Computational Issues in Simple and Influence Games

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1 Definitions, games and problems

2 IsStrong and IsProper

3 IsWeighted

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Framework

- Topics
  - Coalitional Game Theory
  - Decision/Voting/Social Choice Theory
  - Social Network Analysis
  - Algorithms and Complexity

- Models
  - Simple Games
  - Directed Graphs and Collective Choice Models

- Focus
  - Subfamilies of simple games
  - Complexity study of some properties of simple games.
Simple Games

- Simple Games (Taylor & Zwicker, 1999)
Simple Games (Taylor & Zwicker, 1999)

A simple game is a pair \((N, \mathcal{W})\):

- \(N\) is a set of players,
- \(\mathcal{W} \subseteq \mathcal{P}(N)\) is a monotone set of winning coalitions,
- \(\mathcal{L} = \mathcal{P}(N) \setminus \mathcal{W}\) is the set of losing coalitions.
Simple Games

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  A **simple game** is a pair \((N, \mathcal{W})\):  
  
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  - \(\mathcal{L} = \mathcal{P}(N) \setminus \mathcal{W}\) is the set of *losing coalitions*. 

- Members of \(N = \{1, \ldots, n\}\) are called **players** or **voters**. 
  
  Any set of voters is called a **coalition**  
  - \(N\) is the **grand coalition** 
  - \(\emptyset\) is the **null coalition** 
  - the subsets of \(N\) that are in \(\mathcal{W}\) are the **winning coalitions** 
  - A subset of \(N\) that is not in \(\mathcal{W}\) is a **losing coalition**.
Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players $N$:

- **winning coalitions** $\mathcal{W}$.
- **losing coalitions** $\mathcal{L}$.
- **minimal winning coalitions** $\mathcal{W}^m$
  \[ \mathcal{W}^m = \{ X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X \} \]
- **maximal losing coalitions** $\mathcal{L}^M$
  \[ \mathcal{L}^M = \{ X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq L \} \]

This provides us with many representation forms for simple games. We concentrate on the explicit representation of those set families.
Voting Games

Weighted voting games (WVG)

A simple game for which there exists a quota $q$ and it is possible to assign to each $i \in N$ a weight $w_i$, so that $X \in W$ iff $\sum_{i \in X} w_i \geq q$.

WVG can be represented by a tuple of integers $(q; w_1, \ldots, w_n)$. Any weighted game admits such an integer realization, [Carreras and Freixas, Math. Soc. Sci., 1996]
Voting Games

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Decision is taken without interplay of the participants
Influence Games: influence spreading model

An influence graph is a tuple $(G, f)$, where:

$G = (V, E)$ is a labeled and directed graph, and $f: V \to \mathbb{N}$ is a labeling function that quantify how influenceable each node, player or agent is.
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  From an *initial activation* \( X \subseteq V \), activate every node \( u \) having at least \( f(u) \) predecessors in \( X \).
  Repeat until no more nodes are activated.

\[
X = \{a\}
\]

\[
\begin{array}{ccc}
a & 1 & \rightarrow & 1 & \rightarrow & b \\
\downarrow & & & & & \\
1 & \rightarrow & 2 & \rightarrow & d \\
\end{array}
\]
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```
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```

```
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```

```
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>
```

```
1 -- a -- 2
|     |     |
|     |     |
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```
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  From an *initial activation* $X \subseteq V$, activate every node $u$ having at least $f(u)$ predecessors in $X$.
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$$F^1(X) = \{a, c\}$$
Influence Games: influence spreading model

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  From an initial activation $X \subseteq V$, activate every node $u$ having at least $f(u)$ predecessors in $X$.
  Repeat until no more nodes are activated.

---

**Diagram:**

```
1  a  1  b
\downarrow \quad \quad \quad \quad \downarrow
1  c  2  d
```

$F^2(X) = \{a, c, d\}$
Influence Games: influence spreading model

- **Spread of Influence** (Linear threshold model: Chen, 2009; ....)
  From an *initial activation* \( X \subseteq V \),
  activate every node \( u \) having at least \( f(u) \) predecessors in \( X \).
  Repeat until no more nodes are activated.
  The final set of activated nodes \( F(X) \) is the spread of influence from \( X \).

\[
\begin{align*}
F^2(X) &= \{a, c, d\} \\
F^1(X) &= \{a, c\} \\
F^0(X) &= \{a\}
\end{align*}
\]
Influence Games: influence spreading model

- **Spread of Influence** (Linear threshold model: Chen, 2009; ....)
  
  From an *initial activation* $X \subseteq V$, activate every node $u$ having at least $f(u)$ predecessors in $X$.
  
  Repeat until no more nodes are activated.
  
  The final set of activated nodes $F(X)$ is the spread of influence from $X$.
  
  $F(X)$ is polynomial time computable.
An influence game is a tuple \((G, f, q, N)\), where:

- \((G, f)\) is an influence graph,
- \(N \subseteq V(G)\) is the set of players, and
- \(q > 0\) is an integer, the quota.

\(X \subseteq V(G)\) is winning iff \(|F(X)| \geq q\).

\(F\) is monotonic, for any \(X \subseteq N\) and \(i \in N\), if \(|F(X)| \geq q\) then \(|F(X \cup \{i\})| \geq q\), and if \(|F(X)| < q\) then \(|F(X \setminus \{i\})| < q\).

Influence games are simple games. Participants can being influenced to adopt a new trend but have negative "initial" disposition.
Influence Games

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  - Influence games are simple games.

Participants can be influenced to adopt a new trend but have negative "initial" disposition.
Input representations

- Simple Games
  \((N, \mathcal{W}), (N, \mathcal{W}^m), (N, \mathcal{L}), (N, \mathcal{L}^M)\)

- Influence games
  \((G, w, f, q, N)\)

- Weighted voting games
  \((q; w_1, \ldots, w_n)\)

All numbers are integers
Problems on simple games

In general we state a property \( P \), for simple games, and consider the associated decision problem which has the form:

**Name:** IsP  
**Input:** A simple/influence/weighted voting game \( \Gamma \)  
**Question:** Does \( \Gamma \) satisfy property \( P \)?
A simple game \((N, W)\) is

- **strong** if \(S \notin \mathcal{W}\) implies \(N \setminus S \in \mathcal{W}\).
- **proper** if \(S \in \mathcal{W}\) implies \(N \setminus S \notin \mathcal{W}\).

- a **weighted voting game**.
- an **influence game**.
IsStrong: Simple Games

\( \Gamma \) is **strong** if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)
IsStrong: Simple Games

\( \Gamma \) is strong if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

**Theorem**

The IsStrong problem, when \( \Gamma \) is given in explicit losing or maximal losing form can be solved in polynomial time.
IsStrong: Simple Games

Γ is strong if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

**Theorem**

The IsStrong problem, when \( \Gamma \) is given in explicit losing or maximal losing form can be solved in polynomial time.

- First observe that, given a family of subsets \( F \), we can check, for any set in \( F \), whether its complement is not in \( F \) in polynomial time.
- Therefore, the IsStrong problem, when the input is given in explicit losing form is polynomial time solvable.
IsStrong: Simple Games loosing forms

\( \Gamma \) is strong if \( S \notin W \) implies \( N \setminus S \in W \)

- A simple game is not strong iff

\[ \exists S \subseteq N : S \in L \land N \setminus S \in L \]
IsStrong: Simple Games loosing forms

$\Gamma$ is strong if $S \notin W$ implies $N \setminus S \in W$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in L \land N \setminus S \in L$$

which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in L^M : S \subseteq L_1 \land N \setminus S \subseteq L_2$$
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- which is equivalent to there are two maximal losing coalitions \( L_1 \) and \( L_2 \) such that \( L_1 \cup L_2 = N \).
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- This can be checked in polynomial time, given \( L^M \).
IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

**Theorem**

The IsStrong problem is coNP-complete when the input game is given in explicit minimal winning form.
**IsStrong: minimal winning forms**

\( \Gamma \) is strong if \( S \not\in \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

**Theorem**

The \textbf{IsStrong} problem is coNP-complete when the input game is given in explicit minimal winning form.

- The property can be expressed as

\[
\forall S \ [(S \in \mathcal{W}) \text{ or } (S \not\in \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]
\]

- Observe that the property \( S \in \mathcal{W} \) can be checked in polynomial time given \( S \) and \( \mathcal{W}^m \).

- Thus the problem belongs to coNP.
We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.

An instance of the set splitting problem is a collection $C$ of subsets of a finite set $N$. The question is whether it is possible to partition $N$ into two subsets $P$ and $N \setminus P$ such that no subset in $C$ is entirely contained in either $P$ or $N \setminus P$. 

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IsStrong: minimal winning forms

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- We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P$$
IsStrong: minimal winning forms

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- We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P$$

We associate to a set splitting instance $(N, C)$ the simple game in explicit minimal winning form $(N, C^m)$. 
IsStrong: minimal winning forms

- $C^m$ can be computed in polynomial time, given $C$. Why?
IsStrong: minimal winning forms

- $C^m$ can be computed in polynomial time, given $C$. Why?
- Now assume that $P \subseteq N$ satisfies

\[ \forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P \]
IsStrong: minimal winning forms

- $C^m$ can be computed in polynomial time, given $C$. Why?
- Now assume that $P \subseteq N$ satisfies
  \[
  \forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P
  \]
  This means that $P$ and $N \setminus P$ are losing coalitions in the game $(N, C^m)$. 
IsStrong: minimal winning forms

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- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$. 
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- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in $C$ contains a set in $C^m$. 

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Computational Issues in Simple and Influence Games
IsStrong: minimal winning forms

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- This means that $P$ and $N \setminus P$ are losing coalitions in the game $(N, C^m)$.
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in $C$ contains a set in $C^m$.
- Therefore, $(N, C)$ has a set splitting iff $(N, C^m)$ is not proper.
IsProper: winning forms

Γ is proper if \( S \in \mathcal{W} \) implies \( N \setminus S \notin \mathcal{W} \).

**Theorem**

The IsProper problem, when the game is given in explicit winning or minimal winning form, can be solved in polynomial time.
IsProper: winning forms

Γ is proper if \( S \in \mathcal{W} \) implies \( N \setminus S \notin \mathcal{W} \).

**Theorem**

The IsProper problem, when the game is given in explicit winning or minimal winning form, can be solved in polynomial time.

- As before, given a family of subsets \( F \), we can check, for any set in \( F \), whether its complement is not in \( F \) in polynomial time. Therefore, Taking into account the definitions, the IsProper problem is polynomial time solvable.
IsProper: winning forms

- \( \Gamma \) is not proper iff

\[
\exists S \subseteq N : S \in W \land N \setminus S \in W
\]
IsProper: winning forms

- $\Gamma$ is not proper iff

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- which is equivalent to

$$\exists S \subseteq N : \exists W_1, W_2 \in W^m : W_1 \subseteq S \land W_2 \subseteq N \setminus S.$$
IsProper: winning forms

- \( \Gamma \) is not proper iff

\[ \exists S \subseteq N : S \in W \wedge N \setminus S \in W \]

- which is equivalent to

\[ \exists S \subseteq N : \exists W_1, W_2 \in W^m : W_1 \subseteq S \wedge W_2 \subseteq N \setminus S. \]

- equivalent to there are two minimal winning coalitions \( W_1 \) and \( W_2 \) such that \( W_1 \cap W_2 = \emptyset. \)
IsProper: winning forms

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- equivalent to there are two minimal winning coalitions $W_1$ and $W_2$ such that $W_1 \cap W_2 = \emptyset$.

- Which can be checked in polynomial time when $W_1^m$ is given.
**IsProper: maximal losing form**

\( \Gamma \) is proper if \( S \in \mathcal{W} \) implies \( N \setminus S \notin \mathcal{W} \).

**Theorem**

The **IsProper** problem is coNP-complete when the input game is given in extensive maximal losing form.
IsProper: maximal losing form

Γ is proper if \( S \in \mathcal{W} \) implies \( N \setminus S \notin \mathcal{W} \).

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The IsProper problem is coNP-complete when the input game is given in extensive maximal losing form.

- A game is not proper iff

  \[
  \exists S \subseteq N : S \notin L \land N \setminus S \notin L
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- which is equivalent to

  \[
  \exists S \subseteq N : \forall T_1, T_2 \in L^M : S \not\subseteq T_1 \land N \setminus S \not\subseteq T_2
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**IsProper: maximal losing form**

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

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- A game is not proper iff
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- which is equivalent to
  \[ \exists S \subseteq N : \forall T_1, T_2 \in L^M : S \not\subset T_1 \land N \setminus S \not\subset T_2 \]
- Therefore IsProper belongs to coNP.
To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.
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- If a family $C$ of subsets of $N$ is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, W^m)$, in minimal winning form, we provide its dual game $\Gamma' = (N, \{N \setminus L : L \in W^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.
IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

- If a family $C$ of subsets of $N$ is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, W^m)$, in minimal winning form, we provide its dual game $\Gamma' = (N, \{N \setminus L : L \in W^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.
- Besides, a game is strong iff its dual is proper.
Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem Partition:

Name: Partition

Input: $n$ integer values, $x_1, \ldots, x_n$

Question: Is there $S \subseteq \{1, \ldots, n\}$ for which $\sum_{i \in S} x_i = \sum_{i / \in S} x_i$.

Observe that, for any instance of the Partition problem in which the sum of the $n$ input numbers is odd, the answer must be no.
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$$

Observe that, for any instance of the **Partition** problem in which the sum of the $n$ input numbers is odd, the answer must be **NO**.
Theorem

The $\text{IsStrong}$ and the $\text{IsProper}$ problems, when the input is described by an integer realization of a weighted game $(q; w)$, are $\text{coNP}$-complete.
Theorem

The **IsStrong** and the **IsProper** problems, when the input is described by an integer realization of a weighted game \((q; w)\), are coNP-complete.

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
Weighted voting games

Theorem

The \textit{IsStrong} and the \textit{IsProper} problems, when the input is described by an integer realization of a weighted game \((q; w)\), are coNP-complete.

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
- Observe that the weighted game with integer representation \((2; 1, 1, 1)\) is both proper and strong.
Hardness

We transform an instance $x = (x_1, \ldots, x_n)$ of $\textsc{Partition}$ into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$
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(q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\
(2; 1, 1, 1) & \text{otherwise.}
\end{cases}$$

- Function $f$ can be computed in polynomial time provided $q$ does.
We transform an instance $x = (x_1, \ldots, x_n)$ of Partition into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even}, \\ (2; 1, 1, 1) & \text{otherwise}. \end{cases}$$

- Function $f$ can be computed in polynomial time provided $q$ does.
- Independently of $q$, when $x_1 + \cdots + x_n$ is odd, $x$ is a NO input for partition, but $f(x)$ is a YES instance of IsStrong or IsProper.
Assume that $x_1 + \cdots + x_n$ is even.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$. 
Assume that $x_1 + \cdots + x_n$ is even.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both $S$ and $N \setminus S$ are losing coalitions and $f(x)$ is not strong.
IsStrong

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- If $S$ and $N \setminus S$ are losing coalitions in $f(x)$.
  If $\sum_{i \in S} x_i < s$ then $\sum_{i \notin S} x_i \geq s + 1$, $N \setminus S$ should be winning.
  Thus $\sum_{i \in S} x_i = \sum_{i \notin S} x_i = s$, and there exists a partition of $x$. 
Assume that $x_1 + \cdots + x_n$ is even.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s$. 
Assume that $x_1 + \cdots + x_n$ is even. Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$. Set $q(x) = s$.

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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both $S$ and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.
- When $f(x)$ is not proper

$$\exists S \subset N : \sum_{i \in S} x_i \geq s \land \sum_{i \notin S} x_i \geq s,$$

and thus $\sum_{i \in S} x_i = s$. 
Influence games: $\Gamma(G)$

Let's consider a particular type of influence games.

**Definition**

Given an undirected graph $G = (V, E)$, $\Gamma(G)$ is the influence game $(G, f, |V|, V)$ where, for any $v \in V$, $f(v) = d_G(v)$. 

GTA School, Campione d'Italia
Influence games: $\Gamma(G)$

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**Definition**

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Recall that a set $S \subseteq V$ is a vertex cover of a graph $G$ if and only if, for any edge $(u, v) \in E$, $u$ or $v$ (or both) belong to $S$. From the definitions we get the following result.
Recall that a set $S \subseteq V$ is a vertex cover of a graph $G$ if and only if, for any edge $(u, v) \in E$, $u$ or $v$ (or both) belong to $S$. From the definitions we get the following result.

**Lemma**

Let $G$ be an undirected graph. $X$ is winning in $\Gamma(G)$ if and only if $X$ is a vertex cover of $G$. Furthermore, the influence game $\Gamma(G)$ can be obtained in polynomial time, given a description of $G$. 
Isproper and IsStrong

Theorem

For unweighted influence games IsProper and IsStrong are coNP-complete.
Isproper and IsStrong

**Theorem**

*For unweighted influence games IsProper and IsStrong are coNP-complete.*

- Membership in coNP follows from the definitions.
- To get the hardness results, we provide reductions from problems related to Vertex Cover.
- Assume that a graph $G$ has $n$ vertices and $m$ edges.
Let $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Set $\alpha = m + n + 4$ and consider the influence graph $(G_1, f_1)$:
\[ \Delta_1(G, k) \]

\[ \alpha = m + n + 4, \ G_1 = (V_1, E_1) \text{ and } f_1 \text{ define } (G_1, f_1) \]

- \( V_1 = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\} \).
- \( E_1 \) has edge \((z, y)\) and
  - \((e, v_i), (e, v_j), (e, y)\), for \(e = (v_i, v_j) \in E\)
  - \((v_i, x)\), for \(1 \leq i \leq n\) and \((x, s_j), (y, s_j)\), for \(1 \leq j \leq \alpha\).
- The labeling function \(f_1\) is:
  - \(f_1(v_i) = m + 2, 1 \leq i \leq n; f_1(e_j) = 1, 1 \leq j \leq m;\)
  - \(f_1(s_\ell) = 1, 1 \leq \ell \leq \alpha; \text{ and}\)
  - \(f_1(z) = 2, f_1(x) = k + 1, f_1(y) = m + 1.\)
\[ \Delta_1(G, k) \]

\[ \alpha = m + n + 4, \quad G_1 = (V_1, E_1) \] and \( f_1 \) define \((G_1, f_1)\)

- \( V_1 = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\} \).
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  - \((e, v_i), (e, v_j), (e, y)\), for \( e = (v_i, v_j) \in E \)
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- The labeling function \( f_1 \) is:
  - \( f_1(v_i) = m + 2, \quad 1 \leq i \leq n; \)
  - \( f_1(e_j) = 1, \quad 1 \leq j \leq m; \)
  - \( f_1(s_\ell) = 1, \quad 1 \leq \ell \leq \alpha; \) and
  - \( f_1(z) = 2, \quad f_1(x) = k + 1, \quad f_1(y) = m + 1. \)

\[ \Delta_1(G, k) = (G_1, f_1, q_1, N_1) \] where \( q_1 = \alpha \) and \( N_1 = \{v_1, \ldots, v_n, z\} \).
To prove hardness of IsProper, we provide a reduction from the following variation of the Vertex Cover problem:

**Name:** Half vertex cover

**Input:** Given a graph with an odd number of vertices $n$.

**Question:** Is there a vertex cover with size $\leq (n - 1)/2$?

which is also NP-complete.
Let $G$ be an instance of \textsc{Half vertex cover} with $n = 2k + 1$ vertices, for some value $k \geq 1$.

Consider the influence game $\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$.
Let $G$ be an instance of \textsc{Half Vertex Cover} with $n = 2k + 1$ vertices, for some value $k \geq 1$.

Consider the influence game $\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$

Trivially $\Delta_1(G, k)$ can be obtained in polynomial time,
If $G$ has a vertex cover $X$ with $|X| \leq k$,

- $F(X \cup \{z\}) \geq q_1$.
- But as $n + 1 - |X \cup \{z\}| > k$, $F(N \setminus (X \cup \{z\})) \geq q_1$.
- Hence $\Delta_1(G, k)$ is not proper.
When all the vertex covers of $G$ have more than $k$ vertices,

- to have $F(Y) \geq q_1$ we need $|Y \cap \{v_1, \ldots, v_n\}| > k$, i.e., $|Y \cap \{v_1, \ldots, v_n\}| \geq k + 1$.

- For a $Y$, with $F(Y) \geq q_1$ we have two cases:
  - $z \in Y$, then $N \setminus Y \subseteq \{v_1, \ldots, v_n\}$ and $|N \setminus Y| \leq n - k - 1 = k$. Thus, $F(N \setminus Y) < q_1$.
  - $z \notin Y$, then $|N \setminus (Y \cup \{z\})| \leq k$ and $F(N \setminus Y) < q_1$

So, we conclude that $\Delta_1(G, k)$ is proper.

Thus the IsProper problem is coNP-hard.
To finish the proof we show hardness for the **IsStrong** problem. We need another problem.

**Name:** Half Independent set

**Input:** Given a graph with an even number of vertices $n$.

**Question:** Is there an independent set with size $\geq n/2$?

The Half Independent set trivially belongs to NP. Hardness follows from a simple reduction from Half Vertex Cover.
Now we show that the complement of the **Half Independent Set** problem can be reduced to the **IsStrong** problem. We define first an influence graph \((G_3, f_3)\):

![Diagram](image-url)
We associate to an input to \textsc{Half Independent Set} the game

$$\Delta_3(G) = (G_3, f_3, n + m + 5, N_3)$$

where $N_3 = V \cup \{z\}$ and $(G_3, f_3)$ is the influence graph described before.
When $G$ has an independent set with size at least $n/2$, $G$ also has an independent set $X$ with $|X| = n/2$.

It is easy to see that both $X \cup \{z\}$ and its complement are losing coalitions in $\Delta_3(G)$. Therefore, $\Delta_3(G)$ is not strong.
• When $G$ has an independent set with size at least $n/2$, $G$ also has an independent set $X$ with $|X| = n/2$.

• It is easy to see that both $X \cup \{z\}$ and its complement are losing coalitions in $\Delta_3(G)$. Therefore, $\Delta_3(G)$ is not strong.

• When all the independent sets in $G$ have less than $n/2$ vertices.
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When all the independent sets in $G$ have less than $n/2$ vertices.

- When $|X \cap V| < n/2$, its complement has at least $n/2 + 1$ elements in $V$ and thus it is winning.
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- When $|X \cap V| < n/2$, its complement has at least $n/2 + 1$ elements in $V$ and thus it is winning.
- When $|X \cap V| > n/2$, $X$ wins and we have to consider only those teams with $|X \cap V| = n/2$.
- But now neither $X \cap V$ nor $V \setminus (X \cap V)$ are independent sets. Then, $X$ or $N \setminus X$ must contain $z$ and is winning while its complement is losing.
When $G$ has an independent set with size at least $n/2$, $G$ also has an independent set $X$ with $|X| = n/2$.

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- But now neither $X \cap V$ nor $V \setminus (X \cap V)$ are independent sets. Then, $X$ or $N \setminus X$ must contain $z$ and is winning while its complement is losing.

So, $\Delta_3(G)$ is strong.
Subfamilies of Influence Games

Maximum Influence Game
\( \Gamma = (G, f, |V|, V) \) where \( f(v) = d_G(v) \), for \( v \in V \) (\( \Gamma = \Gamma(G) \))

Minimum Influence Game
\( \Gamma = (G, 1_V, q, N) \) where \( 1_V(v) = 1 \), for \( v \in V \).
Lemma

In a maximum influence game $\Gamma$ on a connected graph $G$ the following properties hold.

- $\Gamma$ is proper if and only if $G$ is either not bipartite or a singleton.
- $\Gamma$ is strong if and only if $G$ is either a star or a triangle.
Observe that in

- We know that winning coalitions of $\Gamma = \Gamma(G)$ coincide with the vertex covers of $G$.
- Recall that the complement of a vertex cover is an independent set.
Maximum Influence games: IsProper

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  Recall that the complement of a vertex cover is an independent set.
- If $G$ is a singleton $\Gamma(G)$ is proper. Otherwise,
- If $G = (V, E)$ is bipartite, let $(V_1, V_2)$ be a partition of $V$ so that $V_1$ and $V_2$ are independent sets.
  Now $V_1$ and $V_2 = N \setminus V_1$ are winning and $\Gamma$ is not proper.
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- We know that winning coalitions of $\Gamma = \Gamma(G)$ coincide with the vertex covers of $G$.
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- If $G = (V, E)$ is bipartite, let $(V_1, V_2)$ be a partition of $V$ so that $V_1$ and $V_2$ are independent sets.
  Now $V_1$ and $V_2 = N \setminus V_1$ are winning and $\Gamma$ is not proper.

- If $\Gamma$ is not proper, then the game admits two disjoint winning coalitions i.e, two disjoint vertex covers of $G$, and hence both of them must be independent sets. Thus $G$ is bipartite.
Maximum Influence games: \textit{IsStrong}

Now we prove that $\Gamma$ is not strong if and only if $G$ has at least two non-incident edges.
Maximum Influence games: IsStrong

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Maximum Influence games: IsStrong

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- A graph where all edges are incident is either a triangle or a star.
- If $G$ has at least two non-incident edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, $\{u_1, v_1\}$ and $N \setminus \{u_1, v_1\}$ are both winning and $\Gamma$ is not strong.
Maximum Influence games: IsStrong

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- A graph where all edges are incident is either a triangle or a star.
- If $G$ has at least two non-incident edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, $\{u_1, v_1\}$ and $N \setminus \{u_1, v_1\}$ are both winning and $\Gamma$ is not strong.
- When the game is not strong, there is $X$ such that both $X$ and $N \setminus X$ are losing. For this to happen there must be an edge uncovered by $X$ and another edge uncovered by $N \setminus X$. Thus $G$ must have two non-incident edges.
Minimum Influence

\[ \Gamma = (G, 1_V, q, N) \text{ where } 1_V(v) = 1 \text{ for any } v \in V. \]

- Observe that, if \( G \) is connected, the game has a trivial structure as any non-empty vertex subset of \( N \) is a successful team.
- For the disconnected case we can analyze the game with respect to a suitable weighted game.
Minimum Influence

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- Observe that, if \( G \) is connected, the game has a trivial structure as any non-empty vertex subset of \( N \) is a successful team.

- For the disconnected case we can analyze the game with respect to a suitable weighted game.

- Assume that \( G \) has \( k \) connected components, \( C_1, \ldots, C_k \). Without loss of generality, we assume that all the connected components of \( G \) have non-empty intersection with \( N \). For \( 1 \leq i \leq k \), let \( w_i = |V(C_i)| \) and \( n_i = |V(C_i) \cap N| \).
Lemma

If a winning coalition is minimal then it has at most one node in each connected component. Minimal winning coalitions are in a many-to-one correspondence with the minimal winning coalitions of the weighted game \([q; w_1, \ldots, w_k]\).
Minimum Influence: Knapsack

We consider now two problems (all numbers are integers):

**Name:** KNAPSACK  
**Input:** Given $n$ objects, for $1 \leq i \leq n$, $w_i$ and $v_i$, and $k$.  
**Question:** Find a subset $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} w_i \leq k$ and maximum $\sum_{i \in S} v_i$.

**Name:** 0-1-KNAPSACK  
**Input:** Given a finite set $U$, for each $i \in U$, a weight $w_i$, and a positive integer $k$.  
**Question:** Is there a subset $S \subseteq U$ with $\sum_{i \in S} w_i = k$?

Both problems can be solved in pseudo polynomial time: when all the weights are at most $p(n)$. 
Minimum Influence

Theorem

For unweighted influence games with minimum influence, the problems \textsc{IsProper} and \textsc{IsStrong} belong to \textit{P}.
Minimum Influence: IsProper

Let $\Gamma = (G, 1_V, q, N)$ be an unweighted influence game with minimum influence.

- For the IsProper problem it is enough to check whether there is a winning coalition whose complement is also winning and answer accordingly.
Let $\Gamma = (G, 1_V, q, N)$ be an unweighted influence game with minimum influence.

- For the \textbf{IsProper} problem it is enough to check whether there is a winning coalition whose complement is also winning and answer accordingly.

- We separate the connected components in two sets: those containing one player and those containing more than one player.

Let $A = \{ i \mid n_i = 1 \}$ and $B = \{ i \mid n_i > 1 \}$.

Let $N_A = \bigcup_{i \in A} (N \cap V(C_i))$ and $N_B = N \setminus N_A$.

Let $w_A = \sum_{i \in A} w_i$ and $w_B = w_N - w_A$. 
As all the components in $B$ have at least two vertices, we can find a set $X \subseteq N_B$ such that $|F(X)| = |F(N_B \setminus X)| = w_B$.

- If $w_B \geq q$ the game is not proper.
- If $w_B < q$ the game is proper iff the influence game $\Gamma'$ played on the graph formed by the connected components belonging to $A$ and quota $q' = q - w_B$ is proper.

Observe that $\Gamma'$ is equivalent to the weighted game with a player for each component in $i \in A$ with associated weight $w_i$ and quota $q'$.

Let $\alpha_{\text{min}}$ be the minimum $\alpha \in \{q', \ldots, w_A\}$ for which there is a set $S \subseteq A$ with $\sum_{i \in S} w_i = \alpha$.

Observe that $\Gamma'$ is proper if and only if $w_A - \alpha_{\text{min}} < q'$.

The value $\alpha_{\text{min}}$ can be computed by solving several instances of the 0-1-Knapsack with weights polynomial in $n$. 
Now we prove that the \texttt{IsStrong} problem belongs to \texttt{P}.

- Observe that in order to minimize the influence of the complement of a team $X$ it is enough to consider only those teams $X$ that contain all or none of the players in a connected component.

- Let $w_N = \sum_{i=1}^{k} w_i$, and let $\alpha_{\max}$ be the maximum $\alpha \in \{0, \ldots, q - 1\}$ for which there is a set $S \subseteq \{1, \ldots, k\}$ with $\sum_{i \in S} w_i = \alpha$.

  Note that $\alpha$ can be zero and thus $S$ can be the empty set.
Minimum Influence: IsStrong

Now we prove that the IsStrong problem belongs to P.

- Observe that in order to minimize the influence of the complement of a team \( X \) it is enough to consider only those teams \( X \) that contain all or none of the players in a connected component.

- Let \( w_N = \sum_{i=1}^{k} w_i \), and let \( \alpha_{\text{max}} \) be the maximum \( \alpha \in \{0, \ldots, q - 1\} \) for which there is a set \( S \subseteq \{1, \ldots, k\} \) with \( \sum_{i \in S} w_i = \alpha \).

- Note that \( \alpha \) can be zero and thus \( S \) can be the empty set.

- \( \Gamma \) is strong iff \( w_N - \alpha_{\text{max}} \geq q \).

- The value \( \alpha_{\text{max}} \) can be computed by solving several instances of the 0-1-Knapsack problem with weights \( \leq n \).
Definitions, games and problems

IsStrong and IsProper

IsWeighted

IsInfluence

Decision systems

Conclusions
Lemma

The IsWeighted problem can be solved in polynomial time when the input game is given in explicit winning or losing form.
The \textbf{IsWeighted} problem can be solved in polynomial time when the input game is given in explicit winning or losing form.

We can obtain $W^m$ and $L^M$ in polynomial time. Once this is done we write, in polynomial time, the LP

$$\min q$$
subject to

$$w(S) \geq q \quad \text{if } S \in W^m$$
$$w(S) < q \quad \text{if } S \in L^M$$
$$0 \leq w_i \quad \text{for all } 1 \leq i \leq n$$
$$0 \leq q$$
Lemma

The \textit{IsWeighted} problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.
Lemma

The **IsWeighted** problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.

For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the $i$'th coordinate equal to 1 if and only if $i \in C$. 
Lemma

The **IsWeighted** problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.

- For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the $i$’th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function $\Phi_{W^m}$ given by the DNF:

$$
\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\bigwedge_{i \in S} x_i)
$$
IsWeighted: Minimal and Maximal

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \iff C \text{ is winning in the game given by } (N, W^m)$$
IsWeighted: Minimal and Maximal

By construction we have the following:

\[ \Phi_{W^m}(x_C) = 1 \iff C \text{ is winning in the game given by } (N, W^m) \]

- It is well known that \( \Phi_{W^m} \) is a threshold function iff the game given by \((N, W^m)\) is weighted.
IsWeighted: Minimal and Maximal

By construction we have the following:

\[ \Phi_{W^m}(x_C) = 1 \iff C \text{ is winning in the game given by } (N, W^m) \]

- It is well known that \( \Phi_{W^m} \) is a threshold function iff the game given by \( (N, W^m) \) is weighted.
- Further \( \Phi_{W^m} \) is monotonic (i.e. positive)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.
On the other hand, we can prove a similar result given \((N, L^M)\) just taking into account that a game \(\Gamma\) is weighted iff its dual game \(\Gamma'\) is weighted.

Thus we can compute a minimal winning representation of the dual of \((N, L^M)\) in polynomial time and use the previous result.
Open

The complexity of the **IsWeighted** problem for influence graphs has not been addressed yet.
1 Definitions, games and problems
2 IsStrong and IsProper
3 IsWeighted
4 IsInfluence
5 Decision systems
6 Conclusions
Theorem

Every simple game can be represented by an influence game. Furthermore, when the simple game $\Gamma$ is given by either $(N, W)$ or $(N, W^m)$, an unweighted influence game representing $\Gamma$ can be obtained in polynomial time.
Theorem

Every simple game can be represented by an influence game. Furthermore, when the simple game $\Gamma$ is given by either $(N, W)$ or $(N, W^m)$, an unweighted influence game representing $\Gamma$ can be obtained in polynomial time.

- It is already well known that given $(N, W)$, the family $W^m$ can be obtained in polynomial time.
- Thus we assume in the following that the set of players and the set $W^m$ are given.
Define the graph $G = (V, E)$ as:
- $V$ contains $V_N = \{v_1, \ldots, v_n\}$, one vertex for player and,
- for $X \in \mathcal{W}^m$, a set $V_X$ with $n + 1 - |X|$ nodes.
- We connect vertex $v_i$ with all the vertices in $V_X$ whenever $i \in X$.

For any $1 \leq i \leq n$, $f(v_i) = 1$ and, for any $X \in \mathcal{W}^m$ and any $v \in V_X$, $f(v) = |X|$.

Observe that in the influence game $(G, f, n + 1, V_N)$ a coalition is winning iff its players form a winning coalition in $\Gamma$.

Given $(N, \mathcal{W}^m)$ a description of $(G, f, n + 1, N)$ can be computed in polynomial time.
Generalized opinion leader-follower model (gOLF) A gOLF is a triple $\mathcal{M} = (G, r, q)$ where
Generalized opinion leader-follower model (gOLF) A gOLF is a triple \( M = (G, r, q) \) where

- \( G = (V, E) \) is a two layered bipartite digraph.

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Opinion Leader-Follower

- **Generalized opinion leader-follower model (gOLF)** A gOLF is a triple $\mathcal{M} = (G, r, q)$ where
  - $G = (V, E)$ is a two layered bipartite digraph.
  - $V$ is divides as $L$: leaders, $F$: followers and $I$: independent.
Opinion Leader-Follower

- Generalized opinion leader-follower model (gOLF)
Opinion Leader-Follower

- **Generalized opinion leader-follower model (gOLF)**
  A gOLF is a triple $\mathcal{M} = (G, r, q)$ where
  - $G = (V, E)$ is a two layered bipartite digraph.
  - $r, 1/2 \leq r \leq 1$, is the fraction value.
  - The quota $q, 0 < q \leq n$, is the quota.
Opinion Leader-Follower

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  A gOLF is a triple $\mathcal{M} = (G, r, q)$ where
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- An initial decision vector $x \in \{0, 1\}^n$ is mapped to a final decision vector $c = c^\mathcal{M}(x)$
Opinion Leader-Follower

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- An initial decision vector $x \in \{0, 1\}^n$ is mapped to a final decision vector $c = c^\mathcal{M}(x)$ for $1 \leq i \leq n$, has

$$c_i = \begin{cases} 
1 & \text{if } |\{j \in P_G(i) \mid x_j = 1\}| \geq \lceil r \cdot |P_G(i)| \rceil \\
& \text{and } |\{j \in P_G(i) \mid x_j = 0\}| < \lceil r \cdot |P_G(i)| \rceil \\
0 & \text{if } |\{j \in P_G(i) \mid x_j = 0\}| \geq \lceil r \cdot |P_G(i)| \rceil \\
& \text{and } |\{j \in P_G(i) \mid x_j = 1\}| < \lceil r \cdot |P_G(i)| \rceil \\
x_i & \text{otherwise.}
\end{cases}$$
Opinion Leader-Follower

Generalized opinion leader-follower model (gOLF)

A gOLF is a triple \( M = (G, r, q) \) where

- \( G = (V, E) \) is a two layered bipartite digraph.
- \( r, 1/2 \leq r \leq 1 \), is the fraction value.
- The quota \( q, 0 < q \leq n \), is the quota.

An initial decision vector \( x \in \{0, 1\}^n \) is mapped to a final decision vector \( c = c^M(x) \).
Generalized opinion leader-follower model (gOLF)

A gOLF is a triple $\mathcal{M} = (G, r, q)$ where

- $G = (V, E)$ is a two layered bipartite digraph.
- $r, \frac{1}{2} \leq r \leq 1$, is the fraction value.
- The quota $q$, $0 < q \leq n$, is the quota.

An initial decision vector $x \in \{0, 1\}^n$ is mapped to a final decision vector $c = c^\mathcal{M}(x)$.

The collective decision function $C_\mathcal{M} : \{0, 1\}^n \rightarrow \{0, 1\}$ is defined as

$$C_\mathcal{M}(x) = \begin{cases} 1 & \text{if } |\{ i \in V \mid c_i(x) = 1\}| \geq q \\ 0 & \text{otherwise.} \end{cases}$$
Opinion Leader-Follower

- **Opinion leader-follower model (OLF)** (van den Brink et al. 2011)
  - $n$ must be odd and $q$ is set to $(n + 1)/2$
  - Decision follows single majority rule on final decision.

- **odd-opinion leader-follower model (odd-OLF)**
  - $r = 1/2$ and for all $i \in F$, $\delta^-(i)$ is odd.
  - Change of decision follows single majority rule on an odd set.
Let \((G, f, q, N)\) be an influence game and \(x \in \{0,1\}^n\). Compute \(F(x)\) and let \(y_i = 1\) iff \(i \in F(x)\)

- **non-oblivious influence model**
  The final decision vector is \(y\) and a collective decision is taken with quota \(q\).

- **non-oblivious influence model**
  The final decision vector considers \(y\) and if a follower \((V \setminus N)\) detects a tie on yes-no retracts to its initial decision.
Relationship among decision systems

- collective decision-making models
  - non-oblivious influence
    - gOLF
    - odd-OLF
    - OLF
  - oblivious influence
Satisfaction in influence decision systems

Introduced for OLF in (van den Brink et. al., 2011)

- Let $\mathcal{M}$ be a collective decision-making model over a set of $n$ actors.
- For an initial decision vector $x \in \{0, 1\}^n$, an actor $i$ is satisfied when $C_\mathcal{M}(x) = x_i$.
- The Satisfaction Measure of the actor $i$ corresponds to the number of initial decision vectors for which the actor is satisfied, i.e.,

$$SAT_{\mathcal{M}}(i) = |\{x \in \{0, 1\}^n | C(x) = x_i\}|.$$
Satisfaction and power indices

Let $\Gamma = (N, W)$ be a simple game.

- The **Banzhaf value** of player $i \in N$ is
  \[ Bz_\Gamma(i) = |\{X \in W | X \setminus \{i\} \notin W\}|. \]
- The **Rae index** of player $i \in N$ is
  \[ Rae_\Gamma(i) = |\{X \in W | i \in X\}| + |\{X \notin W | i \notin X\}|. \]
Satisfaction and power indices

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- The **Banzhaf value** of player $i \in N$ is
  $$BZ_\Gamma(i) = |\{X \in \mathcal{W} \mid X \setminus \{i\} \notin \mathcal{W}\}|.$$

- The **Rae index** of player $i \in N$ is
  $$RAE_\Gamma(i) = |\{X \in \mathcal{W} \mid i \in X\}| + |\{X \notin \mathcal{W} \mid i \notin X\}|.$$

- $RAE_\Gamma(i) = 2^{n-1} + BZ_\Gamma(i)$. (Dubey and Shapley 1979)
Satisfaction and power indices

Let $\Gamma = (N, W)$ be a simple game.

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  $$Rae_\Gamma(i) = 2^{n-1} + Bz_\Gamma(i). \text{ (Dubey and Shapley 1979)}$$

**Lemma**

Let $\mathcal{M}$ be a monotonic decision-making model on a set of actors $V$, for $i \in V$, $\text{SAT}_\mathcal{M}(i) = RAE_{\Gamma, \mathcal{M}}(i)$.
Satisfaction and power indices

Let $\Gamma = (N, W)$ be a simple game.

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  $$RAE_{\Gamma}(i) = 2^{n-1} + BZ_{\Gamma}(i). \text{ (Dubey and Shapley 1979)}$$

**Lemma**

Let $\mathcal{M}$ be a monotonic decision-making model on a set of actors $V$, for $i \in V$, $SAT_{\mathcal{M}}(i) = RAE_{\Gamma,\mathcal{M}}(i)$.

**Lemma**

Oblivious and non-oblivious decision models are monotonic.
Computing satisfaction: hardness

Theorem (Molinero, Riquelme, Serna 2015)
The satisfaction measure
The satisfaction problem: easy cases

Computing the Banzhaf value for influence games is #P-hard.
However, the graphs in the reduction are not bipartite.

Theorem
Computing the Satisfaction measure for odd-OLF is #P-hard.

Corollary
Computing the Satisfaction measure, the Banzhaf value or the Rae index if #P-hard for two layered bipartite oblivious and non-oblivious influence models.
Theorem (Molinero, Riquelme, Serna 2015)

Computing the Banzhaf value for influence games is \#P-hard
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Theorem

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**Theorem**

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*Computing the Satisfaction measure, the Banzhaf value or the Rae index if \#P-hard for two layered bipartite oblivious and non-oblivious influence models.*
Strong Hierarchical digraphs: Graph operations

- Disjoint union
  Given two graphs $H_1$ and $H_2$ with $V(H_1) \cap V(H_2) = \emptyset$, $H_1 + H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$.

- One layer extension
  Given a graph $H$, an a set $V' \neq \emptyset$ with $V(H) \cap V' = \emptyset$, $H \otimes V' = (V(H) \cup V', E(H) \cup \{(u, v) \mid u \in \text{FI}(H), v \in V'\}$.

$\text{FI}(H)$ is formed by all nodes with out degree 0.
$\text{LI}(H)$ is formed by all nodes with in degree 0.
In addition to leaders and follower we have mediators.
The family of strong hierarchical graphs is defined recursively as follows.

- The graph $I_a$, for $a > 0$, is a strong hierarchical graph.
- If $H_1$ and $H_2$ are disjoint strong hierarchical graphs, their disjoint union $H_1 + H_2$ is a strong hierarchical graph.
- If $H$ is a strong hierarchical graph and $V' \neq \emptyset$ is a set of vertices with $V(H) \cap V' = \emptyset$, the graph $H \otimes V'$ is a strong hierarchical graph.
Strong Hierarchical digraphs

1 - 2 - 3

4 - 5 - 6 - 7 - 8

9 - 10 - 11

12 - 13 - 14 - 15 - 16
A strong hierarchical influence graph is an influence graph $(G, f)$ where $G$ is a strong hierarchical graph.

A strong hierarchical influence game is an influence game $(G, f, q, N)$ where $G$ is a strong hierarchical graph and $N = L(G) \cup I(G)$. 
Let \((G, f, q, N)\) be an influence game. Let \(|F_k(N, G, f)|\) be the number of \(X \subseteq N\) with \(|F(X)| = k\).
Let \((G, f, q, N)\) be an influence game. Let \(|F_k(N, G, f)|\) be the number of \(X \subseteq N\) with \(|F(X)| = k\).

**Lemma**

Let \((G, f, q, N)\) be a strong hierarchical influence game, for \(1 \leq k \leq n\), the values \(|F_k(N, G, f)|\) can be computed in polynomial time.
Computing satisfaction

Let \((G, f, q, N)\) be an influence game. Let \(|F_k(N, G, f)|\) be the number of \(X \subseteq N\) with \(|F(X)| = k\).

**Lemma**

Let \((G, f, q, N)\) be a strong hierarchical influence game, for \(1 \leq k \leq n\), the values \(|F_k(N, G, f)|\) can be computed in polynomial time.

**Theorem**

The **Satisfaction** problem, for oblivious and non-oblivious models corresponding to strong hierarchical influence games, is polynomial time solvable.
Star influence systems

- A **star influence graph** is an influence graph \((G, f)\), where
  \[ V(G) = L \cup I \cup R \cup \{c\} \cup F \] and
  \[ E(G) = \{(u, c) \mid u \in L \cup R\} \cup \{(c, v) \mid v \in R \cup F\}. \]

![Star Influence Graph Example](image-url)
Star influence systems

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  \[ E(G) = \{(u, c) \mid u \in L \cup R\} \cup \{(c, v) \mid v \in R \cup F\}. \]

- A **star influence game** is a game \(\Gamma = (G, f, q, N)\), where
  \(N = L \cup R \cup I\) and \((G, f)\) is a star influence graph.
Computing the satisfaction measure

An extended star influence graph is obtained from a star influence graph \((G', f')\), by selecting one vertex \(u \in R(G')\) and adding a set of vertices \(F_u\) with label 1 and the set of edges \(\{(u, v) \mid v \in F_u\}\).
Computing the satisfaction measure

An **extended star influence graph** is obtained from a star influence graph \((G', f')\), by selecting one vertex \(u \in R(G')\) and adding a set of vertices \(F_u\) with label 1 and the set of edges \(\{(u, v) \mid v \in F_u\}\).

**Lemma**

Let \((G, f)\) be an extended star influence graph and \(N = L(G) \cup I(G) \cup R(G)\), for \(1 \leq k \leq N\), \(|F_k(N, G, f)|\) can be computed in polynomial time.
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Let \((G, f)\) be an extended star influence graph and \(N = L(G) \cup I(G) \cup R(G)\), for \(1 \leq k \leq N\), \(|F_k(N, G, f)|\) can be computed in polynomial time.

**Theorem**

*The Satisfaction* problem, for oblivious and non-oblivious models corresponding to star influence games, is polynomial time solvable.
1 Definitions, games and problems

2 IsStrong and IsProper

3 IsWeighted

4 IsInfluence

5 Decision systems

6 Conclusions
Conclusions

- We have analyzed simple, weighted and influence games.
- We have concentrated on the study of four computational problems.
- Each of the problems requires different tools for the analysis.
- We had a glimpse to decision systems and models inspired by process in social networks.
- This later aspects provide an interesting area for further research.
Contents taken from a subset of the results in

References

Further suggested reading

- H. Aziz: Algorithmic and complexity aspects of simple coalitional games
  PhD. Thesis, CS Dept. University of Warwick

- F. Riquelme: Structural and computational aspects of simple and influence games
  PhD. Thesis, CS Dept, UPC
Thanks!

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