Sensible Intersection Type Theories

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Finitary/static semantics in the form of intersection type assignments have become a paradigm for analysing the fine structure of all sorts of λ-models. The key step is the construction of a filter model isomorphic to a given λ-model. A property of great interest of filter λ-models is *sensibility*, *i.e.* the interpretation of all unsolvable terms in the least element. The flexibility of intersection type assignments derives from their parametrisation on intersection type theories. We construe intersection type theories as special meet-semilattices and show that appropriate morphisms, in the opposite category of meet-semilattices, preserve sensibility of the induced λ-models. Interestingly the set of saturated sets together with the set of λ-terms is such a meet-semilattice, thus showing that arguments based on Tait-Girards's computability amount to the construction of a morphism. We characterise two classes of intersection type theories which induce sensible filter models. The first is non-effective while the second is effective and it amounts to the generalisation of Mendler's criterion to intersection types and head normalising terms. The complete characterisation of sensible filter models however still escapes.

1 Introduction

The present paper addresses the problem of *head normalisation* for *intersection type theories*. Intersection type theories [10] were invented in Torino by the first author, together with Mario Coppo, in the late '70's of the last century of the previous millennium. As already noticed in [11] intersection types are information systems in the sense of [36]. Since then, intersection types have been widely generalised and utilised for providing useful characterisations for several classes of λ-terms, most notably weak head normalising [16], head normalising [12], normalising [12] and their persistent versions [16], strongly normalising [33], closable [25], and invertible λ-terms [39, 40]. The flexibility of intersection types lies in their correspondence with *clopen sets* in Scott's topological models of λ-calculus, which can thus be understood as models whose points are, in fact, filters of properties of programs. Intersection type theories therefore permit to express the dynamics of programs as filters of their static properties [7, 11]. This correspondence has been nicely expressed categorically as a duality in [1]. Since their introduction, intersection types have become a paradigm for expressing statically all sorts of execution properties of programming languages [8, Part III].

Intersection type theories being so flexible, which in fact is the very reason which makes them successful, are far from having a complete theory. More specifically, in the present paper we address the problem of characterising *sensible* intersection type theories, namely type theories which generate *sensible* filter models, *i.e.* models which assign only the trivial intersection type to an unsolvable term. To this end we construe intersection type theories as meet-semilattices, enriched with an arrow constructor, and

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show that appropriate morphisms in the opposite category of meet-semilattices preserve sensibility of the induced λ-models. This permits us to transfer profusely sensibility results between filter models, thus providing alternatives to the existing proofs of sensibility for many models [16]. The very set of saturated sets, together with the set of λ-terms, being such a meet-semilattice, permits us to reduce to the existence of a morphism all arguments based on Tait-Girards's computability. This is in effect a generalisation of Girard's reducibility candidates. We characterise two classes of sensible intersection type theories. The first is non-effective and it applies to the special class of $\to$-sound intersection type theories. The second is effective and it amounts to the generalisation of Mendler's criterion [30], originally given for recursive second order λ-calculus and strong normalisation, to intersection types and head normalisation. The complete characterisation of sensible filter models however still escapes.

The present paper is a follow up of [14], where the complementary problem of studying non-sensible intersection type theories was addressed. Reading both papers can be beneficial since the two papers have a number, albeit small, of cross-references.

Synopsis

In Sections 2 and 3 we recall basic facts on λ-calculus and the theory of intersection types and filter models. In Section 4 we introduce the algebraic setting of meet-semilattices and establish the transfer results of sensibility between theories and give examples. In Section 5 we characterise two classes of intersection type theories which are sensible. Difficulties in providing complete characterisation of sensible intersection type theories appear in Section 6, where we also discuss the λ-theories of sensible filter models and raise some open questions. Concluding remarks appear in Section 7.

2 λ-calculus

In this section we recall some basic notions and properties of untyped λ-calculus following Chapters 2, 3, and 8 of [6]. Readers familiar with λ-calculus can skip this subsection.

We start by defining λ-terms and β-reduction.

Definition 2.1 (λ-terms [6, Definition 2.1.1]) *The set Λ of* pure λ-terms *is defined by:*

$$M ::= x \mid \lambda x.M \mid MM.$$

We write λ-terms with the usual notational conventions. In particular we write $\lambda\overrightarrow{x}.M$ as short for $\lambda x_1 \cdots x_n.M$ assuming $\overrightarrow{x} = x_1 \cdots x_n$ for $n \in \mathbb{N}$. Free and bound occurrences of variables are defined in the standard way. In particular we assume Barendregt's convention, *i.e.* that different variables have different names [6, Convention 2.1.12]. It is handy and standard to associate names to some closed λ-terms (combinators) [6, Definition 2.1.17(ii)].

Definition 2.2 (β-rule and β-reduction [6, Definitions 2.1.15, 3.1.3 and 3.1.5])

1. *The* β-rule *replaces $(\lambda x.M)N$ with $M[x := N]$, where $M[x := N]$ denotes the λ-term obtained by the (capture free) substitution of x by N in M.*
2. *The* one step β-reduction $\to_\beta$ *is defined as the contextual closure of the β-rule.*
3. *The* β-reduction $\to^*_\beta$ *is defined as the reflexive and transitive closure of $\to_\beta$.*
4. *The* β-convertibility $=_\beta$ *is defined as the equivalence relation generated by $\to^*_\beta$.*

Crucial to our development are the notions of solvability and unsolvability of λ-terms.

Definition 2.3 (Solvable and unsolvable λ-terms [6, Definition 2.2.10])

1. *A λ-term M is* solvable *if there are n λ-terms $N_1,\ldots,N_n$ such that*

$$(\lambda\overrightarrow{x}.M)N_1\ldots N_n \to_\beta^* \mathbf{I},$$

where $\overrightarrow{x}$ are the variables which occur free in M and $\mathbf{I}=\lambda x.x$ is the identity combinator.

2. *A λ-term is* unsolvable *if it is not solvable.*

As in [26] our study of unsolvable terms is based on the notion of head reduction.

Definition 2.4 (Head normal form and head redex [6, Definition 8.3.9])

1. *If $M=\lambda\overrightarrow{x}.xM_1\cdots M_m$, then M is in* head normal form *and x is the* head variable *of M.*
2. *If $M=\lambda\overrightarrow{x}.(\lambda x.N)PM_1\cdots M_m$, then $(\lambda x.N)P$ is the* head redex *of M.*

Every λ-term either is in head normal form or has a head redex.

Proposition 2.5 (Shape of λ-terms [6, Corollary 8.3.8]) *Every λ-term is either of the form $\lambda\overrightarrow{x}.xM_1\cdots M_m$ or of the form $\lambda\overrightarrow{x}.(\lambda x.N)PM_1\cdots M_m$, where $m\geq 0$.*

Definition 2.6 (Head reduction [6, Definition 8.3.10]) *We write $M\to_h N$ if N is obtained from M by reducing its head redex. The* head reduction *of M is the finite or infinite sequence of terms M_0, $\ldots$, M_n, $\ldots$ such that $M=M_0$ and $M_n\to_h M_{n+1}$ with $n\in\mathbb{N}$.*

We use $\to_h^*$ to denote the reflexive and transitive closure of $\to_h$.

In our development we take advantage of the characterisation of unsolvability by means of head reduction.

Theorem 2.7 ([6, Fact 2.2.12]) *A λ-term M is unsolvable iff its head reduction is infinite.*

3 Intersection Types and Filter Models

This section is devoted to the definitions of intersection types, type theories, type assignment systems and filter models.

Up to Definition 3.3 (included) we essentially follow Sections 13.1 and 13.2 of [8]. The only differences are that, in defining intersection types and subtyping, we require the constant U, which is optional in [8], and our subtyping relation has the additional Axiom (U_{top}) and Rule ($\to^\sim$).

Definition 3.1 (Intersection Type Theories)

1. *Given a set of constants $\mathbb{A}$ and a distinguished constant U, the set $\mathbb{T}_\mathbb{A}$ of* intersection types *over $\mathbb{A}\cup\{\mathsf{U}\}$ is generated by the grammar:*

$$A ::= \mathsf{c} \mid \mathsf{U} \mid A\to A \mid A\cap A,$$

where $\mathsf{c}\in\mathbb{A}$.

2. *A* subtyping relation $\leq$ *is a binary relation on* $\mathbb{T}_{\mathbb{A}}$ *closed under the following axioms and rules:*

$$A \leq A \ (\text{Refl}) \qquad B \cap A \leq B \ (\text{IncL}) \qquad B \cap A \leq A \ (\text{IncR}) \qquad A \leq \mathsf{U} \ (\mathsf{U}_{top})$$

$$\frac{B \leq A \quad B \leq A'}{B \leq A \cap A'} \ (\text{Glb}) \qquad \frac{B \leq A \quad A \leq A'}{B \leq A'} \ (\text{Trans}) \qquad \frac{B' \sim B \quad A \sim A'}{B \to A \sim B' \to A'} \ (\to^{\sim})$$

3. *An* intersection type theory (itt) $\mathcal{T}$ *is determined by a set of type constants* $\mathbb{A}$ *and a subtyping relation on the set* $\mathbb{T}_{\mathbb{A}}$, i.e. $\mathcal{T} = \langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$.

We adopt the convention that $\cap$ has precedence over $\to$. We write $A \sim_{\mathcal{T}} B$ as short for $A \leq_{\mathcal{T}} B$ and $B \leq_{\mathcal{T}} A$. The above rules permit to show that $\cap$ and $\to$ are congruences w.r.t. $\sim_{\mathcal{T}}$ and moreover that $\cap$ is idempotent, commutative and associative with neutral element U. We assume that $\bigcap_{i \in \emptyset} A_i = \mathsf{U}$. We summarise this with a proposition which will be useful in Section 4.

Proposition 3.2 *The equivalence classes of an itt* $\mathcal{T}$ *w.r.t. the equivalence* $\sim_{\mathcal{T}}$ *define a meet-semilattice enriched with a binary arrow constructor.*

Definition 3.3 (Type Assignment System) *The* intersection type assignment system *induced by an itt* $\mathcal{T} = \langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$ *is a formal system deriving judgements of the shape* $\Gamma \vdash_{\mathcal{T}} M : A$*, where* $A \in \mathbb{T}_{\mathbb{A}}$ *and a* basis Γ *is a finite mapping from term variables to types in* $\mathbb{T}_{\mathbb{A}}$*:*

$$\Gamma ::= \emptyset \mid \Gamma, x : A.$$

The axioms and rules of the type system are the following

$$\frac{}{\Gamma, x : A \vdash x : A} \ (\text{Ax}) \qquad \frac{}{\Gamma \vdash M : \mathsf{U}} \ (\mathsf{U})$$

$$\frac{\Gamma, x : B \vdash M : A}{\Gamma \vdash \lambda x.M : B \to A} \ (\to\text{I}) \qquad \frac{\Gamma \vdash M : B \to A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A} \ (\to\text{E})$$

$$\frac{\Gamma \vdash M : B \quad \Gamma \vdash M : A}{\Gamma \vdash M : B \cap A} \ (\cap\text{I}) \qquad \frac{\Gamma \vdash M : B \quad B \leq_{\mathcal{T}} A}{\Gamma \vdash M : A} \ (\leq)$$

It is easy to verify that the following rules are admissible

$$\frac{\Gamma, x : B \vdash M : A \quad C \leq_{\mathcal{T}} B}{\Gamma, x : C \vdash M : A} \ (\leq\text{-L}) \qquad \frac{\Gamma \vdash M : A \quad x \notin \Gamma}{\Gamma, x : B \vdash M : A} \ (\text{Weak})$$

where $x \notin \Gamma$ is short for x does not occur in Γ.

The main properties of intersection type assignment systems are the Inversion Lemma and Subject Expansion, which are proved by induction on type derivations.

Lemma 3.4 (Inversion Lemma [8, Theorem 14.1.1])

1. *If* $\Gamma \vdash_{\mathcal{T}} x : A$ *and* $A \not\sim_{\mathcal{T}} \mathsf{U}$*, then* $\Gamma(x) \leq_{\mathcal{T}} A$*;*
2. *If* $\Gamma \vdash_{\mathcal{T}} MN : A$ *and* $A \not\sim_{\mathcal{T}} \mathsf{U}$*, then there are* I *and* B_i*,* C_i *for* $i \in I$ *such that* $\bigcap_{i \in I} C_i \leq_{\mathcal{T}} A$ *and* $\Gamma \vdash_{\mathcal{T}} M : B_i \to C_i$ *and* $\Gamma \vdash_{\mathcal{T}} N : B_i$ *for all* $i \in I$*;*

$$\mathsf{U} \sim A \to \mathsf{U}\ (\to \mathsf{U}) \qquad (B \to A) \cap (B \to A') \sim B \to A \cap A'\ \ (\to \cap)$$

$$\frac{B' \leq B \quad A \leq A'}{B \to A \leq B' \to A'}\ (\to) \qquad \frac{\mathsf{U} \leq B \to A}{\mathsf{U} \leq A}\ (\mathsf{U} \leq)$$

Figure 1: Some axioms and rules for itt's.

3. *If* $\Gamma \vdash_{\mathcal{T}} \lambda x.M : A$*, then there are* I *and* B_i*,* C_i *for* $i \in I$ *such that* $\bigcap_{i\in I}(B_i \to C_i) \leq_{\mathcal{T}} A$ *and* $\Gamma, x : B_i \vdash_{\mathcal{T}} M : C_i$ *for all* $i \in I$*.*

Theorem 3.5 (Subject Expansion [8, Corollary 14.2.5(ii)]) $M \to_{\beta} M'$ *and* $\Gamma \vdash_{\mathcal{T}} M' : A$ *imply*

$$\Gamma \vdash_{\mathcal{T}} M : A.$$

Also crucial is the property of Subject Reduction, which however holds only with a proviso.

Theorem 3.6 (Subject Reduction [8, Proposition 14.2.1(ii)])

$$M \to_{\beta} M' \textit{ and } \Gamma \vdash_{\mathcal{T}} M : A \textit{ imply } \Gamma \vdash_{\mathcal{T}} M' : A$$

if and only if

$$\Gamma \vdash_{\mathcal{T}} \lambda x.N : B \to C \textit{ implies } \Gamma, x : B \vdash_{\mathcal{T}} N : C.$$

In fact, not all type systems induced by itt's enjoy Subject Reduction. Consider $\mathcal{T}_0 = \langle\{\mathsf{c}_0, \mathsf{c}_1\}, \leq_{\mathcal{T}_0}\rangle$, where $\mathcal{T}_0$ has only the axiom $\mathsf{c}_0 \to \mathsf{c}_0 \leq \mathsf{c}_1 \to \mathsf{c}_0$, then $\vdash_{\mathcal{T}_0} \lambda x.x : \mathsf{c}_1 \to \mathsf{c}_0$, but $x : \mathsf{c}_1 \not\vdash_{\mathcal{T}_0} x : \mathsf{c}_0$. Subject Reduction fails since $y : \mathsf{c}_1 \vdash_{\mathcal{T}_0} (\lambda x.x)y : \mathsf{c}_0$, but $y : \mathsf{c}_1 \not\vdash_{\mathcal{T}_0} y : \mathsf{c}_0$.

A sufficient but not necessary condition for Subject Reduction is β-soundness.

Definition 3.7 (β-soundness [8, Definition 14.1.4]) *An itt* $\mathcal{T}$ *is* β-sound *if* $A \not\sim_{\mathcal{T}} \mathsf{U}$ *and* $\bigcap_{i\in I}(B_i \to A_i) \leq_{\mathcal{T}} B \to A$ *imply that there is* $J \subseteq I$ *such that* $B \leq_{\mathcal{T}} \bigcap_{j\in J} B_j$ *and* $\bigcap_{j\in J} A_j \leq_{\mathcal{T}} A$*.*

For example the itt $\mathcal{T}_1 = \langle\{\mathsf{c}_0, \mathsf{c}_1\}, \leq_{\mathcal{T}_1}\rangle$, where $\leq_{\mathcal{T}_1}$ has no other axioms and rules, is β-sound. In contrast, the itt $\mathcal{T}_0$ defined above is not β-sound. The itt $\mathcal{T}_0$ can be made β-sound by adding the axiom $\mathsf{c}_1 \leq \mathsf{c}_0$. Two itt's which are not β-sound but still satisfy Subject Reduction are defined in [11, 4].

Following [14] we consider two important classes of itt's.

Definition 3.8 (Set condition) *An itt* $\mathcal{T} = \langle\mathbb{A}, \leq_{\mathcal{T}}\rangle$ *satisfies the* set condition *if*

$$\bigcap_{i\in I} A_i \leq_{\mathcal{T}} B_1 \to \ldots \to B_n \to C \textit{ with } C \not\sim_{\mathcal{T}} \mathsf{U} \textit{ implies } A_j \sim_{\mathcal{T}} B_1 \to \ldots \to B_n \to D$$

for some $j \in I$ *and some* $D \not\sim_{\mathcal{T}} \mathsf{U}$ *such that* $C \cap D \sim_{\mathcal{T}} C$*.*

Definition 3.9 *Consider axioms and rules in Figure 1.*

1. *An itt is* set-like *if it satisfies the set condition and at least Axioms* $(\to \mathsf{U})$ *and* $(\to \cap)$ *hold.*
2. *An itt is* $\to$-sound *if it satisfies at least Axioms* $(\to \mathsf{U})$*,* $(\to \cap)$*, and Rules* $(\to)$*,* $(\mathsf{U} \leq)$ *hold.*

In order to discuss λ-models over itt's we recall the definition of λ-model. An *environment* in the set $\mathscr{D}$ is a total mapping from term variables to elements of $\mathscr{D}$. Let ρ range over environments. As usual, we denote by $\rho[x := \mathsf{d}]$ the environment which returns d when applied to x and $\rho(y)$ when applied to $y \neq x$.

Definition 3.10 (λ-model [8, Definition 16.1.2]) *A* λ-model *is a triple* $\langle \mathscr{D}, \cdot, [\![\ ]\!]^{\mathscr{D}} \rangle$*, where* $\cdot$ *is a binary operation on* $\mathscr{D}$ (application)*,* $[\![\ ]\!]^{\mathscr{D}}$ *is a mapping from* λ*-terms and environments in* $\mathscr{D}$ *to elements of* $\mathscr{D}$ (term interpretation)*, and* $[\![\ ]\!]^{\mathscr{D}}$ *satisfies:*

1. $[\![x]\!]^{\mathscr{D}}_{\rho} = \rho(x)$;
2. $[\![MN]\!]^{\mathscr{D}}_{\rho} = [\![M]\!]^{\mathscr{D}}_{\rho} \cdot [\![N]\!]^{\mathscr{D}}_{\rho}$;
3. $[\![\lambda x.M]\!]^{\mathscr{D}}_{\rho} = [\![\lambda y.M[x := y]]\!]^{\mathscr{D}}_{\rho}$;
4. $\forall \mathsf{d} \in \mathscr{D}. [\![M]\!]^{\mathscr{D}}_{\rho[x:=\mathsf{d}]} = [\![N]\!]^{\mathscr{D}}_{\rho[x:=\mathsf{d}]}$ *implies* $[\![\lambda x.M]\!]^{\mathscr{D}}_{\rho} = [\![\lambda x.N]\!]^{\mathscr{D}}_{\rho}$;
5. $\rho(x) = \rho'(x)$ *for all variables* x *which occur free in* M *implies* $[\![M]\!]^{\mathscr{D}}_{\rho} = [\![M]\!]^{\mathscr{D}}_{\rho'}$;
6. $[\![\lambda x.M]\!]^{\mathscr{D}}_{\rho} \cdot \mathsf{d} = [\![M]\!]^{\mathscr{D}}_{\rho[x:=\mathsf{d}]}$.

This definition of λ-model was first formulated by Hindley and Longo [22].

We can build λ-models whose domains are sets of filters of types according to the following definition.

Definition 3.11 (Filter [8, Definition 13.4.1]) *Let* $\mathcal{T} = \langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$ *be an itt and* $F \subseteq \mathbb{T}_{\mathbb{A}}$*. The set* F *is a* $\mathcal{T}$-filter *if:*

- $\mathsf{U} \in F$;
- $A, B \in F$ *imply* $A \cap B \in F$;
- $A \in F$ *and* $A \leq_{\mathcal{T}} B$ *imply* $B \in F$.

We use F and G as metavariables for filters and $\mathcal{F}_{\mathcal{T}}$ to denote the set of $\mathcal{T}$-filters. If $X \subseteq \mathbb{T}_{\mathbb{A}}$ we denote by $\uparrow_{\mathcal{T}} X$ the smallest $\mathcal{T}$-filter which contains X. If $X = \{A\}$ we use $\uparrow_{\mathcal{T}} A$ as short for $\uparrow_{\mathcal{T}} \{A\}$.

Filters can be endowed with an applicative structure as follows:

Definition 3.12 (Filter Structure) *Let* $\mathbb{E}_{\mathcal{T}}$ *be the set of environments in* $\mathcal{F}_{\mathcal{T}}$*. The* filter structure over $\mathcal{T}$ *is the triple* $\langle \mathcal{F}_{\mathcal{T}}, \cdot, [\![\ ]\!]^{\mathcal{F}_{\mathcal{T}}} \rangle$ *where*

- application*,* $\cdot : \mathcal{F}_{\mathcal{T}} \times \mathcal{F}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}$*, is defined by*

$$F \cdot G = \{A \mid \exists B \in G.\ B \to A \in F\};$$

- term interpretation*,* $[\![\ ]\!]^{\mathcal{F}_{\mathcal{T}}} : \Lambda \times \mathbb{E}_{\mathcal{T}} \to \mathcal{F}_{\mathcal{T}}$*, is defined by*

$$[\![M]\!]^{\mathcal{F}_{\mathcal{T}}}_{\rho} = \{A \in \mathbb{T}_{\mathbb{A}} \mid \exists \Gamma \models \rho.\ \Gamma \vdash_{\mathcal{T}} M : A\},$$

where ρ *ranges over* $\mathbb{E}_{\mathcal{T}}$ *and* $\Gamma \models \rho$ *if and only if* $x : A \in \Gamma$ *implies* $A \in \rho(x)$.

It is easy to verify that $[\![\ ]\!]^{\mathcal{F}_{\mathcal{T}}}$ satisfies all conditions required to be a λ-model (Definition 3.10), but the last one, which is essential when d is the interpretation of a λ-term. We always have $[\![(\lambda x.M)N]\!]^{\mathcal{F}_{\mathcal{T}}}_{\rho} \subseteq [\![M[x := N]]\!]^{\mathcal{F}_{\mathcal{T}}}_{\rho}$, since Subject Expansion always holds by Theorem 3.5.

Theorem 3.13 ([8, Proposition 16.2.4]) *The filter structure over* $\mathcal{T}$ *is a* λ*-model (dubbed* filter model*) iff*

$$[\![(\lambda x.M)N]\!]^{\mathcal{F}_{\mathcal{T}}}_{\rho} \supseteq [\![M[x := N]]\!]^{\mathcal{F}_{\mathcal{T}}}_{\rho}$$

for all λ*-terms* $M, N \in \Lambda$*, all variables* x *and all environments* ρ *in* $\mathcal{F}_{\mathcal{T}}$*.*

The condition $[\![(\lambda x.M)N]\!]^{\mathcal{F}_\mathcal{T}}_\rho \supseteq [\![M[x := N]]\!]^{\mathcal{F}_\mathcal{T}}_\rho$ means that all types of $(\lambda x.M)N$ are also types of $M[x := N]$, *i.e.* that the type system $\vdash_\mathcal{T}$ enjoys Subject Reduction. Then the following theorem follows naturally, being β-soundness a sufficient condition for Subject Reduction.

Theorem 3.14 ([8, Corollary 16.2.9(i)]) *If $\mathcal{T}$ is a β-sound itt, then the filter structure over $\mathcal{T}$ is a filter model.*

All set-like itt's generate filter models, since the set condition implies β-soundness.

As mentioned after Definition 3.7, in [11, 4] there are filter models over itt's which are not β-sound.

It is interesting to notice that all continuous functions are representable in a filter model over a β-sound itt. This generalises Theorem 2.13(iii) in [11].

Notably graph models [**?**, **?**] are isomorphic to filter models over set-like itt's and inverse limit models [**?**] are isomorphic to filter models over $\rightarrow$-sound itt's, see Example 35 in [14].

We conclude this section giving a crucial definition in this paper:

Definition 3.15 (Sensible Itt, Sensible Filter Model) *An itt $\mathcal{T}$ is* sensible *if all unsolvable terms are typed only by types equivalent to* U. *A filter model $\mathcal{F}_\mathcal{T}$ is* sensible *if $\mathcal{T}$ is sensible,* i.e. *all unsolvable terms are interpreted in the bottom filter, $\uparrow_\mathcal{T} \mathsf{U}$. Otherwise the itt and the filter model are said to be* non-sensible.

4 Transfer Theorems

To the best of our knowledge the original proofs of head-normalisation for itt's, either historically and logically, are based on three methodologies: proof-normalisation [34, 32, 37], indexed reductions [29], or Tait-Girard reducibility arguments [38, 20]. For the purposes of studying when itt's are sensible, once a given itt is proved to be sensible it is sensible to try and design a setting where this result can be transferred easily to similar itt's. To this end it appears convenient to take a more abstract, and less language-dependent, view of itt's. We therefore introduce below a notion of type structure, called *generalised intersection type theory* (*gitt*), together with a notion of morphism between such structures, which will allow for transferring directly properties, such as sensibility, between type systems. We reckon this extension satisfactory, since the very proofs by Tait-Girard reducibility will appear as transfer results from the set of reducibility candidates viewed as generalised types, as will become apparent in Theorem 4.6 and in the next Section.

Definition 4.1 (Generalised Intersection Type Theory) *A* generalised intersection type theory *(shortly gitt) is a meet-semilattice $\langle \Theta, \sqsubseteq_\Theta \rangle$ with a top $\top_\Theta$ and closed under an arrow type constructor $\leadsto_\Theta$. We denote by $\sqcap_\Theta$ the meet, by $\sim_\Theta$ the equivalence induced by $\sqsubseteq_\Theta$, and we use α, β to range over the elements of Θ.*

Proposition 3.2 shows that an itt yields naturally a gitt.

We introduce the following notion of morphism between gitt's.

Definition 4.2 (Embedding) *Let $\langle \Theta, \sqsubseteq_\Theta \rangle$ and $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$ be two gitt's, then $\langle \Theta, \sqsubseteq_\Theta \rangle$ is* embeddable *in $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$ if there is a function $\kappa : \Theta \to \Theta'$ such that:*

1. *$\kappa(\alpha) = \top_{\Theta'}$ if and only if $\alpha = \top_\Theta$;*
2. *$\kappa(\alpha \leadsto_\Theta \beta) = \kappa(\alpha) \leadsto_{\Theta'} \kappa(\beta)$;*

3. $\kappa(\alpha \sqcap_\Theta \beta) = \kappa(\alpha) \sqcap_{\Theta'} \kappa(\beta)$*;*

4. $\alpha \sqsubseteq_\Theta \beta$ *implies* $\kappa(\alpha) \sqsubseteq_{\Theta'} \kappa(\beta)$.

We can naturally extend the notion of Type Assignment Systems to gitt's as follows:

Definition 4.3 (Generalised Type Assignment System) *The* intersection type assignment system *induced by a gitt* $\langle \Theta, \sqsubseteq_\Theta \rangle$ *is a formal system deriving judgements of the shape* $\Upsilon \vdash_\Theta M : \alpha$*, where* $\alpha \in \Theta$ *and a* basis Υ *is a finite mapping from term variables to elements in* Θ*:*

$$\Upsilon ::= \emptyset \mid \Upsilon, x : \alpha.$$

The axioms and rules of the type system are the following

$$\frac{}{\Upsilon, x : \alpha \vdash x : \alpha}\ (\mathrm{Ax}) \qquad \frac{}{\Upsilon \vdash M : \top_\Theta}\ (\top)$$

$$\frac{\Upsilon, x : \beta \vdash M : \alpha}{\Upsilon \vdash \lambda x.M : \beta \rightsquigarrow_\Theta \alpha}\ (\rightsquigarrow \mathrm{I}) \qquad \frac{\Upsilon \vdash M : \beta \rightsquigarrow_\Theta \alpha \quad \Upsilon \vdash N : \beta}{\Upsilon \vdash MN : \alpha}\ (\rightsquigarrow \mathrm{E})$$

$$\frac{\Upsilon \vdash M : \beta \quad \Upsilon \vdash M : \alpha}{\Upsilon \vdash M : \beta \sqcap_\Theta \alpha}\ (\sqcap \mathrm{I}) \qquad \frac{\Upsilon \vdash M : \beta \quad \beta \sqsubseteq_\Theta \alpha}{\Upsilon \vdash M : \alpha}\ (\sqsubseteq)$$

It is now natural to extend to gitt's also the notions of filter and filter model, and then it is straightforward to extend all results on itt's in Section 3 also to gitt's.

A gitt $\langle \Theta, \sqsubseteq_\Theta \rangle$ is *sensible* if any unsolvable term has only types equivalent to $\top_\Theta$.

The following transfer theorem will prove very useful in the sequel:

Theorem 4.4 (Transfer) *Let* $\langle \Theta, \sqsubseteq_\Theta \rangle$ *be* embeddable *in* $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$*. We get:*

1. if $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$ *is sensible, then* $\langle \Theta, \sqsubseteq_\Theta \rangle$ *is sensible.*

2. if $\langle \Theta, \sqsubseteq_\Theta \rangle$ *is non-sensible, then* $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$ *is non-sensible.*

Proof. Let $\Upsilon \vdash_\Theta M : \alpha$, where M is unsolvable and $\alpha \not\sim_\Theta \top_\Theta$. Then $\kappa(\Upsilon) \vdash_{\Theta'} M : \kappa(\alpha)$, where $\kappa(\Upsilon) = \{x : \kappa(\beta) \mid x : \beta \in \Upsilon\}$ by the conditions in Definition 4.2. Condition (1) of 4.2 and $\alpha \not\sim_\Theta \top_\Theta$ imply $\kappa(\alpha) \not\sim_{\Theta'} \top_{\Theta'}$. Therefore $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$ sensible gives $\langle \Theta, \sqsubseteq_\Theta \rangle$ sensible and $\langle \Theta, \sqsubseteq_\Theta \rangle$ non-sensible gives $\langle \Theta', \sqsubseteq_{\Theta'} \rangle$ non-sensible. □

We will now construe Tait-Girard reducibility candidates as a gitt. Let $\mathscr{S}$ denote the set of solvable terms.

Definition 4.5 *A set* $X \subseteq \mathscr{S}$ *is* saturated *if it is closed under* β*-conversion and* $x\overrightarrow{M} \in X$ *for all* x, $\overrightarrow{M}$*.*

The set of saturated sets is a complete lattice w.r.t. set inclusion, with $\mathcal{B} = \{M \in \Lambda \mid M \to_\beta^* x\overrightarrow{M}\}$ as bottom and $\mathscr{S}$ as top. We use $\mathcal{SAT}$ to denote this lattice and $\mathcal{SAT}^{op}$ to denote $\mathcal{SAT}$ with the reverse order having bottom $\mathscr{S}$ and top $\mathcal{B}$. Formally

$$\mathcal{SAT} = \langle \{X \subseteq \mathscr{S} \mid X \text{ is saturated}\}, \subseteq \rangle \text{ and } \mathcal{SAT}^{op} = \langle \{X \subseteq \mathscr{S} \mid X \text{ is saturated}\}, \supseteq \rangle.$$

We define

$$X \Rightarrow Y = \{M \in \Lambda \mid \forall N \in X\ MN \in Y\},$$

where X, Y range over saturated sets and Λ. It is easy to verify that Y saturated implies $X \Rightarrow Y$ saturated and that $X \Rightarrow \Lambda = \Lambda$. Moreover X and Y saturated imply $X \cap Y$ saturated and $X \cap \Lambda = X$ for all X.

The following result is crucial.

Theorem 4.6 *The gitt $\mathscr{S}\Lambda$, namely $\langle \mathcal{SAT} \cup \{\Lambda\}, \subseteq \rangle$ with top Λ, meet $\cap$ and arrow $\Rightarrow$ is sensible.*

Proof. We use the index $\mathscr{S}\Lambda$ instead of the index $\mathcal{SAT} \cup \{\Lambda\}$. We show that $\Upsilon \vdash_{\mathscr{S}\Lambda} M : X$ with $X \not\sim_{\mathscr{S}\Lambda} \Lambda$ implies that $M \in X$ hence solvable. Let $\Upsilon = \{x_n : Y_n \mid n \in \mathbb{N}\}$ and $N_n \in Y_n$ for all $n \in \mathbb{N}$. By induction on the type derivations we can prove that $\Upsilon \vdash_{\mathscr{S}\Lambda} M : X$ with $X \not\sim_{\mathscr{S}\Lambda} \Lambda$ implies

$$M[x_n := N_n \mid n \in \mathbb{N}] \in X.$$

The most interesting case is when the last applied rule is Rule ($\rightsquigarrow$I). In this case $M = \lambda x.M'$ and $X = Y \Rightarrow X'$ and $\Upsilon, x : Y \vdash_{\mathscr{S}\Lambda} M' : X'$. Let $N \in Y$, then by induction hypothesis we have that

$$M'[x_n := N_n \mid n \in \mathbb{N}][x := N] \in X',$$

which implies $M[x_n := N_n \mid n \in \mathbb{N}]N \in X'$, since saturated sets are closed under β-conversion. Since $N \in Y$ is arbitrary by definition we conclude

$$M[x_n := N_n \mid n \in \mathbb{N}] \in Y \Rightarrow X' = X. \qquad \square$$

The rest of this Section is devoted to giving examples of how to apply Theorem 4.4, above, to some itt's and gitt's thereof. It gives a very flexible criterion which cuts both ways, since it can be used both for *reducing* the sensibility of a filter model to that of the embedded filter model, but also for *extending* the non-sensibility of the embedded filter model to that of the filter model in which it embeds.

Example 4.7

1. *The itt $\mathcal{T}_{BCD}$, defined in [7], is shown to be sensible by normalising type derivations. The not β-sound itt's defined in [11, 4] are sensible, since they can be embedded in the itt $\mathcal{T}_{BCD}$ by equating the two distinguished constants in such a way that the resulting subtyping amounts precisely to $\leq_{BCD}$.*

2. *The $\to$-sound itt $\mathcal{T}_*$ with the constant c and the axiom $\mathsf{c} \leq \mathsf{c} \to \mathsf{c}$ is defined in [3]. Similarly we can define the $\to$-sound itt $\mathcal{T}^*$ with the constant c and the axiom $\mathsf{c} \to \mathsf{c} \leq \mathsf{c}$. These itt's are sensible, since they can be appropriately embedded in the sensible itt $\mathcal{T}_{CDZ}$ defined in [13]. The itt $\mathcal{T}_{CDZ}$ has only two totally ordered constants. The embedding is realised by interpreting c as the smaller constant in the former case and as the bigger constant in the latter case.*

3. *The $\to$-sound itt $\mathcal{T}^\flat$ with the constant c and the axiom $\mathsf{c} \sim (\mathsf{c} \to \mathsf{c}) \cap \mathsf{c}$ can be again embedded in the sensible itt $\mathcal{T}_{CDZ}$ defined in 2. This embedding is realised by mapping c in the smaller constant.*

The next proposition makes it possible to build non-sensible filter models starting from sensible ones.

Proposition 4.8 *If $\mathcal{T}$ is a sensible itt we can define a non-sensible itt $\mathcal{T}'$ such that $\mathcal{T}$ is embeddable in $\mathcal{T}'$.*

Proof. Let $\mathcal{T} = \langle \mathbb{A}_{\mathcal{T}}, \leq_{\mathcal{T}} \rangle$. We define $\mathcal{T}'$ by adding a constant $\mathsf{c} \notin \mathbb{A}_{\mathcal{T}}$ and the axiom $\mathsf{c} \sim \mathsf{c} \to \mathsf{c}$. The embedding is the identity. In this way $\mathcal{T}'$ is obtained from $\mathcal{T}$ essentially by adding the itt generating the filter model isomorphic to Park model [31] defined in [25]. $\square$

This proposition permits us to build non-sensible not β-sound filter models starting from the filter models defined in [11, 4].

5 Morphisms Engineering

The power of the Transfer Theorem 4.4 in proving sensibility of itt's, or more generally gitt's, derives from the existence of appropriate embeddings in $\mathcal{SAT}$. Historically, this was done implicitly by defining appropriate type interpretations based on Tait-Girard's computability arguments in [38, 20, 13, 27, 23].

In this section, we discuss two conditions on itt's, or gitt's derived thereof, which ensure that appropriate morphisms, yielding sensibility, exist. The first condition, Definition 5.2, is not effective and it is an almost trivial reformulation of the results in the previous section. Its interest lies in that it can be reversed, Theorem 5.5, for a very large class of itt's, including inverse limit models, thus showing that $\mathcal{SAT}$ is somewhat *universal*. The second condition, Definition 5.7, is a reformulation of Mendler's condition [30] to intersection type theories, and allows for showing constructively the sensibility of many itt's.

Since most of the gitt's in this section arise from itt's, we shall reason directly on itt's.

An $\mathbb{A}$-environment is a mapping from a set of type constants, $\mathbb{A}$, into $\mathcal{SAT}$. We use $\zeta_{\mathbb{A}}$ to range over $\mathbb{A}$-environments.

Definition 5.1 (Type Interpretation) *The* type interpretation *of the set of intersection types* $\mathbb{T}_{\mathbb{A}}$ *induced by the* $\mathbb{A}$*-environment* $\zeta_{\mathbb{A}}$*, notation* $[A]_{\zeta_{\mathbb{A}}}$*, is defined by:*

$$[\mathsf{U}]_{\zeta_{\mathbb{A}}} = \Lambda \qquad [\mathsf{c}]_{\zeta_{\mathbb{A}}} = \zeta_{\mathbb{A}}(\mathsf{c}) \qquad [A \to B]_{\zeta_{\mathbb{A}}} = [A]_{\zeta_{\mathbb{A}}} \Rightarrow [B]_{\zeta_{\mathbb{A}}} \qquad [A \cap B]_{\zeta_{\mathbb{A}}} = [A]_{\zeta_{\mathbb{A}}} \cap [B]_{\zeta_{\mathbb{A}}}.$$

Notice that either $[A]_{\zeta_{\mathbb{A}}}$ is a saturated set or $[A]_{\zeta_{\mathbb{A}}} = \Lambda$.

Definition 5.2 (Saturation) *An itt* $\mathcal{T}$ *is* saturated *if there is a type interpretation which gives rise to a morphism in the sense of Definition 4.2 between the gitt induced by* $\mathcal{T}$ *and* $\mathcal{SAT}$.

It is easy to verify that all conditions of Definition 4.2 are satisfied by type interpretations, but for condition 1 which requires that Axiom $(\to \mathsf{U})$ holds in $\mathcal{T}$. Then, Theorems 4.4(1) and 4.6 immediately imply that:

Theorem 5.3 *A saturated itt is sensible.*

For $\to$-sound itt's, Theorem 5.3 can be reversed. For this we first need an easy auxiliary lemma.

Lemma 5.4 *If* $\Gamma, x : B \vdash_{\mathcal{T}} Mx : A$ *where* x *does not occur in* M *and* $\Gamma' \vdash_{\mathcal{T}} N : B$*, then* $\Gamma'' \vdash_{\mathcal{T}} MN : A$ *for some* Γ''.

Proof. Define

$$\Gamma_1 \Cup \Gamma_2 = \{y : C_1 \cap C_2 \mid y : C_1 \in \Gamma_1 \;\; y : C_2 \in \Gamma_2\} \cup \{y : C_1 \mid y : C_1 \in \Gamma_1 \;\; y \notin \Gamma_2\} \cup \{y : C_2 \mid y : C_2 \in \Gamma_2 \;\; y \notin \Gamma_1\}.$$

We can build a derivation of $\Gamma'' \vdash_{\mathcal{T}} MN : A$ just by replacing the axioms $\hat{\Gamma}, x : B \vdash_{\mathcal{T}} x : B$ with derivations of $\hat{\Gamma} \Cup \Gamma' \vdash_{\mathcal{T}} N : B$ in a derivation of $\Gamma, x : B \vdash_{\mathcal{T}} Mx : A$. □

Theorem 5.5 *Each* $\to$*-sound and sensible itt is saturated.*

Proof. Let $\mathcal{T} = \langle \mathbb{A}, \leq_{\mathcal{T}} \rangle$ be an $\to$-sound and sensible itt. Define the type interpretation

$$\widehat{\zeta_{\mathbb{A}}}(\mathsf{c}) = \{M \mid \exists \Gamma \;\; \Gamma \vdash_{\mathcal{T}} M : \mathsf{c}\}.$$

$$
\begin{array}{rcll}
[B\to A]_{\widehat{\zeta_{\mathbb{A}}}} & = & [B]_{\widehat{\zeta_{\mathbb{A}}}} \Rightarrow [A]_{\widehat{\zeta_{\mathbb{A}}}} & \text{by Definition 5.1}\\
& = & \{M \mid \forall N \in [B]_{\widehat{\zeta_{\mathbb{A}}}}\;\; MN \in [A]_{\widehat{\zeta_{\mathbb{A}}}}\} & \text{by Definition of } \Rightarrow\\
& = & \{M \mid \forall N\;\; \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} N : B\;\; \Gamma \vdash_{\mathcal{T}} MN : A\} & \text{by induction}\\
& = & \{M \mid \exists \Gamma\;\; \Gamma, x : B \vdash_{\mathcal{T}} Mx : A\} & \text{where } x \text{ is fresh by Lemma 5.4}\\
& = & \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : B_i \to A_i\;\; x : B \vdash_{\mathcal{T}} x : B_i\;\; \forall i \in I\;\; \bigcap_{i\in I} A_i \leq_{\mathcal{T}} A\} & \\
& & & \text{by Lemma 3.4(2)}\\
& = & \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : B_i \to A_i\;\; B \leq_{\mathcal{T}} B_i\;\; \forall i \in I\;\; \bigcap_{i\in I} A_i \leq_{\mathcal{T}} A\} & \\
& & & \text{by Lemma 3.4(1)}\\
& = & \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : B \to A_i\;\; \forall i \in I\;\; \bigcap_{i\in I} A_i \leq_{\mathcal{T}} A\} & \\
& & & \text{by Rule } (\leq) \text{ using Rule } (\to)\\
& = & \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : \bigcap_{i\in I}(B \to A_i)\;\; \bigcap_{i\in I} A_i \leq_{\mathcal{T}} A\} & \text{by Rule } (\cap\text{I})\\
& = & \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : B \to \bigcap_{i\in I} A_i\;\; \bigcap_{i\in I} A_i \leq_{\mathcal{T}} A\} & \\
& & & \text{by Rule } (\leq) \text{ using Axiom } (\to\cap)\\
& = & \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : B \to A\} & \text{by Rule } (\leq) \text{ using Rule } (\to).
\end{array}
$$

$$
\begin{array}{rcll}
[B\to \mathsf{U}]_{\widehat{\zeta_{\mathbb{A}}}} & = & [B]_{\widehat{\zeta_{\mathbb{A}}}} \Rightarrow [\mathsf{U}]_{\widehat{\zeta_{\mathbb{A}}}} & \text{by Definition 5.1}\\
& = & \{M \mid \forall N \in [B]_{\widehat{\zeta_{\mathbb{A}}}}\;\; MN \in [\mathsf{U}]_{\widehat{\zeta_{\mathbb{A}}}}\} & \text{by Definition of } \Rightarrow\\
& = & \{M \mid \forall N\;\; \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} N : B\;\; \Gamma \vdash_{\mathcal{T}} MN : \mathsf{U}\} & \text{by induction}\\
& = & \Lambda & \text{by Rule } (\mathsf{U})\\
& = & \{M \mid \vdash_{\mathcal{T}} M : \mathsf{U}\} & \text{by Rule } (\mathsf{U})\\
& = & \{M \mid \vdash_{\mathcal{T}} M : B \to \mathsf{U}\} & \text{by Rule } (\leq) \text{ using Axiom } (\to \mathsf{U}).
\end{array}
$$

Figure 2: Proof of 5.5.

It is enough to show now that $[A]_{\widehat{\zeta_{\mathbb{A}}}} = \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : A\}$, since all conditions of Definition 4.2 hold, and in particular $A \leq_{\mathcal{T}} B$ implies

$$\{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : A\} \subseteq \{M \mid \exists \Gamma\;\; \Gamma \vdash_{\mathcal{T}} M : B\}.$$

The proof is by induction on the definition of type interpretation. The only two interesting cases are proved in Figure 2, where we assume $A \not\sim \mathsf{U}$. □

Notice that Theorem 5.2 does not provide an effective characterisation of sensibility for $\to$-sound itt's, since the definition of $\mathbb{A}$-environment is not constructive *per se*.

The rest of this section is devoted to showing that a special class of itt's, satisfying the *consistent polarity* condition in Definition 5.7, is sensible. In any case, this class of itt's, which we call *natural*, includes essentially all sensible itt's ever used explicitly in the literature.

Definition 5.6 (Natural Itt's) *An $\mathcal{T} = \langle \{\mathsf{c}_i\}_{i\in I}, \leq_{\mathcal{T}}\rangle$ is* natural *if $\leq_{\mathcal{T}}$ satisfies some of the axioms and rules in Figure 1, possibly including the set condition, and is otherwise determined by a set of axioms of a very special form, namely $\mathscr{A} = \{\mathsf{c}_i \sim A_i\}_{i\in I}$. Moreover we assume that each type constant occurs exactly once on the left hand side of an axiom in $\mathscr{A}$, possibly vacuously as an identity. The set $\mathscr{A}$ is the* characteristic set *of $\mathcal{T}$.*

We are now in the position of giving the following crucial definition.

Definition 5.7 (Positive Polarity Condition) *A natural $\mathcal{T} = \langle \{\mathsf{c}_i\}_{i\in I}, \leq_{\mathcal{T}} \rangle$ satisfies the* positive polarity *condition if in all equations of the form $\mathsf{c} \sim A$ derivable in $\leq_{\mathcal{T}}$, from $\mathbf{Pos}(A)$ we cannot derive $\mathbf{Neg}(\mathsf{c})$ applying the following rules:*

$$\frac{\mathbf{Pos}(A \to B)}{\mathbf{Neg}(A)} \qquad \frac{\mathbf{Pos}(A \to B)}{\mathbf{Pos}(B)} \qquad \frac{\mathbf{Neg}(A \to B)}{\mathbf{Pos}(A)} \qquad \frac{\mathbf{Neg}(A \to B)}{\mathbf{Neg}(B)}$$

$$\frac{\mathbf{Pos}(A \cap B)}{\mathbf{Pos}(A)} \qquad \frac{\mathbf{Pos}(A \cap B)}{\mathbf{Pos}(B)} \qquad \frac{\mathbf{Neg}(A \cap B)}{\mathbf{Neg}(A)} \qquad \frac{\mathbf{Neg}(A \cap B)}{\mathbf{Neg}(B)}$$

It is easy to check that this condition essentially amounts to the fact that if $\mathsf{c} \sim_{\mathcal{T}} A$, the constant c does not occur in A nested inside an odd number of arrows.

From now on until the end of the section, unless otherwise stated, we will assume that itt's are natural and satisfy the positive polarity condition in Definition 5.7. Moreover for simplicity, we consider only sets of axioms in which the axioms are of the following three forms: $\mathsf{c} \sim \mathsf{c}$, or $\mathsf{c} \sim \mathsf{c}' \to \mathsf{c}''$, or $\mathsf{c} \sim \mathsf{c}' \cap \mathsf{c}''$. In fact we can always transform sets of axioms in this form by removing renamings and adding new constants and axioms to simplify the right-hand-side of the original axioms.

To prove that $\mathcal{T} = \langle \{\mathsf{c}_i\}_{i\in I}, \leq_{\mathcal{T}} \rangle$ is saturated we have to find an $\mathbb{A}$-environment $\zeta_{\mathbb{A}}$ which induces a morphism. Since $\Rightarrow$ on saturated sets is contra-variant on the domain and covariant on the co-domain, the set condition is harmless since it only allows for less set inclusions that those which hold in every type interpretation. Moreover we have the following proposition.

Proposition 5.8 *Every type interpretation satisfies the axioms and rules in Figure 1.*

Proof. We only consider two interesting cases. Rule $(\to)$ follows from the contra-variance/covariance of $\Rightarrow$. Rule $(\mathsf{U} \leq)$ follows from the fact that $X \Rightarrow Y = \Lambda$ implies $Y = \Lambda$. $\square$

The natural idea to find a type interpretation for a natural itt in $\mathcal{SAT}$, would be to define, out of the characteristic set, a monotone operator and use the fact that $\mathcal{SAT}$ is a complete lattice and hence by Knaster-Tarski's Theorem each monotone operator has a complete lattice of fixed points. But the positive polarity condition, Definition 5.7, yields only an individual constraint on each type constant, which cannot be extended uniformly. Conflicting polarities would naturally arise as in the case of the itt's in the following example.

Example 5.9 *Let $\mathscr{A} = \mathscr{A}' \cup \mathscr{A}''$, where $\mathscr{A}' = \{\mathsf{c}_1 \sim \mathsf{c}_2 \to \mathsf{c}_1, \mathsf{c}_2 \sim \mathsf{c}_1 \to \mathsf{c}_2\}$ and $\mathscr{A}'' = \{\mathsf{c}_3 \sim \mathsf{c}_4 \cap \mathsf{c}_5, \mathsf{c}_4 \sim \mathsf{c}_1 \to \mathsf{c}_3, \mathsf{c}_5 \sim \mathsf{c}_2 \to \mathsf{c}_3\}$. The axioms in $\mathscr{A}'$ require that c_1 and c_2 have opposite polarities, while the axioms in $\mathscr{A}''$ require that c_1 and c_2 have the same polarity.*

In order to be able to define an appropriate type interpretation we need therefore to introduce an appropriate order on the constants appearing in the characteristic set, so that they can be progressively dealt with. To this end we need a number of definitions.

Definition 5.10 (Completion, Closure, Equivalence Class) *Consider an itt $\mathcal{T} = \langle \{\mathsf{c}_j\}_{j\in J}, \leq_{\mathcal{T}} \rangle$ and a subset $\mathscr{A} = \{\mathsf{c}_i \sim A_i\}_{i\in I}$ of its characteristic set.*

1. *We say that axiom $\mathsf{c} \sim A$* defines *the constant c. Hence the* set of constants *defined in $\mathscr{A}$, notation $\mathscr{C}(\mathscr{A})$, is $\{\mathsf{c}_i\}_{i\in I}$.*
2. *The* completion *of a full set of axioms $\mathscr{A}$ is $\mathscr{A} \cup \{\mathsf{c} \sim \mathsf{c} \mid \mathsf{c}' \sim A \in \mathscr{A}$ & c occurs in A & $\mathsf{c} \notin \mathscr{C}(\mathscr{A})\}$. A set of axioms which coincides with its completion is* complete.
3. *Let $\mathscr{A}$ be complete and $c \in \mathscr{C}(\mathscr{A})$.*

(a) *The* closure *of* c *for* $\mathscr{A}$*, notation* $\gamma(\mathsf{c},\mathscr{A})$*, is* $\mathscr{C}(\mathscr{A}')$*, where* $\mathscr{A}'$ *is the smallest complete subset of* $\mathscr{A}$ *such that* $\mathsf{c} \in \mathscr{C}(\mathscr{A}')$*.*

(b) *The* equivalence class *of* c *for* $\mathscr{A}$*, notation* $[\mathsf{c}]^{\mathscr{A}}$*, is defined by*

$$\{\mathsf{c}' \in \mathscr{C}(\mathscr{A}) \mid \gamma(\mathsf{c},\mathscr{A}) = \gamma(\mathsf{c}',\mathscr{A})\}.$$

Clearly equivalence classes for a complete $\mathscr{A}$ induce an equivalence relation on constants parameterised on $\mathscr{A}$, namely, $\mathsf{c} \equiv_{\mathscr{A}} \mathsf{c}'$ if $[\mathsf{c}]^{\mathscr{A}} = [\mathsf{c}']^{\mathscr{A}}$. We can thus define the following relation, which is a well-defined partial order.

Definition 5.11 (Partial Order) *Let* $\mathscr{A}$ *be complete and* $\mathsf{c},\mathsf{c}' \in \mathscr{C}(\mathscr{A})$*. The* partial order *between equivalence classes for* $\mathscr{A}$ *is defined by* $[\mathsf{c}]^{\mathscr{A}} \preceq_{\mathscr{A}} [\mathsf{c}']^{\mathscr{A}}$ *if* $\gamma(\mathsf{c}',\mathscr{A}) \cap [\mathsf{c}]^{\mathscr{A}} \neq \emptyset$*.*

Example 5.12 *Let* $\mathscr{A}$*,* $\mathscr{A}'$ *and* $\mathscr{A}''$ *be as in Example 5.9. Both* $\mathscr{A}$ *and* $\mathscr{A}'$ *are complete, while* $\mathscr{A}''$ *is not. Moreover* $\gamma(\mathsf{c}_1,\mathscr{A}) = \gamma(\mathsf{c}_2,\mathscr{A}) = \mathscr{C}(\mathscr{A}')$ *and* $\gamma(\mathsf{c}_3,\mathscr{A}) = \gamma(\mathsf{c}_4,\mathscr{A}) = \gamma(\mathsf{c}_5,\mathscr{A}) = \mathscr{C}(\mathscr{A})$*. So the axioms in* $\mathscr{A}$ *define two equivalence classes:* $[\mathsf{c}_1]^{\mathscr{A}} = \mathscr{C}(\mathscr{A}')$ *and* $[\mathsf{c}_3]^{\mathscr{A}} = \{\mathsf{c}_3,\mathsf{c}_4,\mathsf{c}_5\}$*, ordered by* $[\mathsf{c}_1]^{\mathscr{A}} \preceq_{\mathscr{A}} [\mathsf{c}_3]^{\mathscr{A}}$*.*

We are now in the position of proving the main result, Theorem 5.17, namely that a natural itt whose characteristic set of axioms satisfies the positive polarity condition, in Definition 5.7, cannot type an unsolvable term. We do this in three steps.

1. We restrict to natural type theories which are *finite*. That this kind of *compactness* result is enough for dealing even with infinite sets of axioms was first noticed by Mendler [30], since all but a number of constants are ever used in any type derivation. Moreover, if the defining equation of a constant is not used in a derivation where that constant appears, then that constant can be safely taken to be equal just to itself.
2. We show how to give a type interpretation for a complete set of axioms $\mathscr{A}$ such that $\mathscr{C}(\mathscr{A})$ consists of a single equivalence class for $\mathscr{A}$, Proposition 5.15.
3. We show how to extend a given type interpretation for a complete set of axioms $\mathscr{A}$ to a type interpretation for the larger complete set of axioms $\mathscr{A}'$ such that the added constants have all the identity axiom in $\mathscr{A}'$, Proposition 5.16.

Both Propositions 5.15 and 5.16 are proved exploiting the fact that complete subsets of $\mathscr{A}$ define appropriate monotone operators on the complete lattice $\Pi_{i\in I}\mathcal{X}_i$, where $\mathcal{X}_i$ can be either $\mathcal{SAT}$ or $\mathcal{SAT}^{op}$. Then any fixed point of these operators, which we know to exist, provides the tuple of saturated sets giving rise to the $\zeta_{\mathbb{A}}$-environment which we need.

In order to define the operators we first need to decorate constants in the axioms $\mathscr{A} = \{\mathsf{c}_i \sim A_i\}_{i\in I}$ with a polarity $p \in \{+,-,\pm\}$. The intuition is that the axiom associated to a constant c^+ should define an operator that is monotone in $\mathcal{SAT}$ on the variable corresponding to that constant, and the one associated to a constant c^- should define an operator that is monotone in $\mathcal{SAT}^{op}$ on the variable corresponding to that constant. The decoration $\pm$ is used for constants whose axiom is the identity. The polarity of constants can be extended in a natural way to all types built using them.

Definition 5.13 (Polarity) *The predicates* **Pos** *and* **Neg** *on types with polarised constants are defined by:*

$$\mathbf{Pos}(\mathsf{c}^+) \quad \mathbf{Pos}(\mathsf{c}^\pm) \qquad\qquad \mathbf{Neg}(\mathsf{c}^-) \quad \mathbf{Neg}(\mathsf{c}^\pm)$$

$$\frac{\mathbf{Neg}(A) \quad \mathbf{Pos}(B)}{\mathbf{Pos}(A \to B)} \quad \frac{\mathbf{Pos}(A) \quad \mathbf{Neg}(B)}{\mathbf{Neg}(A \to B)} \qquad\qquad \frac{\mathbf{Pos}(A) \quad \mathbf{Pos}(B)}{\mathbf{Pos}(A \cap B)} \quad \frac{\mathbf{Neg}(A) \quad \mathbf{Neg}(B)}{\mathbf{Neg}(A \cap B)}$$

Definition 5.14 *A decoration of constants,* $\{\mathsf{c}_i^{p_i}\}_{i\in I}$, agrees *with a set of axioms* $\mathscr{A} = \{\mathsf{c}_i \sim A_i\}_{i\in I}$ *if* $\mathsf{c}_i^{+} \sim A_i$ *implies* $\mathbf{Pos}(A_i)$, $\mathsf{c}_i^{-} \sim A_i$ *implies* $\mathbf{Neg}(A_i)$ *and* $\mathsf{c}_i^{\pm} \sim A_i$ *implies* $A_i = \mathsf{c}_i$.

Let $\mathscr{B} = \{\tilde{\mathsf{c}}_i \sim A_i\}_{i\in I}$ be a complete set of axioms whose type constants are all in the same equivalence class for $\mathscr{B}$, and let $\{\tilde{\mathsf{c}}_i^{p_i}\}_{i\in I}$ be a decoration of the constants which agrees with $\mathscr{B}$. Let $\stackrel{\subseteq}{\supseteq}$ denote $\subseteq$ for $\mathcal{SAT}$ and $\supseteq$ for $\mathcal{SAT}^{op}$. It is easy to see that there exists a decoration, by the positive polarity condition in Definition 5.7, where moreover no constant is decorated with $\pm$. Consider the lattice $(\Pi_{i\in I}\mathcal{X}_i, \stackrel{\subseteq}{\supseteq})$ where $\mathcal{X}_i = \mathcal{SAT}$ if $p_i = +$ and $\mathcal{X}_i = \mathcal{SAT}^{op}$ if $p_i = -$ and $\stackrel{\subseteq}{\supseteq}$ is the order induced on the cartesian product by the order on its components. That is $\langle X_i \mid i\in I\rangle \stackrel{\subseteq}{\supseteq} \langle X_i' \mid i\in I\rangle$, if, for all $i\in I$, X_i and X_i' are saturated sets and $X_i \stackrel{\subseteq}{\supseteq} X_i'$. Let $\mathbb{X}$ range over variables. Define the operator associated to $\mathscr{B}$, $\mathscr{O}_{\mathscr{B}} : \Pi_{i\in I}\mathcal{X}_i \to \Pi_{i\in I}\mathcal{X}_i$, by

$$\mathscr{O}_{\mathscr{B}}(\langle \mathbb{X}_i \mid i\in I\rangle) = \langle A_i^{\star} \mid i\in I\rangle$$

where the mapping $\_^{\star}$ is defined by:

$$A^{\star} = \begin{cases} \mathbb{X}_i & \text{if } A = \tilde{\mathsf{c}}_i^{p} \\ \mathbb{X}_j \Rightarrow \mathbb{X}_k & \text{if } A = \tilde{\mathsf{c}}_j^{p} \to \tilde{\mathsf{c}}_k^{p'} \\ \mathbb{X}_j \cap \mathbb{X}_k & \text{if } A = \tilde{\mathsf{c}}_j^{p} \cap \tilde{\mathsf{c}}_k^{p'}. \end{cases}$$

Then we can easily prove

Proposition 5.15 *Let* $\mathscr{B} = \{\tilde{\mathsf{c}}_i \sim A_i\}_{i\in I}$ *be a complete set of axioms whose type constants are all in the same equivalence class for* $\mathscr{B}$*, then the operator* $\mathscr{O}_{\mathscr{B}}$ *defined above is monotone.*

Let $\mathscr{B} = \{\tilde{\mathsf{c}}_i \sim A_i\}_{i\in I} \subseteq \mathscr{A}$ and let $\mathscr{C}(\mathscr{B})$ be an equivalence class for $\mathscr{A}$ such that the constants in $\mathscr{B}$ are either defined in $\mathscr{B}$ (*i.e.* they belong to $\{\tilde{\mathsf{c}}_i\}_{i\in I}$) or they belong to $\{\tilde{\mathsf{c}}_j\}_{j\in J}$ with $I\cap J = \emptyset$ and we have already a type interpretation in $\mathcal{SAT}$ for them given by $\mathbf{X}_j$ with $j\in J$. Define $\mathscr{B}^{\mathscr{A}} = \mathscr{B} \cup \{\tilde{\mathsf{c}}_j \sim \tilde{\mathsf{c}}_j\}_{j\in J}$. It is easy to see that, by the positive polarity condition in Definition 5.7, there exists a decoration $\{\tilde{\mathsf{c}}_h^{p_h}\}_{h\in I\cup J}$ of the constants in $\mathscr{C}(\mathscr{B}^{\mathscr{A}})$ which agrees with $\mathscr{B}^{\mathscr{A}}$, giving the polarity $\pm$ to all constants in $\{\tilde{\mathsf{c}}_j\}_{j\in J}$. Consider the lattice $(\Pi_{h\in I\cup J}\mathcal{X}_h, \stackrel{\subseteq}{\supseteq})$ where $\mathcal{X}_h = \mathcal{SAT}$ if $p_h = +$ or $p_h = \pm$ and $\mathcal{X}_h = \mathcal{SAT}^{op}$ if $p_h = -$. Let $\langle X_h \mid h\in I\cup J\rangle \stackrel{\subseteq}{\supseteq} \langle X_h' \mid h\in I\cup J\rangle$ and $\mathbb{X}$ be as in previous case. Define the operator associated to $\mathscr{B}^{\mathscr{A}}$, $\mathscr{O}_{\mathscr{B}^{\mathscr{A}}} : \Pi_{h\in I\cup J}\mathcal{X}_h \to \Pi_{h\in I\cup J}\mathcal{X}_h$, by

$$\mathscr{O}_{\mathscr{B}^{\mathscr{A}}}(\langle \mathbb{X}_h \mid h\in I\cup J\rangle) = \langle A_h^{\star} \mid h\in I\cup J\rangle$$

where the mapping $\_^{\star}$ is defined by $\mathbf{X}_j$ if $A = \tilde{\mathsf{c}}_j^{\pm}$ and $\mathbf{X}_j$ is the solution for $\tilde{\mathsf{c}}_j$ with $j\in J$, and as in previous case otherwise. We easily get

Proposition 5.16 *Let* $\mathscr{B} = \{\tilde{\mathsf{c}}_i \sim A_i\}_{I\in I} \subseteq \mathscr{A}$ *and let* $\mathscr{C}(\mathscr{B})$ *consist of an equivalence class for* $\mathscr{A}$ *such that all the constants appearing in* $\mathscr{B}$ *either are in* $\mathscr{C}(\mathscr{B})$ *or are such that we already have a type interpretation for them. Then the operator* $\mathscr{O}_{\mathscr{B}^{\mathscr{A}}}$ *defined above is monotone.*

We can now prove the main result.

Theorem 5.17 *A natural itt with a possibly infinite characteristic set satisfying the condition of positive polarity, in Definition 5.7, is sensible.*

Proof. Consider a finite derivation in a natural itt $\mathcal{T}$. Without loss of generality we can restrict to the finite natural itt $\mathcal{T}'$ whose characteristic set involves only the constants actually used in that derivation, possibly assigning the identity to constants whose defining axioms have not been used in the derivation.

Now use Proposition 5.15 for one of the minimal equivalence classes, according to the partial order in Definition 5.11 on the constants in $\mathcal{T}'$, to derive a first partial type interpretation of the constants. Notice that the set of axioms defining the constants in a minimal equivalence class is complete. Use Proposition 5.16 to extend such a type interpretation to all the constants in $\mathcal{T}'$ adding incrementally an equivalence class such that the solutions for the constants not belonging to that equivalence class have already be found. Since $\mathcal{T}'$ is finite, we can always find such an equivalence class, namely one of the minimal classes in the partial order consisting of the equivalence classes which have not be yet dealt with. Finally, using Theorem 5.3 we conclude the proof. $\square$

We end this section with a few examples. The sensibility of the first theory follows directly by applying Propositions 5.15 and 5.16. The second example deals with a type theory, which was introduced in [14]. Its sensibility can be dealt with either using Theorem 5.17 or even directly taking the fixed points of a monotone operator defined on countable sequences of $\mathcal{SAT}$'s and $\mathcal{SAT}^{op}$'s. Finally, the third example deals with a theory whose sensibility, to our present knowledge, can be dealt with only using Theorem 5.17, through its finite approximations. This is somewhat puzzling because once we know that the theory is sensible, by Theorem 5.5, we can in principle define a type interpretion in $\mathcal{SAT}$.

Example 5.18

1. *Consider the axioms $\mathscr{A}$ of Example 5.9 and let $\mathbb{A} = \mathscr{C}(\mathscr{A})$.*
   - *We start from $\mathbb{A}' = [\mathsf{c}_1]^{\mathscr{A}}$, which is the minimum class of $\mathscr{A}$. Let $\langle \mathbf{X}_1, \mathbf{X}_2 \rangle$ be a fixed point of the operator*
   $$\mathscr{O}_{\mathbb{A}'} : \mathcal{SAT} \otimes \mathcal{SAT}^{op} \to \mathcal{SAT} \otimes \mathcal{SAT}^{op}$$
   defined by
   $$\mathscr{O}_{\mathbb{A}'}(\langle \mathbb{X}_1, \mathbb{X}_2 \rangle) = \langle \mathbb{X}_2 \Rightarrow \mathbb{X}_1, \mathbb{X}_1 \Rightarrow \mathbb{X}_2 \rangle.$$
   - *We then analyse the class $[\mathsf{c}_3]^{\mathscr{A}}$ taking advantage from the solutions for c_1, c_2 already computed. We take as $\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5 \rangle$ the fixed point of the operator*
   $$\mathscr{O}_{\mathbb{A}} : \mathcal{SAT} \otimes \mathcal{SAT} \otimes \mathcal{SAT} \otimes \mathcal{SAT} \otimes \mathcal{SAT} \to \mathcal{SAT} \otimes \mathcal{SAT} \otimes \mathcal{SAT} \otimes \mathcal{SAT} \otimes \mathcal{SAT}$$
   defined by
   $$\mathscr{O}_{\mathbb{A}}(\langle \mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{X}_4, \mathbb{X}_5 \rangle) = (\langle \mathbf{X}_1, \mathbf{X}_2, \mathbb{X}_4 \cap \mathbb{X}_5, \mathbb{X}_1 \Rightarrow \mathbb{X}_3, \mathbb{X}_2 \Rightarrow \mathbb{X}_3 \rangle).$$

   The sensibility of a natural itt with characteristic set $\mathscr{A}$ can be shown by taking $\zeta_{\mathbb{A}}(\mathsf{c}_i) = \mathbf{X}_i$ for $1 \leq i \leq n$.

2. *Consider the axioms $\mathscr{A}_\infty = \{\mathsf{c} \sim \mathsf{c}\} \cup \{\mathsf{c}_n \sim \mathsf{c}_{n+1} \to \mathsf{c} \mid n \in \mathtt{N}\}$ and $\mathbb{A}_\infty = \mathscr{C}(\mathscr{A}_\infty)$. The minimal class of $\mathscr{A}_\infty$ is $[\mathsf{c}]^{\mathscr{A}_\infty}$ and we can take for $\zeta_{\mathscr{A}_\infty}(\mathsf{c})$ an arbitrary saturated set, for example $\mathcal{B}$. But then there is no finite minimal equivalence class from which we can start our procedure for defining a type interpretation in $\mathcal{SAT}$. We could consider the finite approximations of a such a theory, but we can show also that a natural itt with characteristic set $\mathscr{A}_\infty$ is sensible by defining directly the operator*
   $$\mathscr{O}_{\mathbb{A}_\infty} : \mathcal{SAT} \otimes (\mathcal{SAT} \otimes \mathcal{SAT}^{op})^{\mathbb{N}} \to \mathcal{SAT} \otimes (\mathcal{SAT} \otimes \mathcal{SAT}^{op})^{\mathbb{N}}$$
   by
   $$\mathscr{O}_{\mathbb{A}_\infty}(\langle \mathbb{X} \rangle \cdot \langle \mathbb{X}_n \mid n \in \mathbb{N} \rangle) = \langle \mathcal{B} \rangle \cdot \langle \mathbb{X}_{n+1} \Rightarrow \mathbb{X} \mid n \in \mathbb{N} \rangle.$$

   A fixed point of $\mathscr{O}_{\mathbb{A}_\infty}$ exists, since it is monotone. Let $\langle \mathcal{B} \rangle \cdot \langle X_n \mid n \in \mathbb{N} \rangle$ be such a fixed point, then $\zeta_{\mathbb{A}_\infty}(\mathsf{c}) = \mathcal{B}$ and $\zeta_{\mathbb{A}_\infty}(\mathsf{c}_n) = X_n$ for $n \in \mathbb{N}$ is the $\mathbb{A}_\infty$-environment we are looking for.

3. *Consider the itt given by the set of axioms* $\{c_{0,n} \sim c_{1,n} \to c_{0,n+1}, c_{1,n} \sim c_{0,n} \to c_{1,n+1} \mid n \in \mathbb{N}\}$. *This theory can be taken to be* $\to$*-sound and can be proved to be* β*-sound. Moreover finite approximations of this theory can be used to show its sensibility using Theorem 5.5. We ignore how to define inductively an embedding of this theory in* $\mathcal{SAT}$.

6 Towards a Complete Characterisation of Sensible Itt's

Mendler in [30] studied second order λ-calculus with minimal and maximal fixed point type equations. He proved that the system is strongly normalising if and only if the fixed point equations satisfy essentially the positive polarity condition in Definition 5.7. Theorems 5.3 and 5.5 are the analogues, albeit not effective, of Mendler's result, for $\to$-sound intersection type systems and solvable terms. The positive polarity condition on intersection type theories is only a sufficient condition for sensibility. We can indeed build a type interpretation which is, or finitely approximates, an embedding into $\mathscr{S}\Lambda$, for a natural itt's whose characteristic set satisfies the positive polarity condition, but this is not a necessary condition as was the case in [30]. The intersection operator $\cap$ can, in fact, sterilises the contra-variant behaviour of the arrow constructor, as we can see in the following examples. All the itt's considered in these example are assumed to be $\to$-sound and moreover can be proved to be β-sound by induction on their subtypings.

Example 6.1 (Elimination of negative occurrences)

1. *Let* $\mathcal{T}_2$ *be the itt with constants* $\{c_0, c_1\}$ *and axiom*

$$c_0 \sim c_0 \cap c_1 \to c_0.$$

It is immediate to see that the characteristic set of $\mathcal{T}_2$ *does not satisfy the positive polarity condition, in Definition 5.7.*

Nevertheless $\mathcal{T}_2$ *can be shown to be sensible by embedding it in the itt* $\mathcal{T}_2'$ *obtained by adding the axiom* $c_1 \leq c_0$*, which gives*

$$c_0 \sim_{\mathcal{T}_2'} c_1 \to c_0$$

generating a sensible filter model by Proposition 4.4(1). Alternatively, instead of adding the axiom $c_1 \leq c_0$ *we can obtain a sensible filter model, again by Proposition 4.4(1), by adding the axiom*

$$c_1 \sim \mathsf{U} \to c_0$$

since this axiom implies $c_1 \leq c_0$ *by Rule* $(\to)$.

Notice, on the other hand, that if we add to $\mathcal{T}_2$ *the axiom* $c_0 \leq c_1$ *we get* $c_0 \sim_{\mathcal{T}_2} c_0 \to c_0$*. Then the resulting itt is non-sensible by Proposition 4.4(2), because the non-sensible itt generating the filter model isomorphic to Park model [31] defined in [25] is embeddable in it.*

2. *Let* $\mathcal{T}_3$ *be the itt with constants* $\{c_0, c_1, c_2\}$ *together with the axiom*

$$c_0 \sim c_0 \cap (c_1 \to c_2) \to c_1.$$

The itt $\mathcal{T}_{CDZ}$ *considered in Example 4.7(2) has constants* $\{c_3, c_4\}$ *and the axioms*

$$c_3 \sim c_4 \to c_3 \quad c_4 \sim c_3 \to c_4 \quad c_3 \leq c_4.$$

We can show that $\mathcal{T}_3$ *is sensible by embedding it in* $\mathcal{T}_{CDZ}$ *via the structural extension* $\hat{\iota}$ *of* ι *defined by:*

$$\iota(c_0) = c_4 \quad \iota(c_1) = c_4 \quad \iota(c_2) = c_3.$$

Now, since $\iota(\mathsf{c}_0) = \mathsf{c}_4$ *and*

$$\hat{\iota}(\mathsf{c}_0 \cap (\mathsf{c}_1 \to \mathsf{c}_2)) = \iota(\mathsf{c}_0) \cap (\iota(\mathsf{c}_1) \to \iota(\mathsf{c}_2)) = \mathsf{c}_4 \cap (\mathsf{c}_4 \to \mathsf{c}_3) \sim_{\mathcal{T}_{CDZ}} \mathsf{c}_4 \cap \mathsf{c}_3 \sim_{\mathcal{T}_{CDZ}} \mathsf{c}_3$$

which implies $\hat{\iota}(\mathsf{c}_0 \cap (\mathsf{c}_1 \to \mathsf{c}_2) \to \mathsf{c}_1) = \hat{\iota}(\mathsf{c}_0 \cap (\mathsf{c}_1 \to \mathsf{c}_2)) \to \iota(\mathsf{c}_1) \sim_{\mathcal{T}_{CDZ}} \mathsf{c}_3 \to \mathsf{c}_4 \sim_{\mathcal{T}_{CDZ}} \mathsf{c}_4$, *we have as required that*

$$\iota(\mathsf{c}_0) \sim_{\mathcal{T}_{CDZ}} \hat{\iota}(\mathsf{c}_0 \cap (\mathsf{c}_1 \to \mathsf{c}_2) \to \mathsf{c}_1).$$

Achieving an *effective* Mendler-like completeness result appears critical even for natural intersection types and solvable terms, since there are cases where the intersection operator does not prevent the contra-variant behaviour of implication to have the upper hand, as we can see in the following example.

Example 6.2 *Let* $\mathcal{T}_4$ *be the* $\to$*-sound itt with constants* $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ *and with the axiom*

$$\mathsf{c}_0 \sim \mathsf{c}_0 \cap (\mathsf{c}_1 \cap (\mathsf{c}_1 \to \mathsf{c}_2) \to \mathsf{c}_2) \to \mathsf{c}_3.$$

We can type $\Omega_2\Omega_2$ *with* c_3 *since* $\vdash_{\mathcal{T}_4} \Omega_2 : \mathsf{c}_0$ *and* $\vdash_{\mathcal{T}_4} \Omega_2 : \mathsf{c}_1 \cap (\mathsf{c}_1 \to \mathsf{c}_2) \to \mathsf{c}_2$. *The* β*-soundness of* $\mathcal{T}_4$ *can be shown by induction on* $\leq_{\mathcal{T}_4}$ *and hence* $\mathcal{T}_4$ *generates a filter model.*

Theories of Sensible Filter Models

Models give semantics. But what are semantics? In the philosophical tradition crystallised by Leibniz, ontological entities arise once we can tell them apart. So semantics are essentially congruences. Given that there is a plethora of sensible filter models, we could imagine that these would provide a corresponding plethora of semantics for λ-calculus, *i.e.* λ-theories. Formally a λ-theory is just a non-trivial congruence over λ-terms, closed under β-conversion. But this appears not to be immediately the case.

All the λ-theories of sensible filter models which we have considered in this paper appear to equate all λ-terms which have the same Böhm tree, *i.e.* their λ-theories are at least $\mathscr{B}$. We refer to [6, Chapter 16] for more details on λ-theories and Böhm trees. This is the case for the filter model isomorphic to Scott's inverse limit model [11], whose theory is the maximal sensible theory $\mathscr{H}^*$ [6, Definition 16.2.1], the filter model over $\mathcal{T}_{CDZ}$, defined in Example 4.7(2), whose theory is the weaker $\mathscr{B}\eta$ [6, Definition 4.1.4(iii)], and of course the filter model over $\mathcal{T}_{BCD}$, defined in Example 4.7(1), whose theory is $\mathscr{B}$ [6, Definition 16.4.1].

We have not yet found any filter models whose theory is weaker than $\mathscr{B}$. It appears that as soon as we separate two terms with the same Böhm tree, we break the sensibility of the model. But there are many filter models which we know to be sensible and for which we have not yet characterised the λ-theory they define, for instance the filter model generated by the gitt $\mathscr{S}\Lambda$ in Section 4. On the other hand there are itt's which we do not know if they are sensible or not, but which separate fixed points. An interesting example which separates Turing's and Church's fixed point, which we ignore if it is sensible, is the itt given by the only axiom

$$\mathsf{c} \cap (\mathsf{c} \to A) \to B \leq \mathsf{c} \cap (\mathsf{c} \to A),$$

where c is a constant and A and B are arbitrary types such that $B \leq A$.

The minimal sensible theory is $\mathscr{H}$ [6, Definition 4.1.6(ii)]. It is an intriguing open problem whether this theory is precisely the theory of some filter model, or whether filter models have hitherto unknown semantical implications. We hope that this paper will stimulate readers to taking up this intriguing open question, which parallels for sensible theories the open question discussed in [**?**] for general λ-theories.

We conclude this subsection with a last open question. There are two notions of sensible filter model. There is a weaker notion that expresses that all unsolvable terms are equated, and there is a stronger notion that expresses that all unsolvables are equated in the bottom element of the filter model. We ignore if they are equivalent.

7 Related Work and Conclusion

Since the invention in the late seventies, intersection types have revolutionised the approach to semantics of functional programming languages in multiple ways. Firstly, intersection types have reversed the traditional understanding of the relation of specifications to programs, justifying the correctness-oriented approach to program construction. Namely, we should use the specifications themselves to construct a program which meets them, rather than try to prove that an existant program is correct. This has been expressed categorically as a duality, see Abramsky [1], or by means of pointless topology [35]. Secondly, intersection types have made explicit the connection between static and dynamic semantics, namely, the former semantics provides a finitary approximation of the latter. Thirdly, intersection types have allowed for static specifications of a plethora of interesting classes of λ-terms [16]. But, more generally, intersection types have provided, in the past half century, the paradigm for expressing and studying all sorts of semantics of programming languages ranging from quantitative semantics [18, 9, 2, 5] to qualitative semantics [11, 1], from games [24, 17, 19] to power series [21], and for all sorts of domains.

Among the vast number of presentations available today of intersection type theories, in this paper we have built upon the recent comprehensive discussion of filter models and unsolvable terms, which appears in [14]. Actually, the present paper is a counterpart to that paper in that we discuss *sub specie typorum intersectionibus*, sensible filter models or, what is its syntactic analogue, *head normalising* terms.

Intersection type theories are very flexible and hence expressive, but this makes them also rather difficult to classify exhaustively. For instance the nice characterisation given by Mendler [30], of recursive second order type theories which type only strongly normalising terms, cannot be paralleled in the context of itt's and head normalising terms. There are plenty of itt's which do not satisfy any straightforward polarity criterion but nonetheless type non-trivially only head normalising terms. In [15] we argue that this is the case even for intersection-free axioms, contradicting blatantly the simple minded analogue of Mendler's condition. *E.g.*, the theory with the single axiom $c \sim (((c \to c_0) \to c_1) \to c_2$ for c_0, c_1, c_2 generic constants, can type only head-normalising terms [15].

In this paper, we construe itt's as special meet-semilattices and show that morphisms in the opposite category of meet-semilattices preserve sensibility, see Theorem 4.4(1). Moreover we show that the meet-semilattice $\mathcal{SAT}$ is *universal* in the sense that an $\to$-sound itt types non-trivially only head-normalising terms if and only if it can be embedded, as a meet-semilattice in it, see Theorems 5.3 and 5.5. We provide a number of techniques for putting this result into action and give various examples. An immediate consequence is that sensibility transfers transitively in the op-category. Thus once we have a sensible itt, this can play the role of $\mathcal{SAT}$, and sensibility can be easily transferred to all itt's which embed in it. Lacking suitable sensible itt's, we need to define a direct morphism between an itt and $\mathcal{SAT}$. This can be achieved for a large class of natural itt's whose characteristic axioms satisfy a positive polarity condition. This condition essentially amounts to the condition introduced by Mendler in [30] for second order λ-calculus. Thus, by repeatedly solving fixed point equations in $\mathcal{SAT}$, which is a complete lattice, we can prove Theorem 5.17, which amounts to the "if" part of Mendler's result.

Providing a syntactical effective criterion for determining if an itt is sensible does not appear feasible, however, since intersections can produce rather unanticipated consequences, already in natural itt's. See the examples in Example 6.1. We have not studied itt's whose axioms are not equivalences or both whose sides are types.

In conclusion we have explored what was a "seasoned" problem area and provided some advancement both in terms of conjectures and in terms of results.

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