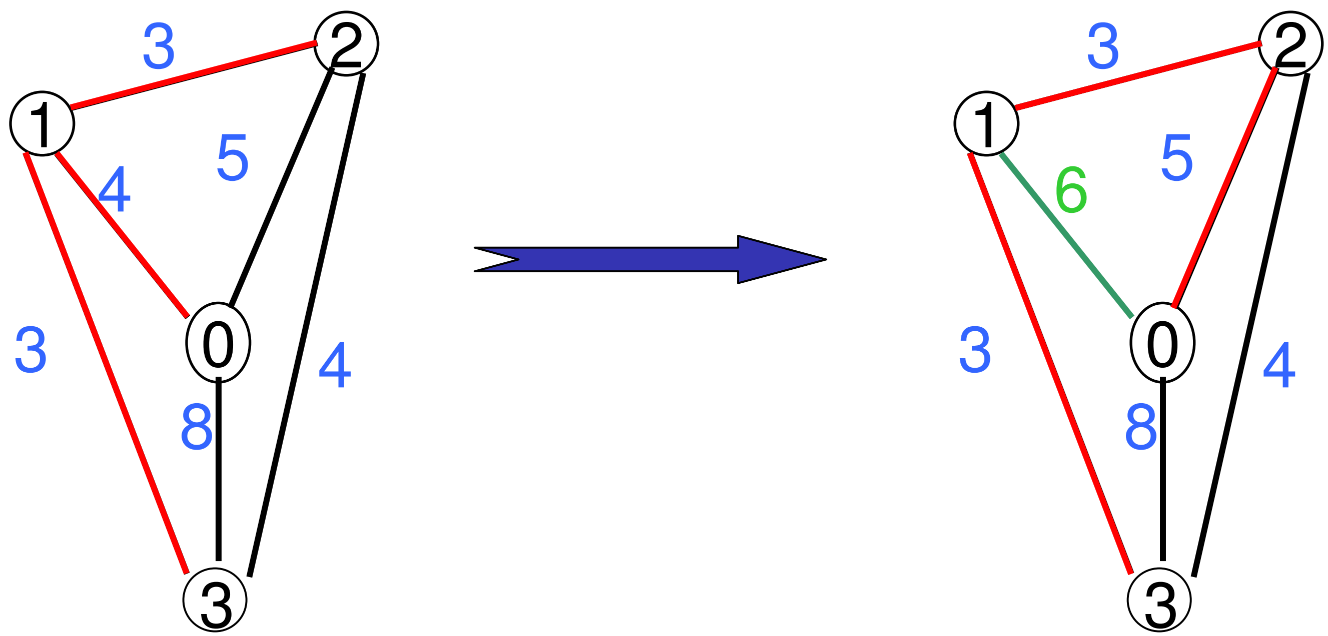
Bird allocation rule

- It always provides an allocation (given a connection situation).
- In general, not a unique allocation (each mcst determines a corresponding Bird allocation…).
- Bird allocations are in the core of mcst games (but are extreme points)

What happens when the structure of the network changes?

- Imagine to use a certain rule to allocate costs.
  - The cost of edges may increase: if the cost of an edge increases, nobody should be better off, according to such a rule (*cost monotonicity*);
  - One or more players may leave the connection situation: nobody of the remaining players should be better off (*population monotonicity*).

Cost monotonicity: Bird allocation behaviour

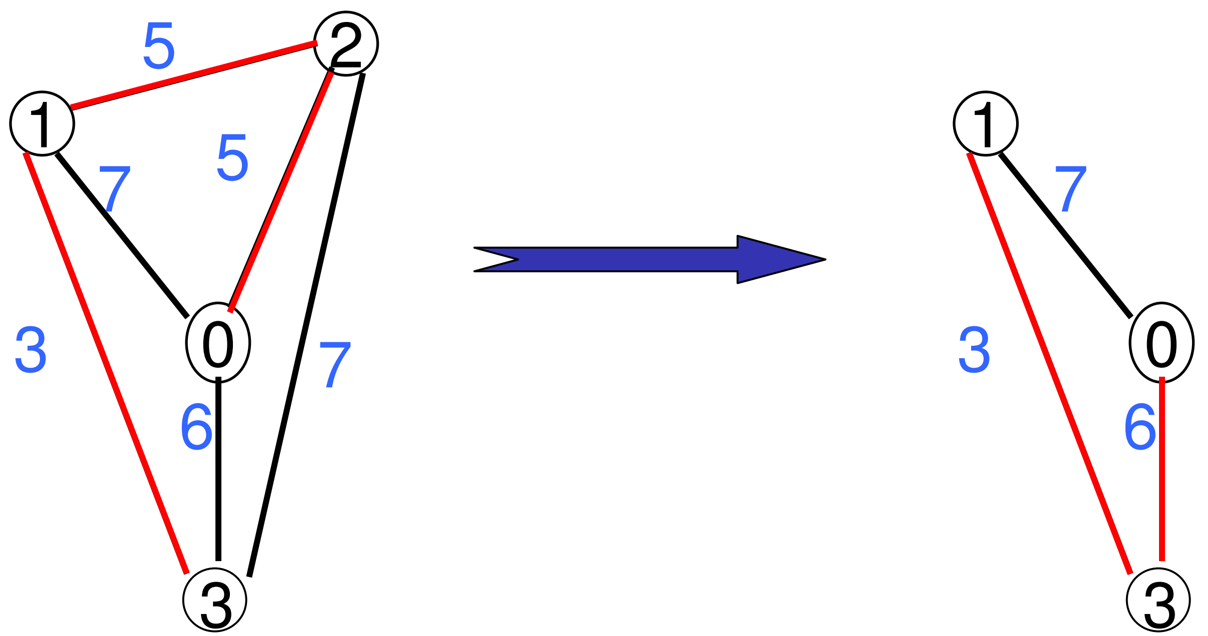


Bird allocation: (4, 3 ,3)

Bird allocation: (3, 5 ,3)

Bird rule does not satisfy cost monotonicity.

Population monotonicity: Bird allocation behaviour



Bird allocation: (5, 5 ,3)

Bird allocation: (3, * ,6)

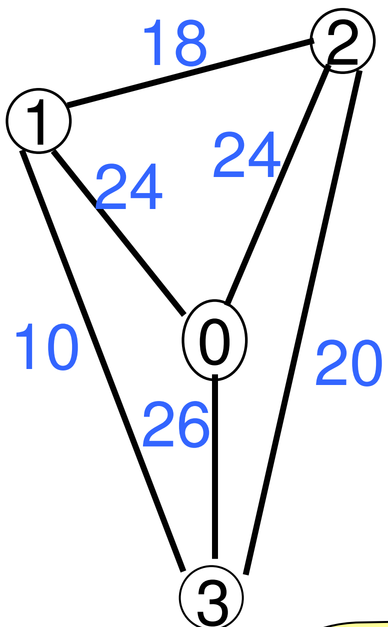
→ Bird rule does not satisfy population monotonicity

Construct & Charge rules

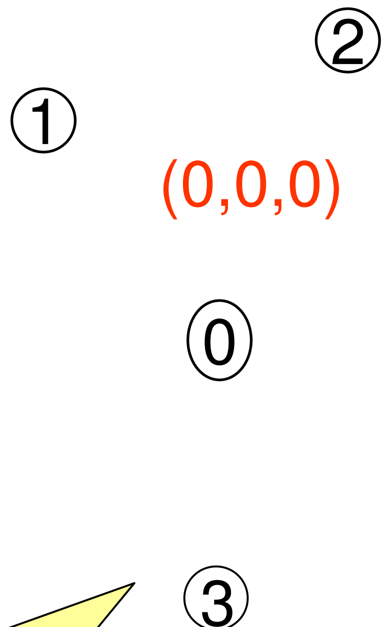
Are based on the following general cost allocation protocol:

- As soon as a link is constructed in the Kruskal algorithm procedure:
  1) it must be totally charged among agents which are not yet connected with the source (*connection property*)
  2) Only agents that are on some path containing the new edge may be charged (*involvement property*)
- when the construction of a mcst is completed, each agent has been charged for a total amount of fractions equal to 1 (*total aggregation property*).

P-value: Feltkamp (1994), Branzei et al. (2004), Moretti (2008)

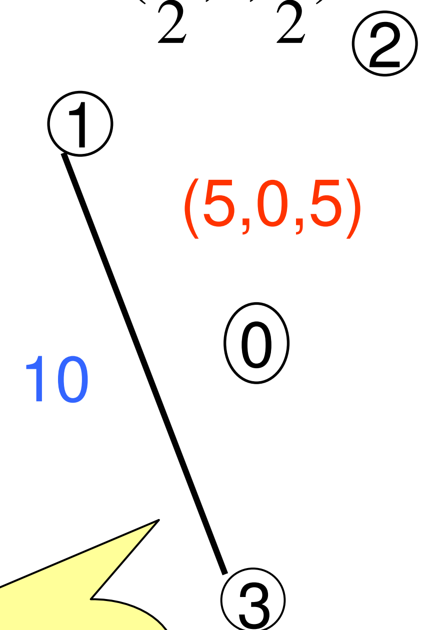


$$b^{\sigma,0} = (1,1,1)^t$$

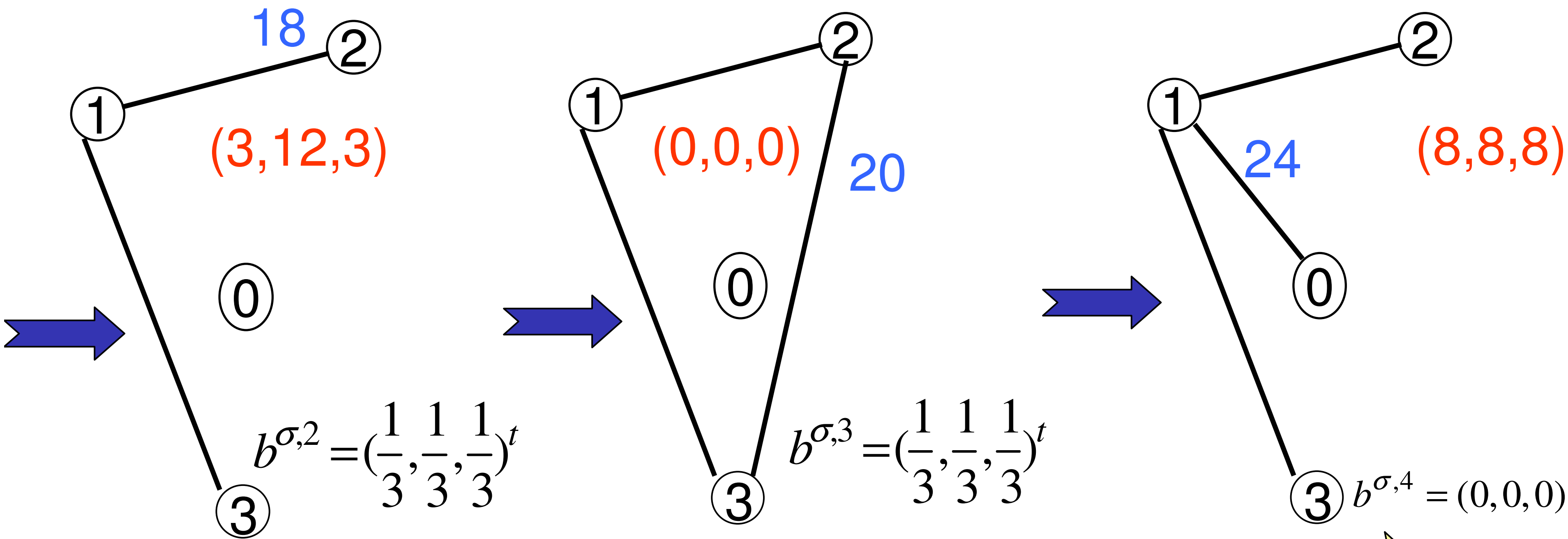


There are no edge costs to share.

$$b^{\sigma,1} = (\frac{1}{2},1,\frac{1}{2})^t$$



1 and 3 share cost 10 equally.



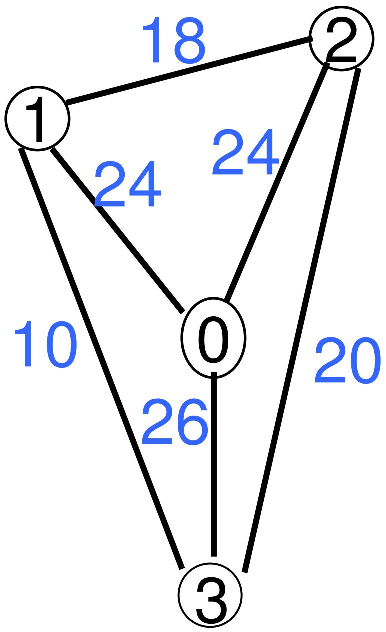
2 is connected to 1 and 3 who were already connected: 2 pays 2/3 of 18 whereas the remaining is shared equally between 1 and 3.

Oops… there is a cycle: nobody want it.

Players are connected to 0: share the total cost of the last edge (=24) equally

P-value

Make the sum of all edge-by-edge allocations:



(0, 0, 0) +
(5, 0, 5) +
(3,12,3) +
(0, 0, 0) +
(8, 8, 8) =

P-value = (16,20,16)

Algorithm to calculate the P-value

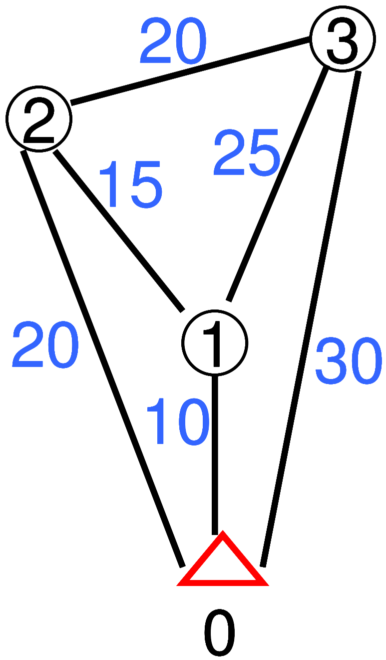
IDEA: charge the cost of an edge constructed during the Kruskal algorithm only between agents involved, keeping into account the cardinality of the connected components at that step and at the previous step of the algorithm:

- At any step of the Kruskal algorithm where a component is connected to some agents, charge the cost of that edge among these agents in the following way:
  - Proportionally to the $\text{cardinality\_current\_step}^{-1}$ if an agent is connected to a component which is connected to the source,
  - Otherwise, charge it proportionally to the difference: $\text{cardinality\_previous\_step}^{-1} - \text{cardinality\_current\_step}^{-1}$

P-value

- Always provides a unique allocation (given a mcst situation).
- It is in the core of the corresponding mcst game.
- Satisfies cost monotonicity.
- Satisfies population monotonicity.
- on a subclass of connection problems it coincides with the Shapley value of mcst games
- …

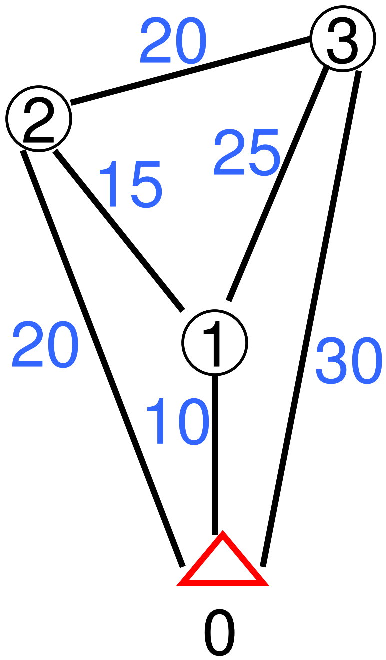
Population Monotonic Allocation Scheme (PMAS)



	1	2	3	
{1,2,3}	10	15	20	c({1,2,3})
{1,2}	10	15	*	c({1,2})
{1,3}	10	*	25	c({1,3})
{2,3}	*	20	20	c({2,3})
{1}	10	*	*	c({1})
{2}	*	20	*	c({2})
{3}	*	*	30	c({3})

$a_{S,i} \geq a_{T,i}$ for all $S,T \in 2^N$ and $i \in N$ with $i \in S \subseteq T$

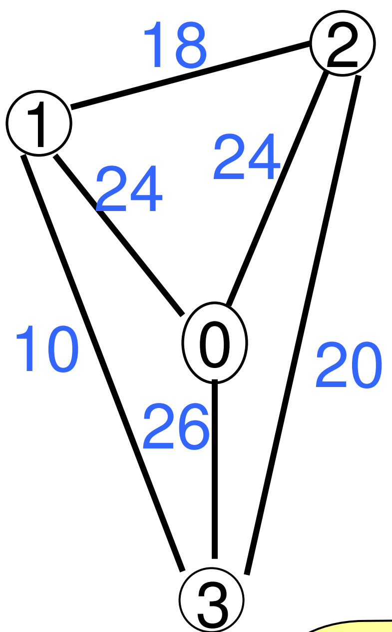
Population Monotonic Allocation Scheme (PMAS)



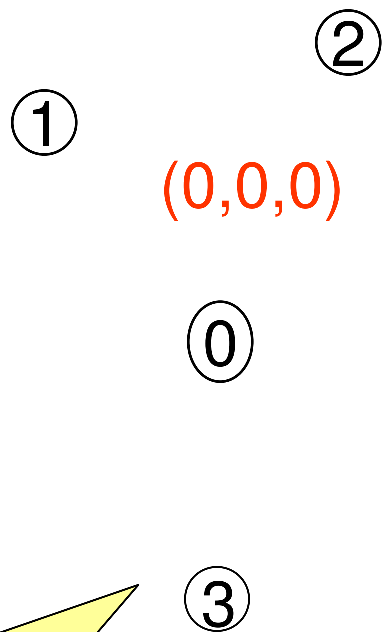
	1	2	3
{1,2,3}	10	15	20
⊆	≤	15	≤
{1,3}	10	*	25
{2,3}	*	20	20
{1}	10	*	*
{2}	*	20	*
{3}	*	*	30

$a_{S,i} \geq a_{T,i}$ for all $S,T \in 2^N$ and $i \in N$ with $i \in S \subseteq T$

Proportional rule: (Feltkamp (1994))

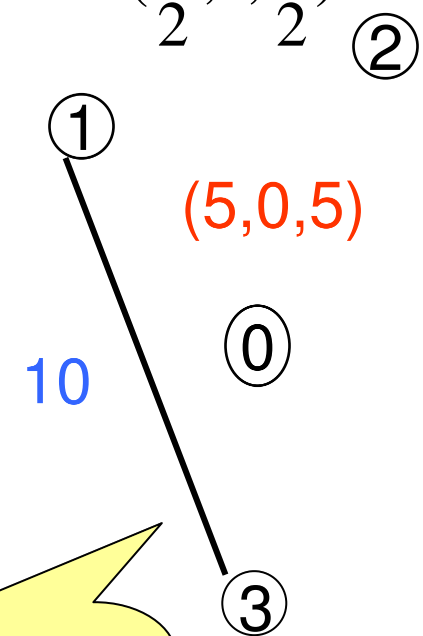


$$b^{\sigma,0} = (1,1,1)^t$$

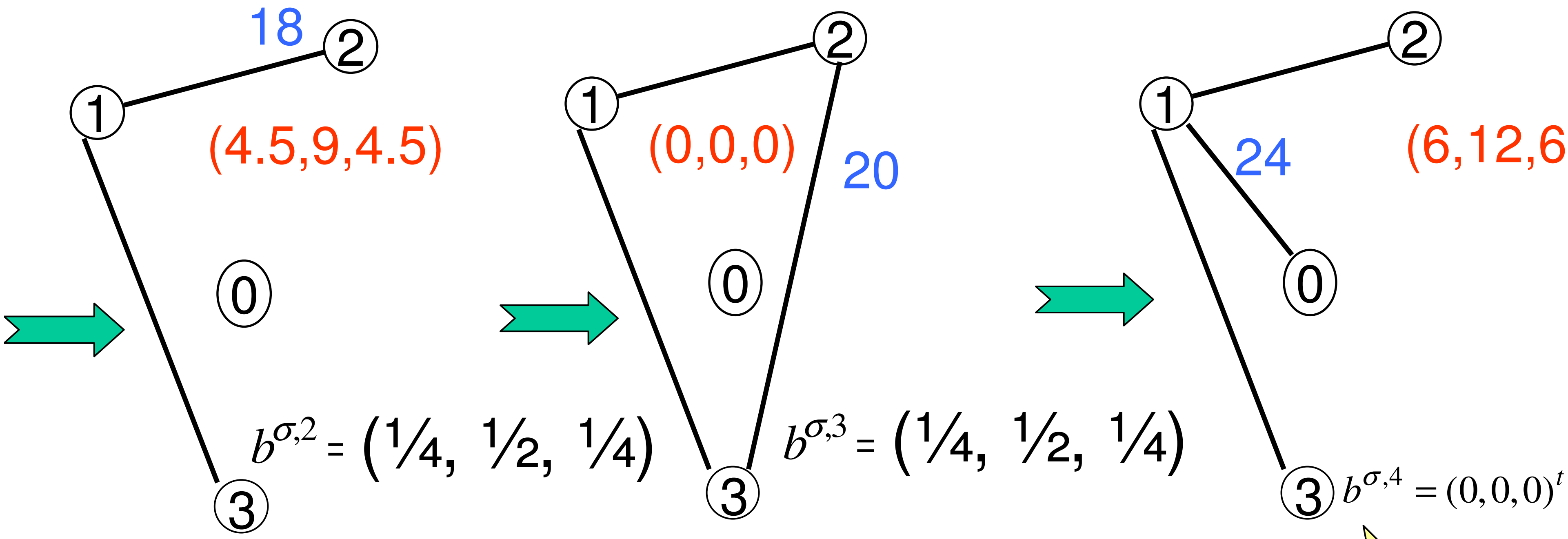


There are no edge costs to share.

$$b^{\sigma,1} = (\frac{1}{2},1,\frac{1}{2})^t$$



1 and 3 share cost 10 equally.



2 is connected to 1 and 3 who were already connected: 2 pays 1/2 of 18 whereas the remaining is shared equally between 1 and 3.

Oops… there is cycle: nobody want it.

Players are connected to 0 and pay the remaining obligations

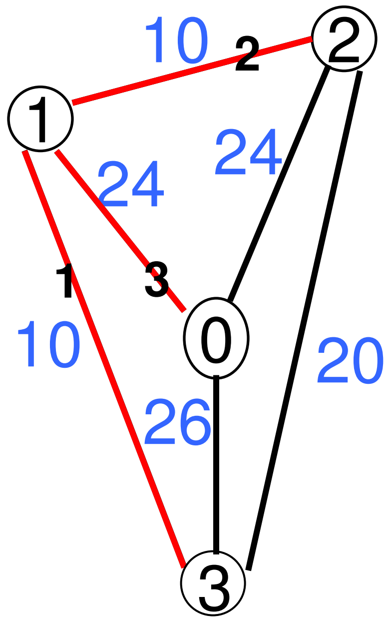
The proportional rule is a Construct & Charge rule

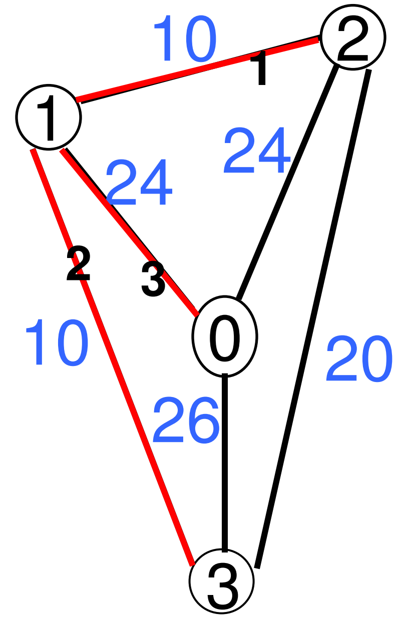
Construct & Charge rules are based on the following general cost allocation protocol:

As soon as a link is constructed in the Kruskal algorithm procedure:

- it must be totally charged among agents which are not yet connected with the source (*connection property*)
- Only agents that are on some path containing the new edge may be charged (*involvement property*)

when the construction of a mcst is completed, each agent has been charged for a total amount of fractions equal to 1 (*total aggregation property*).

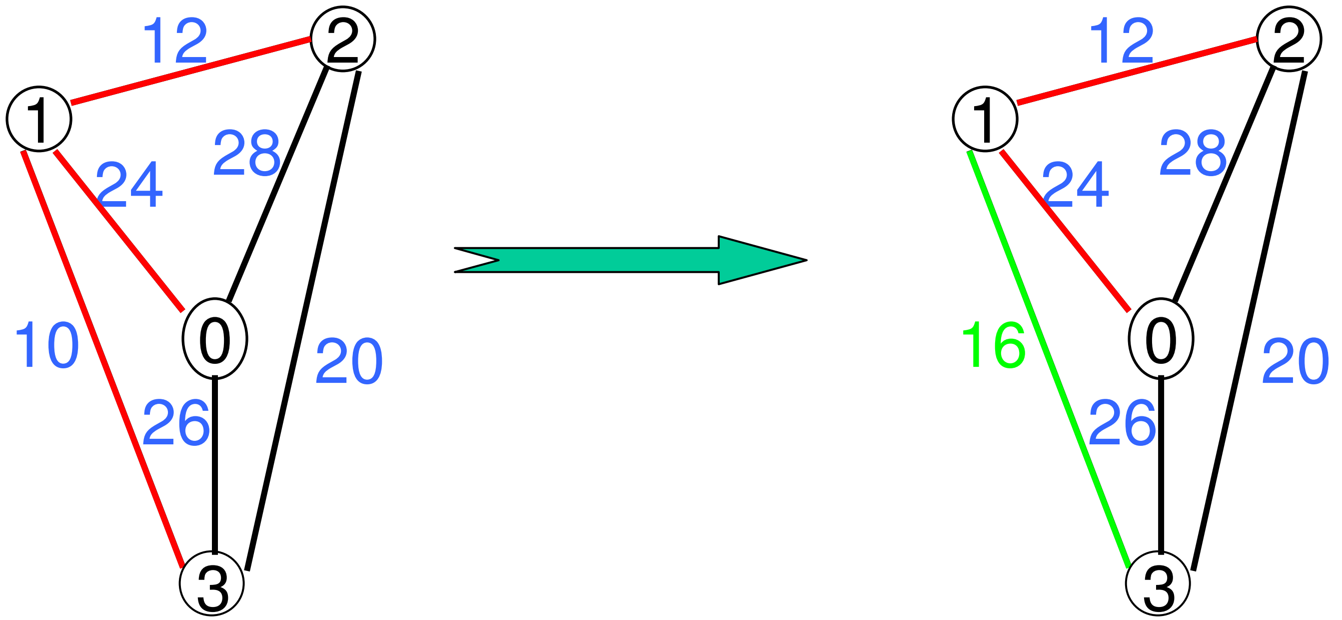




Allocation provided by the proportional rule according to this ordering is (13.5, 17 ,13.5)	Allocation provided by the proportional rule according to this ordering is(13.5, 13.5, 17)

Both allocations are in the core of the corresponding mcst game.

Cost monotonicity: Proportional rule



Proportional rule: (14, 18 ,14) Proportional rule: (16, 16 ,20)

→ The Proportional rule is not cost monotonic.

Axiomatic characterization of the P-value

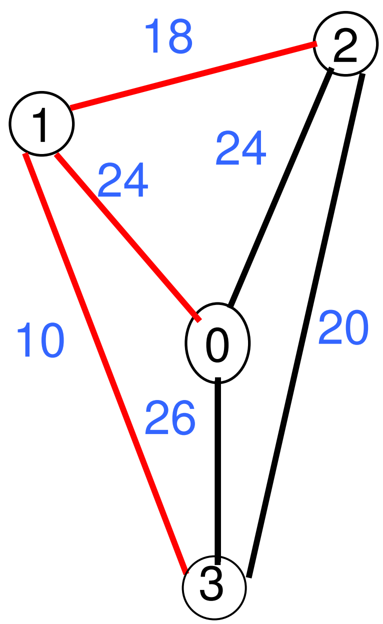
A solution for mcst situations $F : \mathcal{W}^{N'} \rightarrow \Re^N$

Property 1. The solution F is *efficient* (EFF) if for each $w \in \mathcal{W}^{N'}$

$$\sum_{i \in N} F_i(w) = w(\Gamma),$$

where Γ is a minimum cost spanning network on N'.

Example:



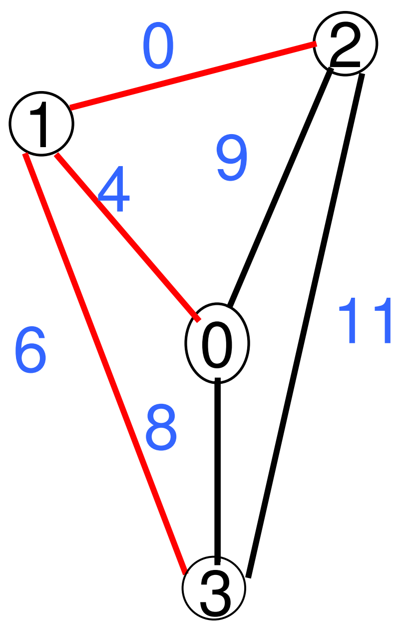
w(Γ)=52

$$P(w) = M^{\sigma} w^{\sigma} = (16, 20, 16)^t$$

Property 2. The solution F has the *Equal Treatment* (ET) property if for each $w \in \mathcal{W}^{N'}$ and for each $i,j \in N$ with $C_i(w) = C_j(w)$

$$F_i(w) = F_j(w).$$

Example:

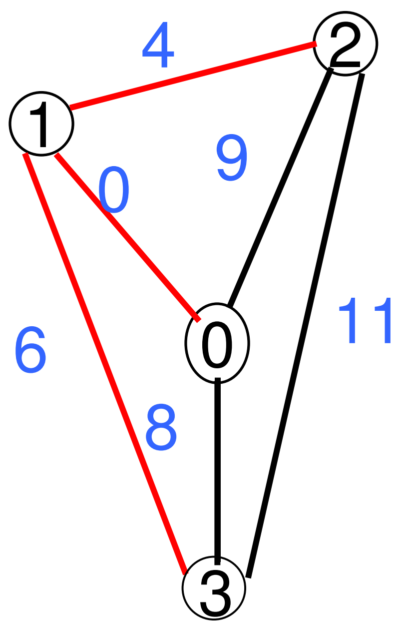


$P(w)=(2,2,6)^t$

Property 3. The solution F has the *Upper Bounded Contribution* (UBC) property if for each $w \in \mathcal{W}^{N'}$ and every (w,N')-component $C \neq \{0\}$

$$\sum_{i \in C \setminus \{0\}} F_i(w) \leq \min_{i \in C \setminus \{0\}} w(\{i,0\}).$$

Example:

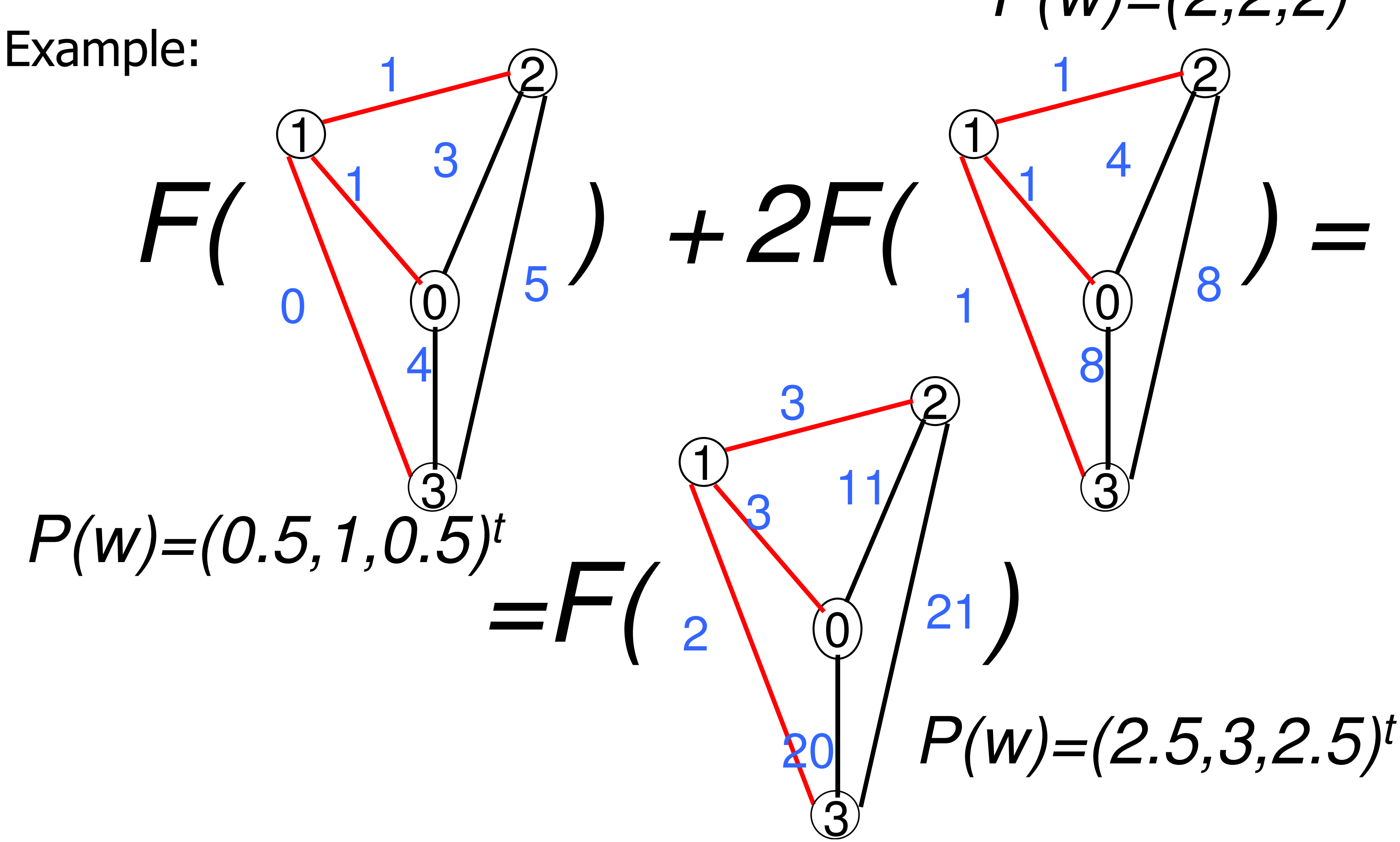


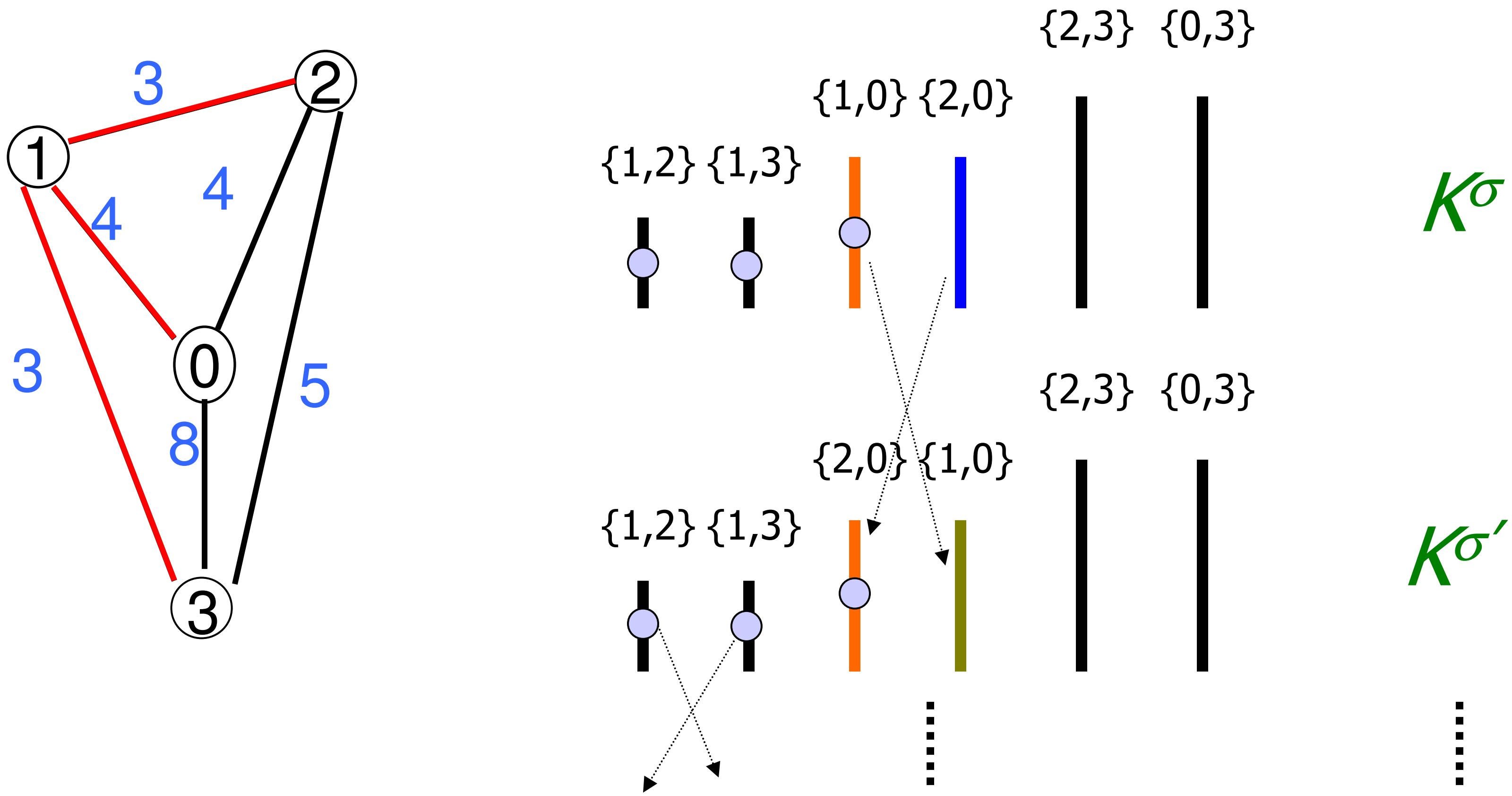
$P(w)=(0,4,6)^t$

Note that 1 is dummy in the corresponding mcst game

Property 4. The solution F has the *Cone-wise Positive Linearity* (CPL) property if for each $\sigma \in \Sigma_{E_{N'}}$, for each pair of mcst situations $w, \hat{w} \in K^{\sigma}$ and for each pair $\alpha, \hat{\alpha} \geq 0$, we have

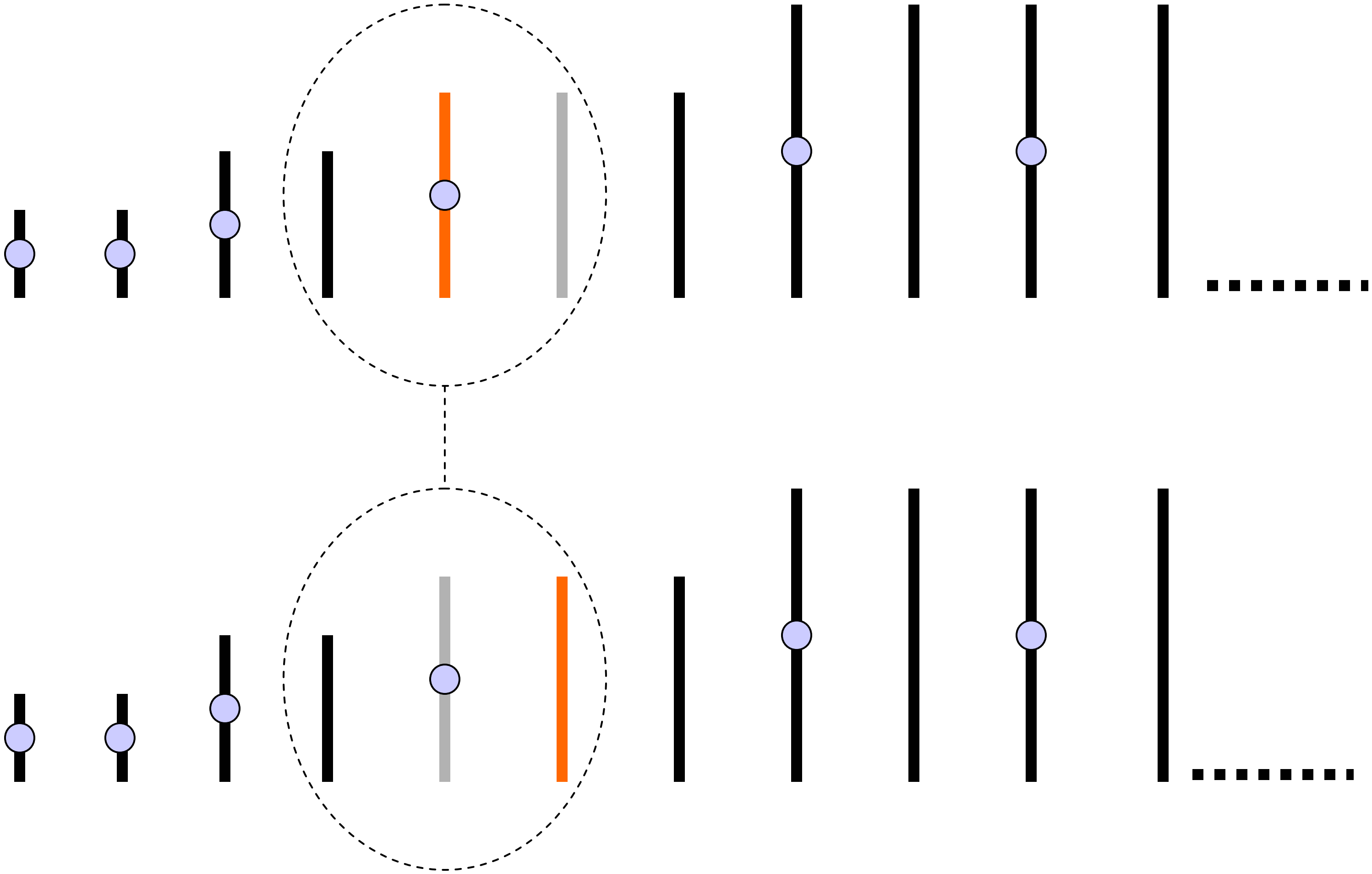
$$F(\alpha w + \hat{\alpha}\hat{w}) = \alpha F(w) + \hat{\alpha} F(\hat{w}).$$





3
4
4
3
8
5
1
2
0
3
{2,3} {0,3}
{1,0} {2,0}
{1,2} {1,3}
K^{σ}
{2,3} {0,3}
{2,0} {1,0}
{1,2} {1,3}
$K^{\sigma'}$

Proposition 1. *If* $w \in K^{\sigma} \cap K^{\sigma'}$ *with* $\sigma, \sigma' \in \Sigma_{E_{N'}}$ *, then* $P^{\sigma}(w) = P^{\sigma'}(w)$.



Definition 3. The P-value is the map $P : \mathcal{W}^{N'} \rightarrow \Re^{N}$, defined by

$$P(w) = P^{\sigma}(w) = M^{\sigma} w^{\sigma}$$

for each $w \in \mathcal{W}^{N'}$ and $\sigma \in \Sigma_{E_{N'}}$ such that $w \in K^{\sigma}$.

Theorem 1. *The P-value is the unique solution which satisfies the properties EFF, ET, UBC and CPL on the class $\mathcal{W}^{N'}$ of mcst situations.*

- It is possible to prove that the P-value satisfies the four properties EFF, ET, UBC and CPL.

- To prove the uniqueness consider a solution for mcst situation F which satisfies EFF, ET, UBC and CPL:

  - first look at the **simple mcst situations** (0-1 cost of edges): on such simple situation, EFF, ET and UBC imply F=P-value;

  - it is possible to decompose each mcst situation as a linear combination of simple mcst problems;

  - by CPL it follows that the F=P-value on each mcst situation.